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# A Hilbert-type Inequality with a non-homogeneous and the Integral in Whole Plane

**Zitian Xie**[1] **and Zheng Zeng**[2]

**Abstract**

By the way of weight function, this paper gives a new Hilbert-type inequality with the integral in whole plane and with some parameters, which is an extension of Hilbert-type inequality with a non-homogeneous, and gives its equivalent form. Also in the paper, by the way of Complex Analysis, the best constant factor is calculated.

**Mathematics Subject Classification:** 26D15
**Keywords:** Hilbert-type integral inequality, weight function, Hölder's inequality.

## 1 Introduction

If $f(x)$, $g(x) \geq 0$, such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$, then [1]

$$\int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x+y} dxdy < \pi \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{1/2}, \qquad (1.1)$$

[1] e-mail: gdzqxzt@163.com
[2] e-mail: zz@sgu.edu.cn

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where the constant factor $\pi$ is the best possible. In 1925, Hardy give an extension of (1.1) by introducing one pair of conjugates (p,q),(i.e.$\frac{1}{p}+\frac{1}{q}=1$) : If $p>1, \frac{1}{p}+\frac{1}{q}=1$, $f(x)$, $g(x)\geq 0$, such that

$$0<\int_0^\infty f^p(x)dx<\infty, \quad \text{and} \quad 0<\int_0^\infty g^q(x)dx<\infty,$$

then we have the following Hardy-Hilbert's integral inequality:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy<\frac{\pi}{\sin(\pi/p)}\left\{\int_0^\infty f^p(x)dx\right\}^{1/p}\left\{\int_0^\infty g^q(x)dx\right\}^{1/q}, \quad (1.2)$$

where the constant factor $\dfrac{\pi}{\sin(\pi/p)}$ also is the best possible.

In recent years,by introducing some parameters and estimating the way of weight function,inequalities (1.1)and(1.2) have many generalizations and variants. (1.1) has been strengthened by Yang and others. ( including double series inequalities ) [2-12].

In 2008 Xie gave a new Hilbert-type Inequality [2] as follows :
If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $f(x), g(x)\geq 0$ such that

$$0<\int_0^\infty x^{-1-p/2}f^p(x)dx<\infty, \quad \text{and} \quad 0<\int_0^\infty x^{-1-q/2}g^q(x)dx<\infty,$$

then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+a^2y)(x+b^2y)(x+c^2y)}dxdy$$
$$<K\left\{\int_0^\infty x^{-1-p/2}f^p(x)dx\right\}^{1/p}\left\{\int_0^\infty x^{-1-q/2}g^q(x)dx\right\}^{1/q}, \quad (1.3)$$

where the constant factor $K=\frac{\pi}{(a+b)(a+c)(b+c)}$ is the best possible.

In this paper, by using the way of weight function and the technic of real analysis and by the way of complex analysis, a new Hilbert-type inequality with the integral in whole plane is given.

In the following, we always suppose that

$$a=\cos\theta, \ \ \theta\in(0,\frac{\pi}{2}); \ \ p>1, \ \ 1/p+1/q=1, \ \ r\in(-3,1).$$

# 2 Some lemmas

**Lemma 2.1.** *If*

$$k_1 := \int_0^\infty \frac{dt}{(t^2+2at+1)^2 t^r}, \quad k_2 := \int_0^\infty \frac{dt}{(t^2-2at+1)^2 t^r},$$

*and*

$$k := \int_{-\infty}^\infty \frac{dt}{(t^2+2at+1)^2 |t|^r},$$

*then*

$$k_1 = k(\theta, r) = \begin{cases} \frac{\pi[\sin r\theta - r\sin\theta\cos(1+r)\theta]}{2\sin^3\theta\sin r\pi}, & if\ r \neq -2, -1, 0, \\ \frac{2\theta - sin2\theta}{4\sin^3\theta}, & if\ r = -2,\ and\ \alpha = 0, \\ \frac{\sin\theta - \theta\cos\theta}{2\sin^3\theta}, & if\ r = -1. \end{cases} \tag{2.1}$$

$$k_2 = k(\pi-\theta, r) = \begin{cases} \frac{\pi[\sin r(\pi-\theta) - r\sin\theta\cos(1+r)(\pi-\theta)]}{2\sin^3\theta\sin r\pi}, & if\ r \neq -2, -1, 0, \\ \frac{2(\pi-\theta)+sin2\theta}{4\sin^3\theta}, & if\ r = -2,\ and\ r = 0, \\ \frac{\sin\theta + (\pi-\theta)\cos\theta}{2\sin^3\theta}, & if\ r = -1. \end{cases} \tag{2.2}$$

