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Proof of the Collatz Conjecture

Bülent Sukuşu[1]

Abstract

The Collatz Conjecture (or 3n+1 problem) has been explored for about 86 years. In this article, we prove the Collatz Conjecture. We will show that this conjecture holds for all positive integers by applying the Collatz inverse operation to the numbers that satisfy the rules of the Collatz Conjecture. Finally, we will prove that there are no positive integers that do not satisfy this conjecture.

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[1]Independent Researcher, Ankara, Turkey.

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1 Introduction

The Collatz Conjecture is one of the unsolved problems in mathematics. Introduced by German mathematician Lothar Collatz in 1937 [1], it is also known as the 3n + 1 problem, 3x + 1 mapping, Ulam Conjecture (Stanislaw Ulam), Kakutani's problem (Shizuo Kakutani), Thwaites Conjecture (Sir Bryan Thwaites), Hasse's algorithm (Helmut Hasse), or Syracuse problem [2–4].

In this paper, $\mathbb{N}$ $=\{0,1,2,3,4,5,\ldots\}$, the symbol $\mathbb{N}$ represents the natural numbers. $\mathbb{N}^+$ $=\{1,2,3,4,\ 5,6,\ldots\}$, the symbol $\mathbb{N}^+$ represents the positive integers. $\mathbb{N}_{odd}$ $=\{1,3,5,7,9,\ 11,13,\ldots\}$, the symbol $\mathbb{N}_{odd}$ represents the positive odd integers.

2 The Conjecture and Related Conversions

Definition 2.1 Let $n,k \in \mathbb{N}^+$ and a function $f : \mathbb{N}^+ \to \mathbb{N}^+$, Collatz defined the following map:

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if n is even} \\ 3n+1, & \text{if n is odd} \end{cases}$$

The Collatz Conjecture states that the orbit formed by iterating the value of each positive integer in the function $f(n)$ will eventually reach 1. The orbit of n under f is $n; f(n), f(f(n)), f(f(f(n))), \ldots\ f^k(n) = 1\ (k \in \mathbb{N}^+)$.
In the following sections, we will call these two arithmetic operations ($n/2$ and $3n + 1$), which we apply to any positive integer n according to the rule of assumption, Collatz operations (CO).

Remark 2.2 According to the definition of the Collatz Conjecture, if the number we choose at the beginning is an even number, then by continuing to divide all even numbers by 2, one of the odd numbers is achieved. So it is sufficient to check whether all odd numbers reach 1 by the Collatz operations.

Therefore, if we prove that it reaches 1 when we apply the Collatz operations to all the elements of the set $\mathbb{N}_{odd}$ $=\{1,3,5,7,9,11,13,15,\ldots\}$, we have proved it for all positive integers.

Remark 2.3 If the Collatz operations are applied to the numbers 2^n $(n \in \mathbb{N}^+)$, then eventually 1 is reached. If we can convert all the elements of the set $\mathbb{N}_{odd}$ into 2^n numbers by applying the Collatz operations, we get the result.

2.1 Collatz Inverse Operation (CIO)

Let $n \in \mathbb{N}^{+}$and $a \in \mathbb{N}_{odd}$; for a to be converted to 2^n by the Collatz operation (CO), it must satisfy the following equation,

$$3.a + 1 = 2^n$$

then,

$$a = \frac{2^n - 1}{3} \tag{1}$$

Lemma 2.4 In (1) $a = \frac{2^n-1}{3}$, a cannot be an integer if n is a positive odd integer.
Proof. If n is a positive odd integer, we can take $n = 2m + 1$ ($m \in \mathbb{N}$), then substituting $2m + 1$ for n in (1) we get,

$$a = \frac{2^{2m+1} - 1}{3} \tag{2}$$

if we factor $2^{2m+1} + 1$,
$2^{2m+1} + 1 = (2 + 1)(2^{2m} - 2^{2m-1} + 2^{2m-2} - \ldots + 1) = \mathbf{3}.k$ ($k \in \mathbb{N}_{odd}$).
Since $(2^{2m+1} + 1)$ is a multiple of 3, $(2^{2m+1} - 1)$ is not a multiple of 3. So in (1) a is not an integer for any number n.

If we substitute $2n$ for n in (1), we get equation

$$a = \frac{2^{2n} - 1}{3} \tag{3}$$

Lemma 2.5 In (3) $a = \frac{2^{2n}-1}{3}$, for each number n there is a different positive odd integer a, ($n \in \mathbb{N}^{+}$).

Proof. When we factorize $2^{2n} - 1$ for $\forall n \in \mathbb{N}^{+}$,

$(2^{2n} - 1) = (2^{x_1} - 1)(2^{x_1} + 1)(2^{x_2} + 1)(2^{x_3} + 1)\ldots.(2^{x_{n-1}} + 1)(2^{x_n} + 1)$ or
$(2^{2n} - 1) = (2^{x_1} - 1)(2^{x_1} + 1)$ in these equations, x_1 is a positive odd integer and x_2, x_3, x_4 … x_n are positive even integers. Since x_1 is a positive odd number,
$(2^{x_1} + 1) = (2 + 1)(2^{x_1-1} - 2^{x_1-2} + 2^{x_1-3} - \ldots + 1) = \mathbf{3}.(\ldots)$ so,
$(2^{2n} - 1) = \mathbf{3}.(\ldots)$
Since each of these numbers has a multiplier of 3, we can find positive odd integers a for all n, and when we apply Collatz operations to these a numbers, we always get 1. $2^{2n} + 1$ is not a multiple of 3, since $2^{2n} - 1$ is a multiple of 3, for $\forall n \in \mathbb{N}^{+}$. In (3), If we replace n with positive integers, we get the set A.

$$a = \frac{2^{2n} - 1}{3};$$

A= { 1,5,21,85,341,1365,5461,21845,87381,... } (Collatz Numbers)

If we can generalize the elements of the set $A = \{1, 5, 21, 85, 341, 1365, 5461, 21845, 87381, \ldots\}$ to all positive odd numbers, we have proved the Collatz Conjecture.

