Theoretical Mathematics & Applications, vol.7, no.4, 2017, 35-52
ISSN: 1792-9687 (print), 1792-9709 (online)
Scienpress Ltd, 2017

Multifractal analysis of local entropies for amenable group actions

Yunping Wang[1] **and Cao Jie**[2]

Abstract

In this paper, we give the multifractal analysis of the weighted local entropies for arbitrary invariant measures for amenable group actions.

Mathematics Subject Classification: 37A35
Keywords: amenable group actions; multifractal analysis

1 Introduction and statement of main result

Let (X, d, T) be a dynamical system, where (X, d) is a compact metric space and $T : X \to X$ is a continuous map. The set $M(X)$ of all Borel probability measures is compact under the weak* topology. Denote by $M(X,T) \subset M(X)$ the subset of all T-invariant measures and $E(X,T) \subset M(X,T)$ the subset

[1] School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, Jiangsu, P.R.China. E-mail: yunpingwangj@126.com
[2] School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, Jiangsu, P.R.China. E-mail: wypcaojie@163.com

Article Info: *Received* : November 2, 2017. *Revised* : November 30, 2017.
Published online : December 1, 2017.

of all ergodic measures. Multifractal analysis is concerned with the study of pointwise dimension of a Borel measure μ (provided the limit exists):

$$d_\mu(x) = \lim_{\epsilon \to 0} \frac{\log \mu(B(x,\epsilon))}{\log \epsilon},$$

where $B(x,\epsilon)$ is an open ϵ-neighborhood of x. Set

$$X_\alpha := \{x \in X : d_\mu(x) = \alpha\}.$$

The purpose is to describe the set X_α. It is worthwhile to mention that the multifractal analysis of Birkhoff average is closely related to the pointwise dimension of the Borel measure. We refer the reader to the references [4, 14, 20, 21, 22]. Here, we introduce the general form of Pesin's multifractal formalism in [12], or [2] as follows. Consider a function $g : Y \to [-\infty, +\infty]$ in a subset Y of X. The level set

$$K^g_\alpha = \{x \in Y : g(x) = \alpha\}$$

are pairwise disjoint, and we obtain a $multifractal\ decomposition$ of X given by

$$X = (X \setminus Y) \cup \bigcup_{\alpha \in [-\infty,+\infty]} K^g_\alpha.$$

Let G be a function defined in the set of subsets of X. The $multifractal\ spectrum$:

$\mathcal{F} : [-\infty, +\infty] \to \mathbb{R}$ of the pair (g, G) is defined by

$$\mathcal{F}(\alpha) = G(K^g_\alpha),$$

where g may denote the Birkhoff averages, Lyapunov exponents, pointwise dimension or local entropies and G may denote the topological entropy, topological pressure or Hausdorff dimension.

Let (X, G) be a $G-$action topological dynamical system, where X is a compact metric space with metric d and G a topological group. In this paper, we assume G is a discrete countable amenable group. Recall that a group G is *amenable* if it admits a left invariant mean(a state on $\ell^\infty(G)$ which is invariant under left translation by G). This is equivalent to the existence of a sequence of finite subsets $\{F_n\}$ of G which are asymptotically invariant, i.e.,

$$\lim_{n\to+\infty}\frac{|F_n \vartriangle gF_n|}{|F_n|}=0, \text{ for all } g\in G.$$

Such sequences are called Følner sequences. For the detail of amenable group actions, one may refer to Ornstein and Weisss pioneering paper [11].

The topological entropy of (X, G) is defined in the following way.

Let $\mathcal{U}$ be an open cover of X, the topological entropy of $\mathcal{U}$ is

$$h_{top}(G,\mathcal{U}) = \lim_{n\to+\infty}\frac{1}{|F_n|}\log N\big(\mathcal{U}_{F_n}\big),$$

where $\mathcal{U}_{F_n} = \bigvee_{g\in F_n} g^{-1}\mathcal{U}$ and $N(\alpha)$ denote the number of sets in a finite subcover of α with smallest cardinality. It is shown that $h_{top}(G,\mathcal{U})$ is not dependent on the choice of the Følner sequences $\{F_n\}$. And the topological entropy of (X, G) is

$$h_{top}(X,G) = \sup_{\mathcal{U}} h_{top}(G,\mathcal{U}),$$

where the supremum is taken over all the open covers of X.

Bowen [1] introduced a definition of topological entropy on subsets inspired by Hausdorff dimension. For an amenable group action dynamical system (X, G), we define the Bowen topological entropy in the following way.

Let $\{F_n\}$ be a Følner sequence in G and $\mathcal{U}$ be a finite open cover of X. Denote $diam(\mathcal{U}) := \max\{diam(U) : U \in \mathcal{U}\}$. For $n \geq 1$ we denote by $\mathcal{W}_{F_n}(\mathcal{U})$ the collection of families $\mathbf{U} = \{U_g\}_{g\in F_n}$ with $U_g \in \mathcal{U}$. For $\mathbf{U} \in \mathcal{W}_{F_n}(\mathcal{U})$ we call the integer $m(\mathbf{U}) = |F_n|$ the length of $\mathbf{U}$ and define

$$\begin{aligned} X(\mathbf{U}) &= \bigcap_{g\in F_n} g^{-1}U_g \\ &= \{x \in X : gx \in U_g \text{ for } g \in F_n\}. \end{aligned}$$

For $Z \subset X$, we say that $\Lambda \subset \bigcup_{n\geq 1} \mathcal{W}_{F_n}(\mathcal{U})$ covers Z if $\bigcup_{\mathbf{U}\in\Lambda} X(\mathbf{U}) \supset Z$. For $s \in \mathbf{R}$, define

$$\mathcal{M}(Z,\mathcal{U}, N, s, \{F_n\}) = \inf_{\Lambda}\{\sum_{\mathbf{U}\in\Lambda}\exp(-sm(\mathbf{U}))\}$$

and the infimum is taken over all $\Lambda \subset \bigcup_{j\geq N} \mathcal{W}_{F_j}(\mathcal{U})$ that covers Z. We note that $\mathcal{M}(\cdot,\mathcal{U}, N, s, \{F_n\})$ is a finite outer measure on X, and

$$\mathcal{M}(Z,\mathcal{U}, N, s, \{F_n\}) = \inf\{\mathcal{M}(C,\mathcal{U}, N, s, \{F_n\}) : C \text{ is an open set that contains } Z\}.$$