*and*

$$k = k_1 + k_2 = \begin{cases} \frac{\pi[\cos r(\frac{\pi}{2}-\theta) + r\sin\theta\cos(\frac{\pi}{2}-\theta)(1+r)]}{2\sin^3\theta\cos\frac{r\pi}{2}}, & if\ r \neq -1, \\ \frac{2\sin\theta + (\pi-2\theta)\cos\theta}{2\sin^3\theta}, & if\ r = -1, \end{cases}$$

**Proof.** Let

$$f(z) = \frac{1}{(1+2bz+z^2)^2 z^r} = \frac{1}{(z-z_1)^2(z-z_2)^2 z^r},$$

then

$$k^* := \int_0^\infty \frac{dt}{(t^2+2bt+1)^2 t^r} = \frac{2\pi i}{1-e^{-2r\pi i}}[Res(f, z_1) + Res(f, z_2)];$$

if $b = \cos\vartheta$ $(0 < \vartheta < \pi)$, and $z_1 = -e^{i\vartheta}$, $z_2 = -e^{-i\vartheta}$, then

$$Res(f, z_1) = \left(\frac{1}{(z-z_2)^2 z^r}\right)'|_{z=z_1} = \frac{1}{(z_1-z_2)^2}\left[\frac{-2}{(z_1-z_2)z^r} - \frac{r}{z_1^{r+1}}\right]$$

$$\begin{aligned} k^* &= \frac{2\pi i}{1-e^{-2r\pi i}} \frac{1}{(-2i\sin\vartheta)^2}\left[\frac{-2}{(-2i\sin\vartheta)(-e^{i\vartheta})^r} + \frac{-2}{(2i\sin\vartheta)(-e^{-i\vartheta})^r}\right. \\ &\quad \left. - \frac{r}{(-e^{i\vartheta})^{r+1}} - \frac{r}{(-e^{-i\vartheta})^{r+1}}\right] \\ &= \begin{cases} \frac{\pi[\sin r\vartheta - r\sin\vartheta\cos(1+r)\vartheta}{2\sin^3\vartheta\sin r\pi}, & \text{if } r \neq -2, -1, 0, \\ \frac{2\vartheta - sin2\vartheta}{4\sin^3\vartheta}, & \text{if } r = -2, \text{and } r = 0, \\ \frac{\sin\vartheta - \vartheta\cos\vartheta}{2\sin^3\vartheta}, & \text{if } r = -1. \end{cases} \end{aligned}$$

Setting $\vartheta=\theta$ and $\vartheta=\pi-\theta$, we have (2.1)and (2.2).

On the other hand,

$$\begin{aligned}
k &= \int_0^\infty \frac{dt}{(t^2+2at+1)^2t^r}+\int_{-\infty}^0 \frac{dt}{(t^2+2at+1)^2(-t)^r}\\
&= k_1+k_2=k(\theta,r)+k(\pi-\theta,r)\\
&= \begin{cases} \frac{\pi[\sin\frac{r\pi}{2}\cos r(\frac{\pi}{2}-\theta)-r\sin\theta\cos\frac{\pi(1+r)}{2}\cos(\frac{\pi}{2}-\theta)(1+r)]}{\sin^3\theta\sin r\pi}, \text{if } r\neq -2,-1,0,\\ \frac{\pi}{2\sin^3\theta}, \quad \text{if } r=-2, \text{and } r=0,\\ \frac{2\sin\theta+(\pi-2\theta)\cos\theta}{2\sin^3\theta}, \quad \text{if } r=-1, \end{cases}\\
&= \begin{cases} \frac{\pi[\cos r(\frac{\pi}{2}-\theta)+r\sin\theta\cos(\frac{\pi}{2}-\theta)(1+r)]}{2\sin^3\theta\cos\frac{r\pi}{2}}, \text{if } r\neq -1,\\ \frac{2\sin\theta+(\pi-2\theta)\cos\theta}{2\sin^3\theta}, \quad \text{if } r=-1, \end{cases}
\end{aligned}$$