2.2 Transformations in the Set A with Infinite Elements

Let the elements of the set $A = \{1, 5, 21, 85, 341, 1365, 5461, 21845, 87381, \ldots\}$ be $\{a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, \ldots\}$ respectively.

Lemma 2.6 In the set A$\setminus\{a_0\}$, if $a_n \equiv 1 \pmod 3$

$$b_n = \frac{2^{2m}.a_n - 1}{3} \tag{4}$$

$m \in \mathbb{N}^+$, if we value m from 1 to infinity, we get B_n set with infinite b_n elements (Collatz numbers) from each a_n.These numbers satisfy the conjecture.

Proof. If $a_n \equiv 1 \pmod 3$, we can take a_n as $3.p + 1$, $(p \in \mathbb{N})$
$a_n = 3.p + 1$ substituting in (4),

$$b_n = \frac{2^{2m}.(3.p + 1) - 1}{3} = \frac{2^{2m}3p + 2^{2m} - 1}{3} = 2^{2m}p + \frac{2^{2m} - 1}{3}$$

$2^{2m} - 1$ is divisible by 3 (Lemma 2.5). So we get an infinite number of different b_n elements, which can be converted to a_n, i.e. 1, by the Collatz operation. The numbers b_n are Collatz numbers and are a sequence of the form $b_{n+1} = 4.b_n + 1$.

Example 2.7 Let $a_1 = 85$, then $a_1 \equiv 1 \pmod 3$,in (4),

$$B = \{113, 453, 1813, 7253, 29013, 116053, \ldots\}$$

Lemma 2.8 In the set A$\setminus\{a_0\}$, if $a_n \equiv 2 \pmod 3$,

$$b_n = \frac{2^{2m-1}.a_n - 1}{3} \tag{5}$$

$m \in \mathbb{N}^+$, if we value m from 1 to infinity, we get B_n set with infinite b_n elements (Collatz numbers) from each a_n. These numbers satisfy the conjecture.

Proof. If $a_n \equiv 2 \pmod 3$, we can take a_n as $3.p+2$ ($p \in \mathbb{N}$)
$a_n = 3.p + 2$ substituting in (5),

$$b_n = \frac{2^{2m-1}.(3p+2)-1}{3} = \frac{2^{2m-1}.3p + 2^{2m} - 1}{3} = 2^{2m-1}p + \frac{2^{2m}-1}{3}$$

$2^{2m}-1$ is divisible by 3 (Lemma 2.5). So we get an infinite number of different b_n elements, which can be converted to a_n, i.e. 1, by the Collatz operation. The numbers b_n are Collatz numbers and are a sequence of the form $b_{n+1} = 4.b_n + 1$.

Example 2.9 Let $a_1 = 5$, then $a_1 \equiv 2 \pmod 3$;

$$B = \{3,\ 13,\ 53,\ 213,\ 853,\ 3413,\ 13653,\ 54613,\ \ldots\}$$

Lemma 2.10 In the set A$\setminus\{a_0\}$, if $a_n \equiv 0 \pmod 3$,

$$b_n = \frac{2^m.a_n - 1}{3} \tag{6}$$

$m \in \mathbb{N}^+$, there is no such integer b_n.

Proof . If $a_n \equiv 0 \pmod 3$, we can take a_n as $3.p$ ($p \in \mathbb{N}$)

$a_n = 3.p$ substituting in (6),

$$b_n = \frac{2^m(3.p)-1}{3} = \frac{2^m 3.p - 1}{3} = 2^m.p - \frac{1}{3},$$

is not integer.
In the following sections, we will call the operations of deriving new Collatz numbers from Collatz numbers by Equations (3), (4) or (5) as Collatz inverse operations (CIO).

2.3 Conversion of all Positive Odd Integers to Collatz Numbers

In the previous sections, when we applied the Collatz operations, we called the numbers that reached 1 as Collatz numbers. Now let's see how all positive integers can be converted to these Collatz numbers.

A= { 1,5,21,85,341,1365,5461,21845,87381 ... } (Collatz Numbers)

If we apply the Collatz inverse operations [Equations (4) or (5)] continuously to each Collatz number, we get infinitely many new Collatz numbers.

$\mathbb{N}_{odd} \rightarrow$ Set of A $\rightarrow 2^{2n} \rightarrow 1$ (Direction of conversion of numbers with CO).
$\mathbb{N}_{odd} \leftarrow$ Set of A $\leftarrow 1$ (Direction of conversion of numbers with CIO).

All positive numbers are obtained by repeatedly applying the Collatz inverse operations to each element of the set A and the Collatz numbers generated from these numbers.
Lemma 2.11 If we apply the Collatz inverse operations $(\frac{2^m.a_n-1}{3})$ $(m \in \mathbb{N}^+)$ to the different Collatz numbers, we obtain new Collatz numbers that are all different from each other.

Proof. Let a_1 and a_2 be arbitrary Collatz numbers and $a_1 \neq a_2$, when we apply the Collatz inverse operations to each of them, the resulting numbers are b_1 and b_2. If $b_1 = b_2$ then,

$b_1 = \frac{2^m.a_1-1}{3} = \frac{2^t.a_2-1}{3} = b_2$ then $2^m.a_1 = 2^t.a_2$ for odd positive integers (a_1 and a_2), must be $a_1 = a_2$ and $m = t$ (contradiction), so if $a_1 \neq a_2$ then $b_1 \neq b_2$.