$\mathcal{M}(Z,\mathcal{U},N,s,\{F_n\})$ increases as N increases. Define

$$\mathcal{M}(Z,\mathcal{U},s,\{F_n\}) = \lim_{N\to+\infty} \mathcal{M}(Z,\mathcal{U},N,s,\{F_n\})$$

and

$$\begin{aligned} h^B_{top}(\{F_n\},Z,\mathcal{U}) &= \inf\{s : \mathcal{M}(Z,\mathcal{U},s,\{F_n\}) = 0\} \\ &= \sup\{s : \mathcal{M}(Z,\mathcal{U},s,\{F_n\}) = +\infty\}. \end{aligned}$$

Set

$$h^B_{top}(\{F_n\},Z) = \sup_{\mathcal{U}} h^B_{top}(\{F_n\},Z,\mathcal{U}),$$

where $\mathcal{U}$ runs over all finite open covers of Z. We call $h^B_{top}(\{F_n\},Z)$ the Bowen topological entropy of (X,G) restricted to Z or the Bowen topological entropy of Z(w.r.t. the Følner sequence $\{F_n\}$).

Similar to the Bowen topological entropy of subsets for $\mathbb{Z}-$actions(see, for example, Pesin [12]), it is easy to show that

$$h^B_{top}(\{F_n\},Z) = \lim_{diam(\mathcal{U})\to 0} h^B_{top}(\{F_n\},Z,\mathcal{U}).$$

So the Bowen topological entropy can be defined in an alternative way.

For a finite subset F in G, we denote by

$$\begin{aligned} B_F(x,\epsilon) &= \{y\in X : d_F(x,y) < \epsilon\} \\ &= \{y \in X : d(gx,gy) < \epsilon,\ \text{for any } g\in F\}. \end{aligned} \tag{1}$$

Definition 1.1. *For $Z\subseteq X, s\geq 0, N\in \mathbf{N}$, $\{F_n\}$ a Følner sequence in G and $\epsilon>0$, define*

$$\mathcal{M}(Z,N,\epsilon,s,\{F_n\}) = \inf \sum_i \exp(-s|F_{n_i}|),$$

where the infimum is taken over all finite or countable families $\{B_{F_{n_i}}(x_i,\epsilon)\}$ such that $x_i\in X, n_i\geq N$ and $\bigcup_i B_{F_{n_i}}(x_i,\epsilon)\supseteq Z$. The quantity $\mathcal{M}(Z,N,\epsilon,s,\{F_n\})$ does not decrease as N increases and ϵ decreases, hence the following limits exists:

$$\begin{aligned} &\mathcal{M}(Z,\epsilon,s,\{F_n\}) = \lim_{N\to+\infty} \mathcal{M}(Z,N,\epsilon,s,\{F_n\}), \\ &\mathcal{M}(Z,s,\{F_n\}) = \lim_{\epsilon\to 0} \mathcal{M}(Z,\epsilon,s,\{F_n\}). \end{aligned}$$

Bowen topological entropy $h_{top}^B(Z,\{F_n\})$ *can be equivalently defined as the critical value of the parameter* s*, where* $\mathcal{M}(Z,s,\{F_n\})$ *jumps from* $+\infty$ *to* 0*, i.e.,*

$$\mathcal{M}(Z,s,\{F_n\}) = \begin{cases} 0, & s > h_{top}^B(Z,\{F_n\}), \\ +\infty, & s < h_{top}^B(Z,\{F_n\}). \end{cases}$$

In [1] Bowen showed that $h_{top}(X,T) = h_{top}^B(X,T)$ for any compact metric dynamical system (X,T). A Følner sequence $\{F_n\}$ in G is said to be *tempered* (see Shulman [17]) if there exists a constant C which is independent of n such that

$$|\bigcup_{k<n} F_k^{-1}F_n| \leq C|F_n|, \text{ for any } n. \tag{2}$$

In Lindenstrauss [6], (2) is also called **Shulman Condition**.

The increasing condition

$$\lim_{n\to+\infty} \frac{|F_n|}{\log n} = \infty. \tag{3}$$

In [23], the authors prove Brin-Katok's entropy formula [3] for amenable group action dynamical systems. The statement of this formula is the following.

Theorem 1.2 (Brin-Katok's entropy formula: ergodic case). *Let* (X,G) *be a compact metric* $G-$*action topological dynamical system and* G *a discrete countable amenable group. Let* μ *be a* $G-$*ergodic Borel probability measure on* X *and* $\{F_n\}$ *a tempered Følner sequence in* G *with the increasing condition* (3)*, then for* μ *almost everywhere* $x \in X$*,*

$$\begin{aligned} &\lim_{\delta\to 0}\liminf_{n\to+\infty} -\frac{1}{|F_n|}\log\mu(B_{F_n}(x,\delta)) \\ =&\lim_{\delta\to 0}\limsup_{n\to+\infty} -\frac{1}{|F_n|}\log\mu(B_{F_n}(x,\delta)) = h_\mu(X,G). \end{aligned}$$

Since this formula gives an alternative definition for metric entropy(known as local entropy), we give the following definition of local entropy in amenable group action case.

Definition 1.3. *Let* (X,G) *be a compact metric* $G-$*action topological dynamical system and* G *a discrete countable amenable group. Denote by* $M(X)$

the collection of Borel probability measures on X. For any $\mu \in M(X)$, $x \in X, n \in \mathbf{N}$, $\epsilon > 0$ and $\{F_n\}$ any Følner sequence in G, denote by

$$\underline{h}_\mu^{loc}(x,\epsilon,\{F_n\}) = \liminf_{n\to+\infty} -\frac{1}{|F_n|}\log\mu(B_{F_n}(x,\epsilon)).$$

Then the (lower) local entropy of μ at x (along $\{F_n\}$) is defined by

$$\underline{h}_\mu^{loc}(x,\{F_n\}) = \lim_{\epsilon\to 0}\underline{h}_\mu^{loc}(x,\epsilon,\{F_n\})$$

and the (lower) local entropy of μ is defined by

$$\underline{h}_\mu^{loc}(\{F_n\}) = \int_X \underline{h}_\mu^{loc}(x,\{F_n\})d\mu.$$

Similarly, we can define the upper local entropy.