The lemma is proved. □

**Lemma 2.2.** *Define the weight functions as follow:*

$$w(x):=\int_{-\infty}^\infty \frac{|x|^{-r+1}}{|y|^r}\frac{1}{(x^2y^2+2xy\cos\theta+1)^2}dy,$$
$$\widetilde{w}(y):=\int_{-\infty}^\infty \frac{|y|^{-r+1}}{|x|^r}\frac{1}{(x^2y^2+2xy\cos\theta+1)^2}dx.$$

*then* $w(x)=\widetilde{w}(y)=k.$ (2.3)

**Proof.** We only prove that $w(x)=k$, for $x\in(-\infty,0)$.

$$\begin{aligned}
w(x) &= \int_{-\infty}^0 \frac{|x|^{-r+1}}{|y|^r}\frac{1}{(x^2y^2+2xy\cos\theta+1)^2}dy\\
&\quad+\int_0^\infty \frac{|x|^{-r+1}}{|y|^r}\frac{1}{(x^2y^2+2xy\cos\theta+1)^2}dy\\
&= \int_{-\infty}^0 \frac{(-x)^{-r+1}}{(-y)^r}\frac{1}{(x^2y^2+2xy\cos\theta+1)^2}dy\\
&\quad+\int_0^\infty \frac{(-x)^{-r+1}}{y^r}\frac{1}{(x^2y^2+2xy\cos\theta+1)^2}dy := w_1+w_2,
\end{aligned}$$

Setting $u = xy$, then

$$w_1 = \int_{-\infty}^{0} \frac{(-x)^{-r+1}}{(-y)^r} \frac{1}{(x^2y^2 + 2xy\cos\theta + 1)^2} dy = \int_0^{\infty} u^{-r} \frac{1}{(u^2 + 2u\cos\theta + 1)^2} du = k_1.$$

Similarly, setting $y = -tx$,

$$w_2 = \int_0^{\infty} u^{-r} \frac{1}{(u^2 - 2u\cos\theta + 1)^2} du = k_2,$$

and

$$w(x) = k_1 + k_2 = k.$$

We have (2.3). □

**Lemma 2.3.** *For* $0 < \varepsilon$, *and* $(r + \max\{\frac{2\varepsilon}{p}, \frac{2\varepsilon}{q}\}) \in (-3, 1)$, *define both functions,* $\widetilde{f}, \widetilde{g}$ *as follow:*

$$\widetilde{f}(x) = \begin{cases} x^{-r-2\varepsilon/p}, & \text{if } x \in (1, \infty), \\ 0, & \text{if } x \in [-1, 1], \\ (-x)^{-r-2\varepsilon/p}, & \text{if } x \in (-\infty, -1); \end{cases}$$

$$\widetilde{g}(x) = \begin{cases} 0, & \text{if } x \in (1, \infty), \\ x^{-r+2\varepsilon/q}, & \text{if } x \in [-1, 1], \\ 0, & \text{if } x \in (-\infty, -1), \end{cases}$$

*then*

$$I(\varepsilon) := 2\varepsilon \left\{ \int_{-\infty}^{\infty} |x|^{pr-1} \widetilde{f}^p(x) dx \right\}^{1/p} \left\{ \int_{-\infty}^{\infty} |x|^{qr-1} \widetilde{g}^q(x) dx \right\}^{1/q} = 1; \quad (2.4)$$

$$\widetilde{I}(\varepsilon) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widetilde{f}(x)\widetilde{g}(y) \frac{1}{(x^2y^2 + 2xy\cos\theta + 1)^2} dxdy$$
$$= k + o(1).(for\ \varepsilon \to 0^+) \quad (2.5)$$

**Proof.** Easily,

$$I(\varepsilon) = 2\varepsilon \left\{ \int_1^{\infty} x^{-1} x^{-2\varepsilon} dx \right\}^{1/p} \left\{ \int_0^{1} x^{-1} x^{-2\varepsilon} dx \right\}^{1/q} = 1;$$