Corollary 2.12 In set theory, the cardinality of a set S represents the number of elements in the set, and is denoted by |S|. The aleph numbers ($\aleph$) indicate the cardinality (size) of well-ordered infinite element sets. $\aleph_0$ is the notation for the cardinality of the set of natural numbers, the next larger cardinality is $\aleph_1$, then $\aleph_2$ and so on. The cardinality of a set is $\aleph_0$ if and only if there is a one-to-one correspondence (bijection) between all elements of the set and all natural numbers. Since there is a one-to-one correspondence between the infinite sets in Figure 1 and the set of natural numbers, the cardinality of each set is $\aleph_0$ [6].
The cardinality of the continuum is $2^{\aleph_0} = \aleph_1$. The order and operations between the cardinality of the sets are as follows: $|\mathbb{N}|=\aleph_0$, $\aleph_1=$ cardinality of the "smallest" uncountably infinite sets;

$\aleph_0<\aleph_1<\aleph_2<\ldots$
$\aleph_0+\aleph_0+\aleph_0+\ldots = \aleph_0.\aleph_0=\aleph_0$
$\aleph_0.\aleph_0.\aleph_0=\aleph_0$
$\aleph_0.\aleph_0.\aleph_0\ldots\ \aleph_0.\aleph_0 = \aleph_0^k = \aleph_0$ (k is a finite positive integer)
$\aleph_0.\aleph_0.\aleph_0\ldots = \aleph_0^{\aleph_0}$

The elements of the set A (Lemma 2.5) are the Collatz numbers. We get new Collatz numbers by applying Collatz inverse operations [Equations (4) or (5)] to each element of this set A. From these new infinite Collatz numbers, infinitely many new Collatz numbers are formed by applying the Collatz inverse operations (CIO) again and again, and this goes on endlessly.
As a result, Collatz numbers fill the Hilbert's Hotel (David Hilbert) until there is no empty room left. The Hilbert Hotel is a thought experiment that has a

countable infinity of rooms with room numbers 1, 2, 3, etc., and demonstrates the properties of infinite sets. In this hotel with an infinite number of guests, an infinite number of new guests (even finite layers of infinite) can be accommodated, provided that only one guest stays in each room [5]. When we fill the odd-numbered rooms of the Hilbert Hotel with Collatz numbers, we also fill the entire hotel with Collatz numbers. Let $n \in \mathbb{N}^+$ and $x, y \in \mathbb{N}_{odd}$, and let the odd-numbered rooms of the Hilbert Hotel be 1, 3, 5, 7, ..., i.e. elements of the set $\mathbb{N}_{odd}$. The result of the Collatz inverse operation is the following equation,

$$\frac{2^n.x - 1}{3} = y \tag{7}$$

In Equation (7), n depends on the values of x. If $x \equiv 1 \pmod 3$ we replace n with all even numbers $n = \{2,4,6,8,\dots\}$, and if $x \equiv 2 \pmod 3$ we replace n with all odd numbers $n = \{1,3,5,7,\dots\}$ respectively (Lemma 2.6 and Lemma 2.8). In (7) we obtain an infinite number of y values as Collatz numbers starting from $x = 1$ (Lemma 2.5). Then, by substituting y values for x in (7), we obtain the Collatz number sets with infinite elements for each y that is not a multiple of 3. [Although we cannot replace x with numbers that are multiples of 3, we get infinite numbers that are multiples of 3 in each Collatz number sets (Figure 1). Because, the numbers in each set give the remainder of 0,1,2 respectively according to $\pmod 3$, as in the $\mathbb{N}_{odd}$ set]. If the same process is repeated and the generated numbers are placed according to the room numbers, there will be no empty rooms left in the Hilbert Hotel. This is because infinite layers of disjoint Collatz number sets[2] are formed without limit by Equation (7), and these sets fill all odd-numbered rooms, i.e. all positive odd integers are obtained (Figure 1). By multiplying these numbers by 2^m ($m \in \mathbb{N}^+$), we find that all even numbers are Collatz numbers (Remark 2.2). Therefore, Collatz numbers fill the Hilbert Hotel and the set of Collatz numbers is equal to the set $\mathbb{N}^+$. Starting with $x = 1$ in (7) and continuing the process to infinity, infinite layers of disjoint Collatz number sets are obtained (Figure 1).

$$\{1\}$$

$$Y_0 = 1^* = [\{1, 5, 21, 85, 341, 1365, 5461, 21845, 87381, \dots\}] \quad |Y_0| = 1$$

$$Y_1 = 1^* = \Big[\, 5^* = \{3,13,53,\dots\}\ 85^* = \{113,453,1813,\dots\}\ 341^* = \{227,909,3637,\dots\}$$
$$5461^* = \{7281,29125,116501,\dots\}\ \dots \Big] \qquad |Y_1| = \aleph_0$$

$$Y_2 = 1^* = \Big[\, 5^* = \{13^* = \{17,69,\dots\}\ 53^* = \{35,141,\dots\}\dots\}\ 85^* = \{\ 113^* = \{75,301\dots\}$$
$$1813^* = \{2417,9669\dots\}\ \dots\ \}\ \dots \Big] \qquad |Y_2| = \aleph_0 + \aleph_0 + \aleph_0\ \dots = \aleph_0.\aleph_0 = \aleph_0^2$$

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[2]Collatz number sets are countably infinite element subsets of the set of positive odd integers.

Y_3= 1*= $\Big[$ 5*={ 13*= {17*={11,45,...} ...} 53*= {35*={23,93,...} ...} ...} 85*={113*={301*={401,1605, ...}...} 1813*={2417*={1611,6445,...}...} ...} ... $\Big]$ $|Y_3| = \aleph_0.\aleph_0.\aleph_0 = \aleph_0^3$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$ $|Y| = \aleph_0.\aleph_0.\aleph_0 \ldots = \aleph_0^{k<\omega}$ $(k \in \mathbb{N})$ [3]

The set of disjoint Collatz number sets:
Y= $\Big[$ {1,5,21,...}{3,13,53,...} {113,453,1813,...}{227,909,3637,...} {7281,29125, 116501,...} {17,69,277,...} {35,141,565,...} ... $\Big]$ $|Y| = \aleph_0.\aleph_0.\aleph_0 \ldots = \aleph_0^{k<\omega}$

Figure 1: Collatz number sets. | | represents the cardinality of the set of Collatz number sets, and * represents conversions of numbers that are not multiples of 3 using Equation (7).
In Figure 1, the infinite layers of Collatz number sets continue to form without restriction until they fill Hilbert's hotel. The restriction occurs only when the hotel is completely filled, that is, when all positive odd numbers are obtained. Imagine buses arriving at Hilbert's Hotel, each carrying an infinite number of passengers. The buses represent disjoint sets, and the passengers represent the elements of these sets. The following buses eventually fill Hilbert's Hotel.