In this case the common value will be denoted by

$$h_\mu^{loc}(x,\{F_n\}) := \underline{h}_\mu^{loc}(x,\{F_n\}) = \overline{h}_\mu^{loc}(x,\{F_n\}).$$

And then, for any G-invariant Borel probability measure μ, and $\alpha \geq 0$, define

$$\widehat{K}_\alpha(\mu) = \{x \in X : h_\mu^{loc}(x,\{F_n\}) = \alpha\}.$$

In [20], Takens and Verbitski defined the (q,μ)-entropy $h_\mu(T,q,\cdot)$ by extending the definition of generalized Hausdorff dimension $\dim_\mu^q(\cdot)$ and showed the following formula

$$h_{top}(\widehat{K}_\alpha(\mu)) = q\alpha + h_\mu(T,q,\widehat{K}_\alpha(\mu)),$$

where $h_{top}(\cdot)$ denotes the topological entropy. Later, in 2007, Yan and Chen [15] considered the multifractal spectra associated with Poincaré recurrences and established an exact formula on multifractal spectrum of local entropies for recurrence time.

2 Preliminary Notes

Let $\mu \in M(X_1,T_1)$ be an invariant Borel measure. For $\alpha \geq 0$, define

$$K_\alpha(\mu) = \{x \in X_1 : h_\mu^{loc}(x,\{F_n\}) = \alpha\}.$$

In this paper, we are interested in local entropies and spectra associated for amenable group actions, we study the size of the set $K_\alpha(\mu)$.

Next, we will try to give our result by defining the weighted (G,q,t)-energy. Let μ be an invariant non-atomic Borel measure. Without loss of generality we may assume that μ is positive on any non-empty open set. For any at most countable collection $\mathcal{G}=\{B_{\{F_n\}}(x,\epsilon)\}$, any $q,t\in\mathbb{R}$ define the (G,q,t)-free energy of $\mathcal{G}$ by

$$F_\mu(\mathcal{G},q,t)=\sum_{B_{F_n}(x,\epsilon)\in\mathcal{G}}\mu(B_{F_n}(x,\epsilon))^q\exp(-t|F_n|).$$

For any given set $Z\subset X_1, Z\neq\emptyset$, and numbers $q,t\in\mathbb{R},\epsilon>0,N\in\mathbb{N}$, put

$$M_{\mu,c}(Z,q,t,\epsilon,N)=\inf_{\mathcal{G}}F_\mu(\mathcal{G},q,t)$$

where the infimum is taken over all finite or countable collections $\mathcal{G}=\{B_{F_{n_i}}(x_i,\epsilon)\}$ with $x_i\in Z$ and $n_i\geq N$ such that $Z\subset\bigcup_{B_{F_n}(x,\epsilon)\in\mathcal{G}}B_{F_{n_i}}(x_i,\epsilon)$. To complete the definition, we assume that

$$M_{\mu,c}(\emptyset,q,t,\epsilon,N)=0$$

for any q,t,ϵ and N. The quantities $M^{\mathbf{a}}_{\mu,c}(Z,q,t,\epsilon,N)$ are non-decreasing in N, hence the following limit exists:

$$M_{\mu,c}(Z,q,t,\epsilon)=\lim_{N\to\infty}M_{\mu,c}(Z,q,t,\epsilon,N)=\sup_{N>1}M_{\mu,c}(Z,q,t,\epsilon,N).$$

Since we consider covers with centers in a given set, the qualities $M_{\mu,c}(Z,q,t,\epsilon)$ are not necessarily monotonic with respect to the set Z. We enforce monotonicity by putting

$$M_\mu(Z,q,t,\epsilon)=\sup_{Z'\subset Z}M_{\mu,c}(Z',q,t,\epsilon).$$

We now state (without proof) some basic facts. And these are standard proofs of Hausdorff dimension type and similar to the properties of topological entropy in [1], topological pressure in [13].

Lemma 2.1. *For any $t\in\mathbb{R}$ the set function $M_\mu(Z,q,t,\epsilon)$ has the following properties:*

(1) $M_\mu(\emptyset,q,t,\epsilon)=0$;

(2) $M_\mu(Z_1,q,t,\epsilon)\leq M_\mu(Z_2,q,t,\epsilon)$ *for any* $Z_1\subset Z_2$;

(3) $M_\mu(\bigcup_{i=1}^{\infty}Z_i,q,t,\epsilon)\leq\sum_{i=1}^{\infty}M_\mu(Z_i,q,t,\epsilon)$ *for any* $Z_i\subset X, i=1,2,\cdots$.

Remark 2.1 It is easily to check that $M_\mu(\cdot, q, t, \epsilon)$ is an outer measure. And $M_\mu(Z, q, t, \epsilon)$ plays a similar role with the $\mathcal{M}(Z, \epsilon, s)$ in Definition 1.1.

Lemma 2.2. *There exists a critical value $h_\mu(\{F_n\}, q, Z, \epsilon) \in [-\infty, \infty]$ such that*

$$M_\mu(Z, q, t, \epsilon) = \begin{cases} 0 & if\ t > h_\mu(\{F_n\}, q, Z, \epsilon) \\ \infty & if\ t < h_\mu(\{F_n\}, q, Z, \epsilon). \end{cases}$$

Lemma 2.3. *The following holds:*

(1) $h_\mu(\{F_n\}, q, \emptyset, \epsilon) = -\infty$;
(2) $h_\mu(\{F_n\}, q, Z_1, \epsilon) \leq h_\mu(\{F_n\}, q, Z_2, \epsilon)$ *for* $Z_1 \subset Z_2$;
(3) $h_\mu(\{F_n\}, q, \bigcup_{i=1}^{\infty} Z_i, \epsilon) = \sup_i h_\mu(\{F_n\}, q, Z_i, \epsilon)$ *where* $Z_i \subset X_1, i = 1, 2, \cdots$.