Let $y = -Y$ ,using $\widetilde{f}(-x) = \widetilde{f}(x)$, $\widetilde{g}(-x) = \widetilde{g}(x)$, and

$$\widetilde{f}(-x) \int_{-\infty}^{\infty} \widetilde{g}(y) \frac{1}{(x^2y^2 - 2xy\cos\theta + 1)^2} dy$$
$$= \widetilde{f}(x) \int_{-\infty}^{\infty} \widetilde{g}(Y) \frac{1}{(x^2Y^2 + 2xY\cos\theta + 1)^2} dY$$

we have that $\widetilde{f}(x)\int_{-\infty}^{\infty}\widetilde{g}(y)\frac{1}{(x^2y^2+2xy\cos\theta+1)^2}dy$ is an even function,then

$$\begin{aligned}
\widetilde{I}(\varepsilon) &= 2\varepsilon\int_0^{\infty}\widetilde{f}(x)\left(\int_{-\infty}^{\infty}\widetilde{g}(y)\frac{1}{(x^2y^2+2xy\cos\theta+1)^2}dy\right)dx\\
&= 2\varepsilon\left[\int_1^{\infty}x^{-r-\frac{2\varepsilon}{p}}\left(\int_{-1}^{0}(-y)^{-r+\frac{2\varepsilon}{q}}\frac{1}{(x^2y^2+2xy\cos\theta+1)^2}dy\right)dx\right.\\
&\qquad\left.+\int_1^{\infty}x^{-r-\frac{2\varepsilon}{p}}\left(\int_0^1 y^{-r+\frac{2\varepsilon}{q}}\frac{1}{(x^2y^2+2xy\cos\theta+1)^2}dy\right)dx\right]\\
&:= I_1+I_2.
\end{aligned}$$

Setting $y=u/x$, then

$$\begin{aligned}
I_1 &= 2\varepsilon\int_1^{\infty}x^{-r-\frac{2\varepsilon}{p}}\left(\int_0^1 y^{-r+\frac{2\varepsilon}{q}}\frac{1}{(x^2y^2-2xy\cos\theta+1)^2}dy\right)dx\\
&= 2\varepsilon\int_1^{\infty}x^{-1-2\varepsilon}\left(\int_0^x u^{-r+\frac{2\varepsilon}{q}}\frac{1}{(u^2-2u\cos\theta+1)^2}du\right)dx\\
&= 2\varepsilon\left[\int_1^{\infty}x^{-1-2\varepsilon}\left(\int_0^1 u^{-r+\frac{2\varepsilon}{q}}\frac{1}{(u^2-2u\cos\theta+1)^2}du\right)dx+\right.\\
&\qquad\left.+\int_1^{\infty}x^{-1-2\varepsilon}\left(\int_1^x u^{-r+\frac{2\varepsilon}{q}}\frac{1}{(u^2-2u\cos\theta+1)^2}du\right)dx\right]\\
&= \int_0^1 u^{-r+\frac{2\varepsilon}{q}}\frac{1}{(u^2-2u\cos\theta+1)^2}du\\
&\qquad+2\varepsilon\int_1^{\infty}u^{-r+\frac{2\varepsilon}{q}}\frac{1}{(u^2-2u\cos\theta+1)^2}du\left(\int_u^{\infty}x^{-1-2\varepsilon}dx\right)du\\
&= \int_0^1 u^{-r+\frac{2\varepsilon}{q}}\frac{1}{(u^2-2u\cos\theta+1)^2}du+\int_1^{\infty}u^{-r-\frac{2\varepsilon}{p}}\frac{1}{(u^2-2u\cos\theta+1)^2}du\\
&= \int_0^{\infty}u^{-r-\frac{2\varepsilon}{p}}\frac{1}{(u^2-2u\cos\theta+1)^2}du+\int_0^1(u^{\frac{2\varepsilon}{q}}-u^{-\frac{2\varepsilon}{p}})u^{-r}\frac{1}{(u^2-2u\cos\theta+1)^2}\\
&= \int_0^{\infty}u^{-r-\frac{2\varepsilon}{p}}\frac{1}{(u^2+2u\cos(\pi-\theta)+1)^2}du\\
&\qquad+\int_0^1(u^{\frac{2\varepsilon}{q}}-u^{-\frac{2\varepsilon}{p}})u^{-r}\frac{1}{(u^2-2u\cos\theta+1)^2}\\
&= k(\pi-\theta,r+\frac{2\varepsilon}{p})+\eta(\varepsilon)
\end{aligned}$$

there $\lim_{\varepsilon\to 0^+}\eta(\varepsilon)=0$, and we have $I_1\to k_2$ (for $\varepsilon\to 0^+$)