- $Y_0 = \{1, 5, 21, 85, 341, \ldots\}$ (**0.layer:** infinite people, card. of buses: 1)
- $Y_1 = \aleph_0$ (**1. layer:** cardinality of buses: $\aleph_0$)
  - $\{(5,3),(5,13),(5,53),\ldots,$
    $(85,113),(85,453),\ldots,$
    $(341,227),(341,909),\ldots$
    $\ldots \}$ (infinite buses each with infinite people)
- $Y_2 = \aleph_0 \cdot \aleph_0$ (**2. layer:** cardinality of buses: $\aleph_0.\aleph_0 = \aleph_0^2$)
  - $\{(5,13,17),(5,13,69),\ldots,$
    $(85,113,75),(85,113,301),\ldots$
    $\ldots \}$ (infinite ferries, each containing infinite buses, infinite people on each bus)
- $Y_3 = \aleph_0 \cdot \aleph_0 \cdot \aleph_0$ (**3. layer:** cardinality of buses:$\aleph_0.\aleph_0.\aleph_0 = \aleph_0^3$)
  - $\{(5,13,17,11),(5,13,17,45),\ldots,$
    $(85,113,301,401),(85,113,301,1605),\ldots$
    $\ldots \}$ (infinite oceans with infinite ferries on each, infinite buses on each ferry, infinite people on each bus)

[3] ω is the ordinal number and represents the first infinite ordinal. The ordinal number ω is the smallest element greater than any natural number.

- $\mathrm{Y} = \aleph_0.\aleph_0.\aleph_0 \ldots = \aleph_0^{k<\omega}$ (**k.layer:**card. of buses: $\aleph_0.\aleph_0.\aleph_0\ldots = \aleph_0^{k<\omega}$)

Since there are different people on the buses, the buses represent disjoint Collatz number sets. As we move from each layer to the next layer as the number of layers increases, the cardinality of the set of disjoint Collatz number sets increases by a factor of $\aleph_0$, so $\aleph_0.\aleph_0.\aleph_0\ldots = \aleph_0^{k<\omega}$ $(k \in \mathbb{N})$ is the cardinality of the set of all disjoint Collatz sets.

Where $k \in \mathbb{N}$ and $k < \omega$, i.e., k can be any natural number. ω is the ordinal number and represents the first infinite ordinal. The ordinal number ω is the smallest element greater than any natural number. Each disjoint Collatz set forming the layers forms a sequence like the set Y_0 such that $a_n = 4 \cdot a_{n-1} + 1$. The elements of these sequences form a loop with the remainders 0, 1, 2 according to $\pmod 3$. New sets, i.e. new layers, are formed from the elements with the remainders 1 and 2 according to $\pmod 3$. We saw above that there are 0. 1. and 2. layers. Therefore, if there is an n-th layer, there is also an (n+1)th layer. Since all layers are generated inductively, all natural number layers can be generated up to the first infinite ordinal number, ω, i.e.

$$0.\ 1.\ 2.\ 3.\ 4.\ \ldots\ k. < \omega \quad (k \in \mathbb{N})$$

Since the elements of the disjoint Collatz sets are positive odd numbers, layers can be constructed corresponding to the number of elements in the $\mathbb{N}_{odd}$ set. As the layers (k) are determined by induction, it does not matter whether k is defined as a natural number or a positive odd integer; the number of layers is the same regardless of the set used in the inductive definition. The same number of layers are indexed differently.[4] The room numbers at the Hilbert hotel are 1. 3. 5.... $k. < \omega$ $(k \in \mathbb{N}_{odd})$. Since $\aleph_0^{k<\omega}$ is the cardinality of the set of disjoint Collatz sets, from the union of these disjoint sets we get the set of positive odd integers with cardinality $\aleph_0^{k+2<\omega}$. $k+2$ is the first number after k in the ordered set of positive odd integers, and since $k+2 < \omega$ as many numbers are generated as there are elements in the set of positive odd integers. The cardinality $\aleph_0^{k<\omega}$ means that a disjoint Collatz set of cardinality $\aleph_0$ can be created for each room in the hotel. Thus, the cardinality $\aleph_0^{k<\omega}$, $(k \in \mathbb{N}_{\mathrm{odd}})$ contains all layers, all disjoint sets and all numbers that can be contained in the set of positive odd numbers. In this way, the Hilbert hotel is filled by creating countably infinite layers (layers can be generated as many as there are rooms in the hotel). This means that as many guests are sent as there are rooms in the hotel, i.e. one guest for each available room.