Definition 2.4. *The $(\{F_n\}, q, \mu)$-entropy of Z is*

$$h_\mu(\{F_n\}, q, Z) = \limsup_{\epsilon \to 0} h_\mu(\{F_n\}, q, Z, \epsilon).$$

Similar to Lemma 2.3, we state (without proof) some basic properties of $h_\mu(G, q, \cdot)$.

Proposition 2.5. *The following holds:*

(1) $h_\mu(\{F_n\}, q, \emptyset) = -\infty$;
(2) $h_\mu(\{F_n\}, q, Z_1) \leq h_\mu(\{F_n\}, q, Z_2)$ *for* $Z_1 \subset Z_2$;
(3) $h_\mu(\{F_n\}, q, \bigcup_{i=1}^{\infty} Z_i) = \sup_i h_\mu(\{F_n\}, q, Z_i)$ *where* $Z_i \subset X_1, i = 1, 2, \cdots$.

In this paper, we will prove

Theorem 2.6. *Let μ be a non-atomic G-invariant measure and positive on any non-empty open set. For any $\alpha \geq 0$ and every $q \in \mathbb{R}$, we have*

$$h_{top}^B(\{F_n\}, K_\alpha(\mu)) = q\alpha + h_\mu(\{F_n\}, q, K_\alpha(\mu)).$$

3 Main Results

Proposition 3.1 Let μ be non-atomic G-invariant measure and positive on any non-empty open set. For any subset $Z \subset X$ one has $h_\mu(\{F_n\}, 0, Z) = h^B_{top}(\{F_n\}, Z)$.

Proof. If $Z = \emptyset$, the statement is obvious, since both sides are equal to $-\infty$. Suppose that $Z \neq \emptyset$, we start by showing $h_\mu(\{F_n\}, 0, Z) \geq h^B_{top}(\{F_n\}, Z)$. Let $\mathcal{U}$ be an open cover of X and choose any $\epsilon < \dfrac{\gamma(\mathcal{U})}{2}$ with $\gamma(\mathcal{U})$ denotes the Lebesgue number of $\mathcal{U}$. Consider an arbitrary collection $\mathcal{G} = \{B_{F_{n_i}}(x_i, \epsilon)\}$ with $n_i > N$ such that $x_i \in Z$ and $Z \subset \bigcup_{B_{F_{n_i}}(x_i,\epsilon)\in\mathcal{G}} B_{F_{n_i}}(x_i, \epsilon)$. For the fixed $\mathcal{U}$, we can choose $\mathbf{U}_{n_i} \in \mathcal{W}_{F_{n_i}}$ such that $B_{F_{n_i}}(x_i, \epsilon) \subset \mathbf{U}_{n_i}$. Let $\Gamma_\mathcal{G} = \{\mathbf{U}_{n_i}\}$. Obviously, $\Gamma_\mathcal{G}$ covers Z and

$$F_\mu(\mathcal{G}, 0, t) = \sum_{B_{F_{n_i}}(x_i,\epsilon)\in\mathcal{G}} \exp(-t|F_{n_i}|) = \sum_{\mathbf{U}_{n_i}\in\Gamma_\mathcal{G}} \exp(-t|F_{n_i}|).$$

Since $\mathcal{G}$ is arbitrary, we conclude that

$$M_{\mu,c}(Z, 0, t, \epsilon, N) = \inf_{\mathcal{G}} F_\mu(\mathcal{G}, 0, t) \geq \mathcal{M}(Z, \mathcal{U}, t, N).$$

Taking limits as $N \to \infty$,

$$\mathcal{M}(Z, \mathcal{U}, t) \leq M_{\mu,c}(Z, 0, t, \epsilon) \leq M_\mu(Z, 0, t, \epsilon).$$

Therefore,

$$h^B_{top}(\{F_n\}, Z, \mathcal{U}) \leq h_\mu(\{F_n\}, 0, Z, \epsilon)$$

for any $\epsilon < \dfrac{\gamma(\mathcal{U})}{2}$. Let $\epsilon \to 0$, we have

$$h^B_{top}(\{F_n\}, Z, \{\mathcal{U}_i\}^k_{i=1}) \leq \limsup_{\epsilon\to 0} h_\mu(\{F_n\}, 0, Z, \epsilon) = h_\mu(\{F_n\}, 0, Z),$$

which yields that

$$h^B_{top}(\{F_n\}, Z) \leq h_\mu(\{F_n\}, 0, Z).$$

Let us now show the opposite inequality. Assume that

$$h_\mu(\{F_n\}, 0, Z) - h^B_{top}(\{F_n\}, Z) > 3\gamma > 0.$$

Then there exists $\epsilon > 0$ such that

$$h_\mu(\{F_n\}, 0, Z, \epsilon) - h^B_{top}(\{F_n\}, Z) > 2\gamma.$$

By definition of topological entropy, there exists an open cover $\mathcal{U}$ with $\mathrm{diam}(\mathcal{U}) < \epsilon$ such that

$$h_\mu(\{F_n\}, 0, Z, \epsilon) - h^B(\{F_n\}, Z, \mathcal{U}) > \gamma. \tag{4}$$

Let Z' be an arbitrary subset of Z and $\Gamma = \{\mathbf{U}_{n_i}\}$ be an arbitrary collection of strings covering Z'. We may assume that $\mathbf{U}_{n_i} \cap Z' \neq \emptyset$ for $\mathbf{U}_{n_i} \in \Gamma$. Otherwise we just delete those strings and obtain a smaller collection of strings, which still covers Z'. For any $\mathbf{U}_{n_i} \in \Gamma$, we choose an arbitrary $x_{\mathbf{U}_{n_i}} \in \mathbf{U}_{n_i} \cap Z'$. Thus,

$$x_{\mathbf{U}_{n_i}} \in \mathbf{U}_{n_i} \subset B_{F_{n_i}}(x_{\mathbf{U}_{n_i}}, \epsilon).$$

Therefore, the collection $\mathcal{G} = \{B_{F_n}(x_{\mathbf{U}_{n_i}}, \epsilon)\}$ is a centered cover of Z'. ¿From the definition of weighted free energies, we obtain

$$M_{\mu,c}(Z', 0, s, \epsilon) \leq \mathcal{M}(Z', \mathcal{U}, s)$$

for any $s \in \mathbb{R}$. Furthermore,

$$M_\mu(Z, 0, s, \epsilon) = \sup_{Z' \subset Z} M_{\mu,c}(Z', 0, s, \epsilon) \leq \mathcal{M}(Z, \mathcal{U}, s).$$

The last inequality holds due to the monotonicity of $\mathcal{M}(., \mathcal{U}, s)$ with respect to the first argument. Finally, we get $h_\mu(\{F_n\}, 0, Z, \epsilon) \leq h(\{F_n\}, Z, \mathcal{U})$ which is contradicted with (4). □

Remark 3.1. If $q = 0$, Theorem 2.6 can be showed by Proposition 3.1 easily. We will prove Theorem 2.6 for each $q \in \mathbb{R}$ in next section.