Similarly $I_2\to k_1$ (for $\varepsilon\to 0^+$). The lemma is proved. □

**Lemma 2.4.** *If* $0 < \int_{-\infty}^{\infty} |x|^{pr-1} f^p(x)dx < \infty$,*we have*

$$J := \int_{-\infty}^{\infty} |y|^{-pr+p-1} \left( \int_{-\infty}^{\infty} f(x) \frac{1}{(x^2y^2 + 2xy\cos\theta + 1)^2} dx \right)^p dy$$
$$\leq k^p \int_{-\infty}^{\infty} |x|^{pr-1} f^p(x)dx. \tag{2.6}$$

**Proof.** By Lemma 2.2, we find

$$\begin{aligned}
&\left( \int_{-\infty}^{\infty} f(x) \frac{1}{(x^2y^2 + 2xy\cos\theta + 1)^2} dx \right)^p \\
&= \left[ \int_{-\infty}^{\infty} \frac{1}{(x^2y^2 + 2xy\cos\theta + 1)^2} \left( \frac{|x|^{r/q}}{|y|^{r/p}} f(x) \right) \left( \frac{|y|^{r/p}}{|x|^{r/q}} \right) dx \right]^p \\
&\leq \int_{-\infty}^{\infty} \frac{1}{(x^2y^2 + 2xy\cos\theta + 1)^2} \frac{|x|^{r(p-1)}}{|y|^r} f^p(x)dx \\
&\quad \times \left( \int_{-\infty}^{\infty} \frac{1}{(x^2y^2 + 2xy\cos\theta + 1)^2} \frac{|y|^{r(q-1)}}{|x|^r} dx \right)^{p-1} \\
&= k^{p-1} |y|^{pr-p+1} \int_{-\infty}^{\infty} \frac{1}{(x^2y^2 + 2xy\cos\theta + 1)^2} \frac{|x|^{r(p-1)}}{|y|^r} f^p(x)dx,
\end{aligned}$$

$$\begin{aligned}
J &\leq k^{p-1} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \frac{1}{(x^2y^2 + 2xy\cos\theta + 1)^2} \frac{|x|^{r(p-1)}}{|y|^r} f^p(x)dx \right] dy \\
&= k^{p-1} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \frac{1}{(x^2y^2 + 2xy\cos\theta + 1)^2} \frac{|x|^{r(p-1)}}{|y|^r} dy \right] f^p(x)dx \\
&= k^p \int_{-\infty}^{\infty} |x|^{pr-1} f^p(x)dx. \qquad (2.7)
\end{aligned}$$

□

# 3 Main results

**Theorem 3.1.** *If both functions,* $f(x)$ *and* $g(x)$ *, are nonnegative measurable functions, and satisfy*

$$0 < \int_{-\infty}^{\infty} |x|^{pr-1} f^p(x)dx < \infty \quad and \quad 0 < \int_{-\infty}^{\infty} |x|^{qr-1} g^q(x)dx < \infty,$$

*then,*

$$I^* := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)g(y) \frac{1}{(x^2y^2 + 2xy\cos\theta + 1)^2} dxdy$$

$$< k\left(\int_{-\infty}^{\infty}|x|^{pr-1}f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{qr-1}g^q(x)dx\right)^{1/q}, \qquad (3.1)$$

*and*

$$J=\int_{-\infty}^{\infty}|y|^{-pr+p-1}(\int_{-\infty}^{\infty}f(x)\frac{1}{(x^2y^2+2xy\cos\theta+1)^2}dx)^p dy$$
$$< k^p\int_{-\infty}^{\infty}|x|^{pr-1}f^p(x)dx. \qquad (3.2)$$

*Inequalities (3.1) and (3.2) are equivalent, and the constant factors in the two forms are all the best possible.*

**Proof.** If (2.7) takes the form of equality for some $y\in(-\infty,0)\cup(0,\infty)$ , then there exist constants $M$ and $N$, such that they are not all zero, and

$$M\frac{|x|^{r(p-1)}}{|y|^r}f^p(x),\times(-\infty,\infty).$$

Hence, there exists a constant $C$, such that

$$M|x|^{rp}f^p(x)=N|y|^{rq}=C, \qquad \text{a.e. in} \quad (-\infty,\infty).$$

We claim that $M=0$. In fact, if $M\neq 0$, then $|x|^{pr-1}f^p(x)=\frac{C}{M|x|}$ a.e. in $(-\infty,\infty)$ which contradicts the fact that $0<\int_{-\infty}^{\infty}|x|^{pr-1}f^p(x)dx<\infty$. In the same way, we claim that $N=0$. This is too a contradiction and hence by (2.7), we have (3.2).