[4]Since Collatz number sets are positive odd numbers, k obtained by induction can be defined as a positive odd number. The value k defined for natural numbers is found by induction in the same way for positive odd integers. Thus, layers can be defined as positive odd numbers: 1. 3. 5.... $k. < \omega$ $(k \in \mathbb{N}_{odd})$

In Figure 1, sets of disjoint Collatz sets represent the layers. The numbers in the layers form the sets in the upper layers with CIO, while the sets in the upper layers become the numbers in the lower layers with CO.

$$1 \leftarrow \mathrm{Y}_0\ \mathrm{Y}_1\ \underset{\leftarrow \mathrm{CO}}{\overset{\mathrm{CIO}\rightarrow}{\mathrm{Y}_2}}\ \mathrm{Y}_3 \ldots \mathrm{Y}_{k<\omega}$$

$\mathrm{Y}_0 = [\{1, 5, 21, 85, 341, 1365, 5461, \ldots\}] \quad |\mathrm{Y}_0| = 1 \quad$ (0. layer)

$\mathrm{Y}_1 = [5^* = \{3, 13, 53, \ldots\}, 85^* = \{113, 453, 1813, \ldots\}, 341^* = \{227, 909, 3637, \ldots\}, 5461^* = \{7281, 29125, 116501, \ldots\}, \ldots] \qquad |\mathrm{Y}_1| = \aleph_0 \quad$ (1. layer)

$\mathrm{Y}_2 = [5^* = \{13^* = \{17, 69, \ldots\}, 53^* = \{35, 141, \ldots\}, \ldots\} \quad 85^* = \{113^* = \{75, 301, \ldots\}, 1813^* = \{2417, 9669, \ldots\}, \ldots\}, \ldots] \quad |\mathrm{Y}_2| = \aleph_0^2 \quad$ (2. layer)

$\mathrm{Y}_3 = [5^* = \{13^* = \{17^* = \{11, 45, \ldots\}, \ldots\} \quad 53^* = \{35^* = \{23, 93, \ldots\}, \ldots\}, \ldots\} \; 85^* = \{113^* = \{301^* = \{401, 1605, \ldots\}, \ldots\}, 1813^* = \{2417^* = \{1611, 6445, \ldots\}, \ldots\}, \ldots\} \ldots] \quad |\mathrm{Y}_3| = \aleph_0^3 \quad$ (3. layer)

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$|\mathrm{Y}_k| = \aleph_0^{k<w} \quad (k \in \mathbb{N})$$

The reason that the number of layers can be defined as $k < \omega,\ k \in \mathbb{N}$ is that since the set of positive odd integers is bounded from below, the initial layers are available and the other layers are obtained as in the inductive method with Equation 7.

$$\mathrm{CIO} = \frac{2^n \cdot x - 1}{3} \quad \text{and} \quad \mathrm{CO} = \frac{3 \cdot x + 1}{2^n} \quad (x \in \mathbb{N}_{\text{odd}})$$

$$\text{Layers: } 0.\ 1.\ 2.\ \underset{\leftarrow \mathrm{CO}}{\overset{\mathrm{CIO}\rightarrow}{3.\ 4.}}\ 5.\ 6. \ldots k. < \omega \quad (k \in \mathbb{N})$$

Collatz inverse operations (CIO) create sets in the next higher layer from numbers in the lower layer. Collatz operations (Collatz function) convert the sets in the upper layer to the numbers in the previous lower layer. In other words, with the Collatz function, the sets (numbers in sets) in each layer are transformed to the previous layer and then to other layers and finally to layer 0 (set Y_0) and reach 1. Therefore, the sequences of positive odd integers generated by the Collatz function have no initial term and, like the set of positive odd integers, are unbounded from above and bounded by 1 from below. Thus, the positive odd numbers in the k-th layer are transformed by the Collatz function into the numbers in the previous $(k-1)$th layer and so on, until they finally become an element of the set Y_0 (0-th layer) and then become

1. All sequences of positive odd numbers generated by the Collatz function (CO) behave similarly. Every number that is a multiple of 3 in the sets in the layers is connected (transformed) to an element of these sequences and follows the same path as the sequence. Any number that is a multiple of 3 is transformed into the number in the previous layer in the same way as the entire set of which it is an element. Therefore, all sequences of positive odd integers generated by the Collatz function are convergent, i.e., the smallest element of the set of positive odd integers converges to 1. In the next section, we will see that there is no divergent sequence generated by the Collatz function. Let's take a number from the 3rd layer, say 11, and the sequence obtained from this number with the Collatz function is as follows, where the parentheses indicate the number of layers.

$$\begin{gathered}\text{CO} \rightarrow \\ 11_{(3.)} \rightarrow 17_{(2.)} \rightarrow 13_{(1.)} \rightarrow 5_{(0.)} \rightarrow 1\end{gathered}$$

The starting number of this sequence is not $11_{(3.)}$ in layer 3, the starting numbers of all sequences are the numbers in the k-th layer generated with CIO, i.e. positive odd integers in the k-th layer $k < \omega, (\ k \in \mathbb{N})$. When the Collatz inverse operations (CIO) are applied to the numbers forming the sequences, they are sequentially transformed into the numbers of the next higher layer. In this way, divergent sequences are formed with CIO.

$$\begin{gathered}\text{CO} \rightarrow \\ 9_{(5.)} \rightarrow 7_{(4.)} \rightarrow 11_{(3.)} \rightarrow 17_{(2.)} \rightarrow 13_{(1.)} \rightarrow 5_{(0.)} \rightarrow 1\end{gathered}$$

When a number in the sequence that is a multiple of 3, such as $9_{(5)}$, is reached, it can be continued with a number that is not a multiple of 3 from the set $\{ 9, 37, 149, 597, 2389, \dots\}$ generated by $7_{(4.)}$ using CIO, e.g. 37. All numbers in this set are converted to 7 with CO and reach 1 by the same path in the sequence. Continuing with 37, if a multiple of 3 occurs again, the same procedure is repeated and the first term of the sequence becomes a number in the k-th layer. This is true for all sequences generated by the Collatz function.

$$\begin{gathered}\text{CO} \rightarrow \\ \dots \rightarrow 65_{(7.)} \rightarrow 49_{(6.)} \rightarrow 37_{(5.)} \rightarrow 7_{(4.)} \rightarrow 11_{(3.)} \rightarrow 17_{(2.)} \rightarrow 13_{(1.)} \rightarrow 5_{(0.)} \rightarrow 1\end{gathered}$$

$$\dots \rightarrow 65_{(7.)} \rightarrow 49_{(6.)} \rightarrow \begin{matrix} 37_{(5.)} \\ 9_{(5.)} \end{matrix} \rightarrow 7_{(4.)} \rightarrow 11_{(3.)} \rightarrow 17_{(2.)} \rightarrow 13_{(1.)} \rightarrow 5_{(0.)} \rightarrow 1$$

Since the sequences generated by the Collatz inverse function are divergent, all sequences generated by the Collatz function have no initial terms; they have as many elements as the number of layers, i.e., $k < \omega$, $(k \in \mathbb{N})$. All sequences of positive odd numbers generated by the Collatz function reach 1 starting from the numbers in the k-th layer, so all sequences are convergent.