Proof of Theorem 2.6 Consider $\alpha \geq 0$ and the corresponding level set

$$\begin{aligned} K_\alpha(\mu) &= \{x \in X_1 : \ h_\mu^{loc}(x, \{F_n\}) = \alpha\} \\ &= \{x \in X_1 : \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{-\log \mu(B_{F_n}^{\mathbf{a}}(x, \epsilon))}{|F_n|} \\ &= \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{-\log \mu(B_{F_n}^{\mathbf{a}}(x, \epsilon))}{|F_n|} = \alpha\}. \end{aligned}$$

Choose some monotonic sequence $\epsilon_M \to 0$ as $M \to \infty$ and this sequence will be fixed for the rest of this section. Let $\delta > 0$ and put

$$K_{\alpha,M} = \Big\{x \in K_\alpha(\mu) : \alpha - \delta < \liminf_{n \to \infty} \frac{-\log \mu(B_{F_n}(x, \epsilon_M))}{|F_n|}\Big\}.$$

Obviously, $K_{\alpha,M} \subset K_{\alpha,M+1}$ and $K_\alpha(\mu) = \bigcup_{M=1}^{\infty} K_{\alpha,M}$. Due to the monotonicity of $\dfrac{-\log \mu(B_{F_n}(x,\epsilon))}{|F_n|}$ with respect to ϵ, for each $x \in K_\alpha(\mu)$ and every $\epsilon > 0$ one has

$$\limsup_{n\to\infty} \frac{-\log \mu(B_{F_n}(x,\epsilon))}{|F_n|} \leq \alpha.$$

Fix $x \in K_{\alpha,M}$, there exists $N_0 = N_0(x,\delta,\epsilon_M)$ such that

$$\alpha - \delta < \frac{-\log \mu(B_{F_n}(x,\epsilon_M))}{|F_n|} \leq \alpha + \delta$$

for all $n \geq N_0$. Put

$$K_{\alpha,M,N} = \{x \in K_{\alpha,M} : N_0 = N_0(x,\delta,\epsilon_M) < N\}.$$

Again, it is easy to see that $K_{\alpha,M,N} \subset K_{\alpha,M,N+1}$ and $K_{\alpha,M} = \bigcup_{N=1}^{\infty} K_{\alpha,M,N}$. Using the properties of weighted topological entropy, we conclude that

$$h_{top}^B(\{F_n\}, K_\alpha(\mu), \mathcal{U}) = \lim_{M\to\infty} \lim_{N\to\infty} h_{top}^B(\{F_n\}, K_{\alpha,M,N}, \mathcal{U}).$$

Lemma 3.1. *Suppose $\mathcal{U}$ is an open cover respect to X. Consider $K_{\alpha,M,N}$ for some $M, N \in \mathbb{N}$ such that $\epsilon_M < \frac{\gamma(\mathcal{U})}{2}$, where $\gamma(\mathcal{U})$ denotes the Lebesgue number of $\mathcal{U}$. Then for $s \geq q\alpha + |q|\delta + t$ one has*

$$\mathcal{M}(\{F_n\}, K_{\alpha,M,N}, \mathcal{U}, s) \leq M_{\mu,c}(\{F_n\}, K_{\alpha,M,N}, q, t, \epsilon_M).$$

Proof. Suppose that $n > N$ and $\mathcal{G}_n = \{B_{F_{n_i}}(x_i, \epsilon_M)\}$ is an arbitrary cover of $K_{\alpha,M,N}$ with $x_i \in K_{\alpha,M,N}$ such that $n_i \geq n \geq N$ for all i. Then for every x_i, we can get some string $\mathbf{U}_{n_i}$ satisfying $B_{n_i}(x_i, \epsilon_M) \subset \mathbf{U}_{n_i}$, i.e., there exists $\Gamma_n := \{\mathbf{U}_{n_i}\}$ such that

$$K_{\alpha,M,N} \subset \bigcup_{B_{F_{n_i}}(x_i,\epsilon_M)\in\mathcal{G}_n} B_{n_i}(x_i,\epsilon_M) \subset \bigcup_{\mathbf{U}_{n_i}\in\Gamma_n} \mathbf{U}_{n_i}.$$

Since $x_i \in K_{\alpha,M,N}$ for all i and $n_i \geq n > N$, we get

$$\exp(-(\alpha+\delta)|F_{n_i}|) \leq \mu(B_{F_{n_i}}(x_i,\epsilon_M)) \leq \exp(-(\alpha-\delta)|F_{n_i}|).$$

If $q \geq 0$, then $\mu(B_{F_{n_i}}(x_i, \epsilon_M))^q \geq \exp(-q(\alpha+\delta)|F_{n_i}|)$ and

$$\begin{aligned}
&\sum_{B_{F_{n_i}}(x_i,\epsilon_M)\in\mathcal{G}_n} \mu(B_{F_{n_i}}(x_i, \epsilon_M))^q \exp(-t|F_{n_i}|) \\
&\geq \sum_{B_{F_{n_i}}(x_i,\epsilon_M)\in\mathcal{G}_n} \exp(-|F_{n_i}|(q\alpha + q\delta + t)) \\
&\geq \sum_{\mathbf{U}_{n_i}\in\Gamma_n} \exp(-|F_{n_i}|s) \\
&\geq \mathcal{M}(K_{\alpha,M,N}, \mathcal{U}, s, n)
\end{aligned} \tag{5}$$