By Hölder's inequality with weight and (3.2), we have,

$$I^* = \int_{-\infty}^{\infty}\left[|y|^{-r+\frac{1}{q}}\int_{-\infty}^{\infty}f(x)\frac{1}{(x^2y^2+2xy\cos\theta+1)^2}dx\right]\left[|y|^{r-\frac{1}{q}}g(y)\right]dy$$
$$\leq (J)^{\frac{1}{p}}\left(\int_{-\infty}^{\infty}|y|^{qr-1}g^q(y)dy\right)^{1/q}. \qquad (3.3)$$

Using (3.2), we have (3.1).

Setting

$$g(y)=|y|^{-pr+p-1}(\int_{-\infty}^{\infty}f(x)\frac{1}{(x^2y^2+2xy\cos\theta+1)^2}dx)^{p-1},$$

then

$$J=\int_{-\infty}^{\infty}|y|^{qr-1}g^q(y)dy$$

by (2.7) we have $J<\infty$.

If $J = 0$ then (3.2) is proved; if $0 < J < \infty$, by (3.1), we obtain

$$\begin{aligned} 0 &< \int_{-\infty}^{\infty} |y|^{qr-1} g^q(y)dy = J = I^* \\ &< k\left(\int_{-\infty}^{\infty} |x|^{pr-1} f^p(x)dx\right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{qr-1} g^q(x)dx\right)^{1/q}, \end{aligned}$$

$$\left(\int_{-\infty}^{\infty} |x|^{qr-1} g^q(x)dx\right)^{1/p} = J^{1/p} < k\left(\int_{-\infty}^{\infty} |x|^{pr-1} f^p(x)dx\right)^{1/p}.$$

Inequalities (3.1) and (3.2) are equivalent.

If the constant factor $k$ in (3.1) is not the best possible, then there exists a positive $h$ (with $h < k$), such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)\frac{1}{(x^2y^2 + 2xy\cos\theta + 1)^2}dxdy$$

$$< h\left(\int_{-\infty}^{\infty} |x|^{pr-1} f^p(x)dx\right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{qr-1} g^q(x)dx\right)^{1/q}. \tag{3.4}$$

For $\varepsilon > 0$, by (3.4), using Lemma 2.3, we have

$$\widetilde{I}(\varepsilon) = k + o(1)$$

$$< \varepsilon h\left(\int_{-\infty}^{\infty} |x|^{pr-1} \widetilde{f}^p(x)dx\right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{qr-1} \widetilde{g}^q(x)dx\right)^{1/q} = h. \tag{3.5}$$

Hence we find, $k + o(1) < h$. For $\varepsilon \to 0^+$, it follows that $k \le h$, which contradicts the fact that $h < k$ . Hence the constant $k$ in (3.1) is the best possible. As (3.1) and (3.2) are equivalent, if the constant factor in (3.2) is not the best possible, then by using (3.2), we can get a contradiction that the constant factor in (3.1) is not the best possible. Thus we complete the prove of the theorem. □

**Remark 3.2.** For $\theta = \frac{\pi}{4}$, in (3.1), we have the following particular result:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)\frac{1}{(x^2y^2 + \sqrt{2}xy + 1)^2}dxdy < \frac{\sqrt{2}\pi[\sin\frac{r\pi}{4} + \frac{\sqrt{2}r}{2}\cos\frac{(1+r)\pi}{4}]}{cos\frac{\pi r}{4}}$$

$$\left(\int_{-\infty}^{\infty} |x|^{pr-1} f^p(x)dx\right)^{1/p} \times \left(\int_{-\infty}^{\infty} |x|^{qr-1} g^q(x)dx\right)^{1/q}. \tag{3.6}$$

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