As can be seen in Figure 1, the cardinality of the set of disjoint sets generated by the CIO from a number that is not a multiple of 3 at any layer is $\aleph_0 \cdot \aleph_0 \cdot \aleph_0 \cdots$. To define this expression, it is necessary to have an initial $\aleph_0$. This is possible only if the sequences generated by the Collatz function are convergent. For example, the cardinality of the set of disjoint sets generated by the CIO from 11 in layer 3 is as follows. As the number of layers increases, the cardinality of the sets increases by a factor of $\aleph_0$ (Figure 1).

$$|{}^{11}\mathrm{Y}| = \aleph_0 \cdot \aleph_0 \cdot \aleph_0 \ldots$$
$$|{}^{17}\mathrm{Y}| = \aleph_0 \cdot |{}^{11}\mathrm{Y}|$$
$$|{}^{13}\mathrm{Y}| = \aleph_0 \cdot |{}^{17}\mathrm{Y}|$$
$$|{}^{5}\mathrm{Y}| = \aleph_0 \cdot |{}^{13}\mathrm{Y}|$$
$$|{}^{1}\mathrm{Y}| = \aleph_0 \cdot |{}^{5}\mathrm{Y}| = \aleph_0 \cdot \aleph_0 \cdot \aleph_0 \cdots = \aleph_0^{k<\omega} \quad (k \in \mathbb{N})$$

The elements of each Collatz number set in Figure 1, obtained by transforming each Collatz number, form a sequence such that each term is 4 times the previous term plus 1. Thus, the elements of each Collatz number set form a loop with remainders 0,1,2 according to $\pmod 3$. New Collatz number sets are generated continuously to infinity from numbers with remainders 1 and 2 according to $\pmod 3$. Therefore, $\aleph_0^0$, $\aleph_0^1$, $\aleph_0^2$ exist (Figure 1), and for $\forall n \in \mathbb{N}^+$, if $\aleph_0^n$ exists, then $\aleph_0^{n+1}$ also exists. Thus, the cardinality of the set of disjoint Collatz number sets in Figure 1 is $\aleph_0.\aleph_0.\aleph_0 \ldots^{\cdot} = \aleph_0^{k<\omega}$ $(k \in \mathbb{N})$. Since all elements of the Collatz number sets form a cycle with the remainders 0,1,2 with respect to $\pmod 3$, all positive odd numbers that are multiples of 3 are obtained from the remainders 0 according to $\pmod 3$.

The elements of the Collatz number sets obtained by Equation 7 form a sequence in which each term is 1 more than 4 times the previous term. The same method is used to cover the set of positive odd integers. From each odd integer in the $\mathbb{N}_{odd}$ set, sets are formed such that the next term is 1 more than 4 times the previous term.

$$\begin{aligned} p_1 = &\quad \{\mathbf{1}, 5, 21, 85, \ldots\} \\ p_2 = &\quad \{\mathbf{3}, 13, 53, 213, \ldots\} \\ &\quad \{5, 21, 85, 341, \ldots\} \\ p_3 = &\quad \{\mathbf{7}, 29, 117, 469, \ldots\} \\ p_4 = &\quad \{\mathbf{9}, 37, 149, 597, \ldots\} \\ &\quad \vdots \end{aligned}$$

The union of sets that are disjoint from sets of the form is equal to the set of positive odd integers $\bigcup_{i=1}^{\infty} p_i = \mathbb{N}_{\text{odd}}$. Since other sets are subsets of disjoint sets, they can be ignored.

$\mathbb{N}_{odd}$ =[{1,5,21,85,...}{3,13,53,213,...}{7,29,117,469,...}{9,37,149,597,...} {11,45,181,725,...} ...]
The union of disjoint Collatz number sets obtained in Figure 1 is equal to the set $\mathbb{N}_{odd}$. This is because the cardinality of the set of disjoint Collatz number sets, by the inductive method described above, it was shown in Figure 1 that $\aleph_0^0$, $\aleph_0^1$, $\aleph_0^2$ exist and $\forall k \in \mathbb{N}^+$, if $\aleph_0^k$ exists, then $\aleph_0^{k+1}$ also exists.

Set of disjoint Collatz Number Sets (Figure 1):
Y= [{1,5,21,...}{3,13,53,...}{113,453,1813,...} {227,909,3637,...} {7281, 29125, 116501,...} {17, 69, 277,...} {35, 141, 565,...} ...]