for $s \geq q\alpha + q\delta + t$. On the other hand, if $q \leq 0$, then $\mu(B_{F_{n_i}}(x_i, \epsilon_M))^q \geq \exp(-(\alpha-\delta)q|F_{n_i}|)$ and

$$\begin{aligned}
\sum_{B_{F_{n_i}}(x_i,\epsilon_M)\in\mathcal{G}_n} & \mu(B_{F_{n_i}}(x_i, \epsilon_M))^q \exp(-t|F_{n_i}|) \\
&\geq \sum_{B_{F_{n_i}}(x_i,\epsilon_M)\in\mathcal{G}_n} \exp(-|F_{n_i}|(q\alpha - q\delta + t)) \\
&\geq \sum_{\mathbf{U}_{n_i}\in\Gamma_n} \exp(-|F_{n_i}|s) \\
&\geq \mathcal{M}(K_{\alpha,M,N}, \mathcal{U}, s, n)
\end{aligned} \tag{6}$$

for $s \geq q\alpha - q\delta + t$. Together (5) with (6), we have

$$\mathcal{M}(K_{\alpha,M,N}, \mathcal{U}, s, n) \leq M_{\mu,c}(K_{\alpha,M,N}, q, t, \epsilon_M, n).$$

Let $n \to \infty$;

$$\mathcal{M}(K_{\alpha,M,N}, \mathcal{U}, s) \leq M_{\mu,c}(K_{\alpha,M,N}, q, t, \epsilon_M).$$

□

Lemma 3.2. *Suppose $K_{\alpha,M,N}$ for some $M, N \in \mathbb{N}$ and $\mathcal{U}$ is an open cover of X satisfy* $\mathrm{diam}(\mathcal{U}) < \frac{\epsilon_M}{2}$. *Then for $s \leq q\alpha - |q|\delta + t$ one has*

$$M_\mu(K_{\alpha,M,N}, q, t, \epsilon_M) \leq \mathcal{M}(K_{\alpha,M,N}, \mathcal{U}, s).$$

Proof. Fix some integers M, N and let $Z \subset K_{\alpha,M,N}, Z$ be a nonempty set. Since the open cover $\mathcal{U}$ satisfy $\mathrm{diam}(\mathcal{U}) < \frac{\epsilon_M}{2}$, we can choose any $n > N$ and let $\Gamma_n = \{\mathbf{U}_{n_i}\}$ be an arbitrary collection of strings covering Z with $n_i \geq n$.

Without loss of generality we may assume that $\mathbf{U}_{n_i} \cap Z \neq \emptyset$ for each $\mathbf{U}_{n_i} \in \Gamma_n$. Pick any $x_{\mathbf{U}_{n_i}} \in \mathbf{U}_{n_i} \cap Z$. It follows from $\mathrm{diam}(\mathcal{U}) < \frac{\epsilon_M}{2}$ that

$$\mathbf{U}_{n_i} \subset B_{F_{n_i}}(x_{\mathbf{U}_{n_i}}, \epsilon_M).$$

The collection $B_{F_n}(x, \epsilon_M)$ is centered cover of Z. Since $x_{\mathbf{U}_{n_i}} \in Z \subset K_{\alpha,M,N}$ and $n > N$, one has

$$\exp(-|F_{n_i}|(\alpha+\delta)) \leq \mu(B_{F_n}(x_{\mathbf{U}_{F_{n_i}}}, \epsilon)) \leq \exp(-|F_{n_i}|(\alpha-\delta))$$

For $q \geq 0$

$$\begin{aligned} M_{\mu,c}(Z,q,t,\epsilon_M,n) &\leq \sum_{\mathbf{U}_{n_i} \in \Gamma_n} \mu(B_n(x_{\mathbf{U}_{n_i}}, \epsilon_M))^q \exp(-n_i t) \\ &\leq \sum_{\mathbf{U}_{n_i} \in \Gamma_n} \exp(-|F_{n_i}|(q\alpha - q\delta + t)) \\ &\leq \sum_{\mathbf{U}_{n_i} \in \Gamma_n} \exp(-|F_{n_i}| s) \end{aligned}$$

for $s \leq q\alpha - q\delta + t$. Since Γ_n is arbitrary, we get

$$M_{\mu,c}(Z,q,t,\epsilon_M,n) \leq \mathcal{M}(Z,\mathcal{U},s,n).$$

Let $n \to \infty$,

$$M_{\mu,c}(Z,q,t,\epsilon_M) \leq \mathcal{M}(Z,\mathcal{U},s) \leq \mathcal{M}(K_{\alpha,M,N},\mathcal{U},s).$$

Moreover,

$$M_\mu(K_{\alpha,M,N},q,t,\epsilon_M) \leq \mathcal{M}(K_{\alpha,M,N},\mathcal{U},s). \tag{7}$$

For $q \leq 0$, we have

$$\mu(B_{F_{n_i}}(x_{\mathbf{U}_{n_i}}, \epsilon_M))^q \leq \exp(-|F_{n_i}| q(\alpha+\delta)).$$

Hence,

$$\begin{aligned} M_{\mu,c}(Z,q,t,\epsilon_M,n) &\leq \sum_{\mathbf{U}_{n_i} \in \Gamma_n} \mu(B_{F_n}(x_{\mathbf{U}^{\mathbf{a}}_{n_i}}, \epsilon_M))^q \exp(-|F_{n_i}| t) \\ &\leq \sum_{\mathbf{U}_{n_i} \in \Gamma_n} \exp(-|F_{n_i}|(q\alpha + q\delta + t)) \\ &\leq \sum_{\mathbf{U}_{n_i} \in \Gamma_n} \exp(-|F_{n_i}| s) \end{aligned}$$

for $s \le q\alpha + q\delta + t$. Similar to the case $q > 0$, we can get

$$M_\mu^{\mathbf{a}}(K_{\alpha,M,N}, q, t, \epsilon_M) \le m(K_{\alpha,M,N}, \mathcal{U}, s). \tag{8}$$

Together (7) with (8), we complete the proof. $\square$

Finally, we prove Theorem 2.6. By the definition of Bowen topological entropy, we only need to show

$$h_{top}^B(\{F_n\}, K_\alpha(\mu)) = q\alpha + h_\mu(\{F_n\}, q, K_\alpha(\mu)).$$