The set Y is equal to the set $\mathbb{N}_{odd}$. The sets Y and $\mathbb{N}_{odd}$ are composed of the same disjoint sets and are equal in number, i.e., they are equal sets. The number of disjoint Collatz sets cannot be less than the number of sets in the set $\mathbb{N}_{odd}$ because, as shown by induction, set formation is continuous, and the number of sets cannot be greater because the Collatz numbers are elements of the set $\mathbb{N}_{odd}$.
The cardinality of the set of disjoint sets in the $\mathbb{N}_{odd}$ set is $\aleph_0$. It was found that the cardinality of the set of disjoint Collatz number sets is $\aleph_0.\aleph_0.\aleph_0 \ldots = \aleph_0^{k<\omega}$ ($k \in \mathbb{N}_{odd}$) (Figure 1). Y is the set of all disjoint Collatz sets and $\mathbb{N}_{odd}$ is the set of all disjoint sets.

$$\aleph_0 \cdot \aleph_0 \cdot \aleph_0 \cdots = \aleph_0^{k<\omega} = \aleph_0 \quad \text{and} \quad k \in \mathbb{N}_{odd} \implies \text{Y} \supseteq \mathbb{N}_{\text{odd}}$$

Thus the set of Collatz numbers certainly covers its universal set $\mathbb{N}_{odd}$, but cannot exceed it, since the positive odd integer k obtained by induction cannot be equal to ω and Collatz numbers are positive odd numbers. The cardinality $\aleph_0^{k<\omega}$ ($k \in \mathbb{N}_{odd}$) means that as many disjoint Collatz sets can be generated as there are disjoint sets in the set Nodd. This is because $k < \omega$ means that as many disjoint Collatz sets can be generated as there are positive odd numbers. [5] The Collatz number set covers the $\mathbb{N}_{odd}$ set, and since the $\mathbb{N}_{odd}$ set covers the Collatz number set, they are equal. Thus, we find that the set of Collatz numbers is equal to the set $\mathbb{N}^+$ (Remark 2.2), and and we prove the Collatz Conjecture for the set $\mathbb{N}^+$.

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[5] The expression $\aleph_0^{k<\omega}$ ($k \in \mathbb{N}_{odd}$), the cardinality of the set of disjoint Collatz number sets, implies that as many as k disjoint sets can be generated, and since k can be all positive odd numbers, it covers all disjoint sets that can exist in the set $\mathbb{N}_{odd}$ generated by the given rule.

3 The Absence of any Positive Integer other than Collatz Numbers

In this section, we prove that there are no positive integers that do not satisfy the conjecture.

Let s_1 be a number that is not a Collatz number and $(s_1 \in \mathbb{N}_{odd})$, then when Collatz operations are applied to s_1, until odd numbers are found.

$$s_1 \rightarrow \ \frac{3.s_1 + 1}{2^n}, \quad \mathrm{s}_2 \rightarrow \mathrm{s}_3 \rightarrow \mathrm{s}_4 \rightarrow \mathrm{s}_5 \rightarrow \mathrm{s}_6 \rightarrow \mathrm{s}_7 \rightarrow \mathrm{s}_8 \rightarrow \mathrm{s}_9 \rightarrow \mathrm{s}_{10} \ldots$$

S$= \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, \ldots\}$ and the elements of the set S are not Collatz numbers $(s_n \in \mathbb{N}_{odd})$.

Lemma 3.1 The elements of the set S do not any loop.
Proof. Suppose such a loop exists.

$$\begin{array}{lccccccccr}
 & s_1 \rightarrow s_2 & \rightarrow s_3 \rightarrow & s_4 & \rightarrow & s_5 \rightarrow & s_6 \rightarrow & s_7 & \rightarrow s_8 & \rightarrow s_9 \rightarrow s_{10} \\
 & \uparrow & & & & & & & & \downarrow \\
\text{CIO} \downarrow & \vdots & & & & & & & & \mathrm{s}_{11} \downarrow \text{CO} \\
 & \uparrow & & & & & & & & \downarrow \\
 & s_n \leftarrow \ldots\ldots\ldots\ldots & \leftarrow & \ldots\ldots & \leftarrow \ldots\ldots\ldots\ldots & \leftarrow & & & & \mathrm{s}_{13} \leftarrow \mathrm{s}_{12}
\end{array}$$

Figure 4

For such a loop to be exist in positive odd integers (Figure 4), all the elements of the loop must be equal, because the infinite set of numbers obtained by applying the CIO to each element of the loop is the same, that is, $\{s_1, s_{11}, s_{12}, \ldots \ s_2, s_{21}, s_{22}, \ldots \ s_3, s_{31}, s_{31}, \ldots \ s_n, s_{n1}, s_{n2}, \ldots\}$. In the positive odd integers, only the number 1 can form a loop with itself, so all elements of the loop are 1.
Lets take s_1 in the loop, $s_1 \not\equiv 0 \pmod 3$ and $(n, m \in \mathbb{N}^+)$, then if $s_1 \rightarrow^{CIO} = s_1 \rightarrow^{CO}$,

$$\frac{2^n s_1 - 1}{3} = \frac{3s_1 + 1}{2^m} \qquad 2^{n+m}.s_1 - 2^m = 9s_1 + 3, \quad s_1 = \frac{2^m + 3}{2^{n+m} - 9}$$

s_1 cannot be any positive odd integer other than 1 in this equation. Therefore, there is no loop other than 1 in the sequences of positive odd integers formed by Collatz operations.
For any positive odd number (s_1) that is not a Collatz number to exist, at least one of the following conditions must be satisfied.

I. When Collatz operations (CO) are applied to s_1, there must be a loop other than 1.
II. When Collatz operations (CO) are applied to s_1, the resulting sequence must be divergent.
It has been shown that case I does not exist, i.e. there are no loops other than 1. Now let us consider case II. If CO is applied to s_1, the resulting sequence of positive odd numbers S= $\{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, \ldots\}$. As shown in Figure 1, applying continuous Collatz inverse operations (CIO) to the element s_1 of the set S (s_2 instead of s_1 if s_1 is a multiple of 3) yields infinite layers of sets without Collatz numbers.