We may assume that $K_\alpha(\mu) \neq \emptyset$. Otherwise, the statement is obvious, since both sides are equal to $-\infty$. When $K_\alpha(\mu) \neq \emptyset$, we divide the proof into two steps:

Step 1: $h_{top}^B(\{F_n\}, K_\alpha(\mu)) \le q\alpha + h_\mu(\{F_n\}, q, K_\alpha(\mu))$. Suppose that the opposite is true: let

$$\gamma = \frac{1}{4}(h_{top}^B(\{F_n\}, K_\alpha(\mu)) - q\alpha - h_\mu(\{F_n\}, q, K_\alpha(\mu))) > 0.$$

Clearly,

$$h_{top}^B(K_\alpha(\mu)) = \lim_{diam(\mathcal{U})\to 0} h_{top}^B(\{F_n\}, K_\alpha(\mu), \mathcal{U}).$$

There exists a family of open covers $\mathcal{U}$ such that

$$h_{top}^B(\{F_n\}, K_\alpha(\mu), \mathcal{U}) > q\alpha + h_\mu(\{F_n\}, q, K_\alpha(\mu)) + 3\gamma.$$

Let $\delta > 0$ be an arbitrary positive number if $q = 0$ and $\delta = \frac{\gamma}{2|q|}$ if $|q| > 0$. Consider $K_{\alpha,M,N}$ defined above, choose sufficiently large M, N such that the following three conditions are satisfied:

$$\begin{aligned} &h_{top}^B(\{F_n\}, K_{\alpha,M,N}, \mathcal{U}) > q\alpha + h_\mu(\{F_n\}, q, K_\alpha(\mu)) + 2\gamma, \\ &\epsilon_M < \delta, h_\mu(\{F_n\}, q, K_\alpha(\mu)) + \frac{\gamma}{2} \ge h_\mu(\{F_n\}, q, K_\alpha(\mu), \epsilon_M). \end{aligned} \tag{9}$$

This is possible because

$$h_{top}^B(\{F_n\}, K_\alpha(\mu), \{\mathcal{U}_i\}_{i=1}^k) = \lim_{M\to\infty}\lim_{N\to\infty} h_{top}^B(\{F_n\}, K_{\alpha,M,N}, \{\mathcal{U}_i\}_{i=1}^k)$$

and

$$h_\mu(\{F_n\}, q, K_\alpha(\mu)) = \limsup_{\epsilon\to 0} h_\mu^B(\{F_n\}, q, K_\alpha(\mu), \epsilon).$$

By the definition of $h^B_{top}(\{F_n\}, K_{\alpha,M,N}, \{\mathcal{U}_i\}_{i=1}^k)$, the inequality (9) implies

$$\mathcal{M}(K_{\alpha,M,N}, \{\mathcal{U}_i\}_{i=1}^k, q\alpha + h_\mu(\{F_n\}, q, K_\alpha(\mu)) + 2\gamma)) = \infty.$$

It follows from $s = q\alpha + h_\mu(\{F_n\}, q, K_\alpha(\mu)) + 2\gamma$, $t = h_\mu(\{F_n\}, q, K_\alpha(\mu)) + \gamma - |q|\delta$ and Lemma 3.1 that

$$M_{\mu,c}(K_{\alpha,M,N}, q, h_\mu(\{F_n\}, q, K_\alpha(\mu)) + \gamma - |q|\delta, \epsilon_M) = \infty.$$

Moreover,

$$M_\mu(K_{\alpha,M,N}, q, h_\mu(\{F_n\}, q, K_\alpha(\mu)) + \gamma - |q|\delta, \epsilon_M) = \infty. \tag{10}$$

Here, we arrive at a contradiction with the assumption above. Indeed,

$$\begin{aligned} h_\mu(\{F_n\}, q, K_\alpha(\mu)) + \gamma - |q|\delta \geq& h_\mu(\{F_n\}, q, K_\alpha(\mu)) + \frac{\gamma}{2} \\ \geq& h_\mu(\{F_n\}, q, K_\alpha(\mu), \epsilon_M) \\ \geq& h_\mu(\{F_n\}, q, K_{\alpha,M,N}, \epsilon_M) \end{aligned}$$

and therefore one must have

$$M_\mu(K_{\alpha,M,N}, q, h_\mu(T_1, q, K_\alpha(\mu)) + \gamma - |q|\delta, \epsilon_M) = 0$$

which contradicts (10).

Step 2: $h^B_{top}(\{F_n\}, K_\alpha(\mu)) \geq q\alpha + h_\mu(\{F_n\}, q, K_\alpha(\mu))$. Suppose that the opposite is true: let

$$\gamma = \frac{1}{4}(q\alpha + h_\mu(\{F_n\}, q, K_\alpha(\mu)) - h^B_{top}(\{F_n\}, K_\alpha(\mu))) > 0.$$

By $h_\mu(\{F_n\}, q, K_\alpha(\mu)) = \limsup_{\epsilon \to 0} h_\mu(\{F_n\}, q, K_\alpha(\mu), \epsilon)$, we can choose a decreasing sequence $\epsilon_M \to 0$ such that

$$h_\mu(\{F_n\}, q, K_\alpha(\mu)) = \lim_{M \to \infty} h_\mu(\{F_n\}, q, K_\alpha(\mu), \epsilon_M).$$

Let $\delta > 0$ be an arbitrary positive number if $q = 0$ and $\delta = \frac{\gamma}{2|q|}$ if $|q| > 0$. Choose sufficiently large M such

$$\epsilon_M < \delta, h_\mu(\{F_n\}, q, K_\alpha(\mu), \epsilon_M) > h_\mu(\{F_n\}, q, K_\alpha(\mu)) - \frac{\gamma}{2}.$$

Since

$$h^B_{top}(\{F_n\}, K_\alpha(\mu)) = \lim_{diam(\mathcal{U}) \to 0} h^B(K_\alpha(\mu), \mathcal{U}).$$

One can find a family of open covers $\mathcal{U}$ such that

$$\mathrm{diam}(\mathcal{U}) < \frac{\epsilon_M}{2}$$

and

$$q\alpha + h_\mu(\{F_n\}, q, K_\alpha(\mu)) > h(\{F_n\}, K_\alpha(\mu), \mathcal{U}) + 3\gamma.$$

Furthermore, consider $K_{\alpha,M,N}$ defined above, we can get

$$q\alpha + h_\mu(\{F_n\}, q, K_\alpha(\mu)) > h^B_{top}(\{F_n\}, K_{\alpha,M,N}, \mathcal{U}) + 2\gamma \tag{11}$$

$$h_\mu(\{F_n\}, q, K_\alpha(\mu), \epsilon_M) - \gamma \leq h_\mu(\{F_n\}, q, K_{\alpha,M,N}, \epsilon_M) \tag{12}$$

for M, N large enough. This is possible because

$$h^B_{top}(\{F_n\}, K_\alpha(\mu), \{\mathcal{U}_i\}_{i=1}^k) = \lim_{M\to\infty} \lim_{N\to\infty} h^B_{top}(\{F_n\}, K_{\alpha,M,N}, \{\mathcal{U}_i\}_{i=1}^k)$$

and

$$h_\mu(\{F_n\}, q, K_\alpha(\mu), \epsilon_M) = \lim_{M\to\infty} \lim_{N\to\infty} h_\mu(\{F_n\}, q, K_{\alpha,M,N}, \epsilon_M).$$

By the definition of $h^B_{top}(\{F_n\}, K_{\alpha,M,N}, \mathcal{U})$, the inequality (11) implies

$$\mathcal{M}(K_{\alpha,M,N}, \mathcal{U}, q\alpha + h_\mu(\{F_n\}, q, K_\alpha(\mu)) - 2\gamma) = 0.$$

It follows from $s = q\alpha + h_\mu(T, q, K_\alpha(\mu)) - 2\gamma$, $t = h_\mu(\{F_n\}, q, K_\alpha(\mu)) - \gamma + |q|\delta$ and Lemma 3.2 that

$$M_\mu(K_{\alpha,M,N}, q, h_\mu(\{F_n\}, q, K_\alpha(\mu)) - \gamma + |q|\delta, \epsilon_M) = 0. \tag{13}$$

Here, we arrive at a contradiction with the assumption above. Indeed, by (12)

$$\begin{aligned} h_\mu(\{F_n\}, q, K_\alpha(\mu)) - \gamma + |q|\delta \leq & h_\mu(\{F_n\}, q, K_\alpha(\mu)) - \frac{\gamma}{2} \\ \leq & h_\mu(\{F_n\}, q, K_\alpha(\mu), \epsilon_M) - \gamma \\ \leq & h_\mu(\{F_n\}, q, K_{\alpha,M,N}, \epsilon_M). \end{aligned}$$

Therefore one must have

$$M_\mu(K_{\alpha,M,N}, q, h_\mu(\{F_n\}, q, K_\alpha(\mu)) - \gamma + |q|\delta, \epsilon_M) = \infty$$

which contracts (13). **ACKNOWLEDGEMENTS.** The authors want to

thank the referees for their valuable suggestions.

References

[1] R. Bowen, Topological entropy for noncompact sets, *Transactions of the American Mathematical Society*, **184**, (1973), 125–136.

[2] L. Barreira, *Dimension and recurrence in Hyperbolic Dynamics*, Birkhäuser, 2008.

[3] M. Brin and A. Katok, On local entropy, *Lecture Notes in Math*, **1007**, (1983), 30–38.

[4] E. Chen,T. Küpper and L. Shu, Topological entropy for divergence points, *Ergod. Th. & Dynam. Sys*, **25**, (2005), 1173–1208.

[5] H. Federer, *Geometric Measure Theory, Springer-Verlag*, New York, 1969.

[6] E. Lindenstrauss, Pointwise theorems for amenable groups, *Invent. Math*, **146**, (2001), 259–295.

[7] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, 1995.

[8] L. Olsen, Self-affine multifractal Sierpinski sponges in $\mathbb{R}^d$, *Pacific J. Math*, **183**, (1998), 143–199.

[9] L. Olsen, A multifractal formalism, *Adv. Math*, **116**, (1995), 82–196.

[10] J. Ollagnier and D. Pinchon, The variational principle, *Studia Math*, **72**, (1982), 151–159.

[11] D. Ornstein and B. Weiss, Entropy and isomorphism theorems for actions of amenable groups, *J. Anal. Math*, **48**, (1987), 1–141.

[12] Y. Pesin, *Dimension theory in Dynamical systems. Contemporary Views and Applications*, The University of Chicago Press, 1997.

[13] Y. Pesin and B. Pitskel, Topological pressure and the variational principle for noncompact sets, *Functional Analysis and its Applications*, **18**, (1984), 307–318.

[14] Y. Pei and E. Chen, On the variational principle for the opological pressure for certain non-compact sets, *Sci. China Math*, **53**, (2010), 1117–1128.

[15] Z. Yan and E. Chen, Multifractal analysis of local entropies for recurrence time, *Chaos, Solitons Fractals*, **33**, (2007), 1584–1591.

[16] X. Raymond and C. Tricot, Packing regularity of sets in n-space,*Math. Proc. Cambridge Phililos. Soc*, **103**,(1998), 133–145.

[17] A. Shulman, Maximal ergodic theorems on groups, *Dep. Lit. NIINTI*, **2184**, (1988).

[18] A. Stepin and A. Tagi-Zade, Variational characterization of topological pressure of the amenable groups of transformations,*Dokl.Akad.Nauk.Sssr*, **254**, (1980), 545–549 .

[19] F. Takens and E. Verbitski,General multifractal analysis of local entropies, *Fundamenta Mathematicae*, **165**, (2000), 203–237.

[20] F. Takens and E. Verbitskiy, On the variational principle for the topological entropy of certain non-compact sets, *Ergod. Th. and Dynam. Sys*, **23**, (2003), 317–348.

[21] D. Thompson, A variational principle for topological pressure for certain non-compact sets, *J. Lond. Math. Soc*, **80**, (2009), 585–602.

[22] X. Zhou and E. Chen, Multifractal analysis for the historic set in topological dynamical systems, *Nonlinearity*, **26**, (2013), 1975–1997.

[23] D. Zheng and E. Chen. Bowen entropy for actions of amenable groups, *Israel Journal of Mathematics*, **212**, (2016), 895–911.