$$\{s_1\}$$

$${}^{s_1}\mathrm{S}_0 = \Big[\ s_1^* = \{s_{11}, s_{12}, s_{13}, s_{14}, s_{15}, s_{16}, s_{17}, s_{18} \ldots\}\ \Big] \qquad |{}^{s_1}\mathrm{S}_0| = 1$$

$${}^{s_1}\mathrm{S}_1 = \Big[\ s_1^* = \{s_{11}^* = \{s_{111}, s_{112}, s_{113} \ldots\}\ s_{12}^* = \{s_{121}, s_{122}, s_{123} \ldots\}\ s_{13}^* = \{s_{131}, s_{132}, s_{133} \ldots\} \ldots\} \ldots\ \Big] \qquad |{}^{s_1}\mathrm{S}_1| = \aleph_0$$

$${}^{s_1}\mathrm{S}_2 = \Big[\ s_{11}^* = \{s_{111}^* = \{s_{1111}, s_{1112}, s_{1113} \ldots\}\ s_{112}^* = \{s_{1121}, s_{1122}, s_{1123} \ldots\} \ldots\}\ s_{12}^* = \{s_{121}^* = \{s_{1211}, s_{1212}, s_{1213} \ldots\}\ s_{122}^* = \{s_{1221}, s_{1222}, s_{1223} \ldots\} \ldots\}\ \ldots\ \Big] \qquad |{}^{s_1}\mathrm{S}_2| = \aleph_0 + \aleph_0 + \aleph_0 \ldots = \aleph_0.\aleph_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \qquad |{}^{s_1}\mathrm{S}| = \aleph_0.\aleph_0.\aleph_0 \ldots$$

The set of disjoint sets that are not Collatz number sets formed by s_1:

$${}^{s_1}\mathrm{S} = \Big[\ \{s_{11}, s_{12}, s_{13}, \ldots\}\ \{s_{111}, s_{112}, s_{113}, \ldots\}\{s_{121}, s_{122}, s_{123}, \ldots\}\ \{s_{131}, s_{132}, s_{133}, \ldots\}\ \{s_{1111}, s_{1112}, \ldots\}\ \{s_{1121}, s_{1122}, \ldots\}\ \ldots\ \Big] \qquad |{}^{s_1}\mathrm{S}| = \aleph_0.\aleph_0.\aleph_0 \ldots$$

Figure 5: Sets that are not Collatz number sets.| | represents cardinality of the set of sets, and * represents conversions of numbers that are not multiples of 3 using Equation (7).

The elements of each set in Figure 5, obtained by converting each number that is not a Collatz number, form a sequence such that the next term is 4 times the previous term plus 1. Thus, the elements of each set form a loop with remainders 0,1,2 according to (mod 3). New sets are formed continuously to infinity from numbers with remainders 1 and 2 according to (mod 3). Therefore, the cardinality of the set of disjoint sets that are not Collatz number sets formed by s_1 in Figure 5 is $|{}^{s_1}\mathrm{S}| = \aleph_0.\aleph_0.\aleph_0 \ldots$

When Collatz operations (CO) are applied to s_1, the resulting sequence of positive odd integers is,

$$s_1 \to s_2 \to s_3 \to s_4 \to s_5 \to s_6 \to s_7 \to s_8 \to s_9 \to s_{10} \ldots$$

Each term of the sequences generated by Collatz operations (CO) represents different layers, as seen in Corollary 2.12. Therefore, when the layers change in this way from s_1 to s_2 or from s_2 to s_3 and so on, the cardinality of the resulting disjoint sets increases by a factor of $\aleph_0$ respectively:

$$\begin{aligned}
|{}^{s_2}\mathrm{S}| &= \aleph_0 \cdot |{}^{s_1}\mathrm{S}| \\
|{}^{s_3}\mathrm{S}| &= \aleph_0 \cdot |{}^{s_2}\mathrm{S}| = \aleph_0 \cdot \aleph_0 \cdot |{}^{s_1}\mathrm{S}| \\
|{}^{s_4}\mathrm{S}| &= \aleph_0 \cdot \aleph_0 \cdot \aleph_0 \cdot |{}^{s_1}\mathrm{S}| \\
\vdots \quad &= \ldots \aleph_0 \cdot \aleph_0 \cdot \aleph_0 \cdot |{}^{s_1}\mathrm{S}| = \ldots \aleph_0 \cdot \aleph_0 \cdot \aleph_0 \cdots = |\mathrm{S}|
\end{aligned}$$

This result implies that the cardinality of the set of disjoint sets cannot be defined as $\aleph_0^{n<\omega}$ $(n \in \mathbb{N})$. The cardinality of the set of disjoint sets formed by the set S is $\ldots \aleph_0^{n}, \aleph_0^{n+1}, \aleph_0^{n+2} \ldots$ $(n \in \mathbb{N})$. The set of natural numbers is a bounded set from below, so an unbounded set of the form $\{\ldots, n, n{+}1, n{+}2, \ldots\}$ cannot be generated in the set of natural numbers. Therefore, positive odd number sequences generated by the Collatz function (CO) cannot be divergent, they must be convergent. If we apply Collatz operations (CO) to any positive odd integer, the resulting sequence is convergent. This is because the set of positive odd integers is a bounded set from below.

As shown in Corollary 2.12, when the Collatz function (CO) is applied to any positive odd integer, the resulting sequence has no initial term, since all sequences can be extended to divergent sequences by applying Collatz inverse operations (CIO). For example, let s_1 be a positive odd integer. If s_1 is not a multiple of 3 (if s_1 is a multiple of 3, start with s_2 instead of s_1), a divergent sequence is constructed from s_1 by the Collatz inverse operations (CIO), i.e., from s_1 to s_{11}, from s_{11} to s_{111}, etc. (if s_{11} is a multiple of 3, replace by s_{12}, if s_{111} is a multiple of 3, replace by s_{112}, etc.). When the Collatz function (CO) is applied to s_1, the result is $s_1, s_2, s_3, s_4, \ldots$

$$\underset{(\mathrm{CO}\,\rightarrow)}{\overset{(\leftarrow\,\mathrm{CIO})}{\ldots, s_{11111}, s_{1111}, s_{111}, s_{11}, s_1, s_2, s_3, s_4, s_5, \ldots}}$$

If the sequence of positive odd integers obtained with the Collatz function (CO) is first a convergent sequence, i.e. $\{\ldots, s_{11111}, s_{1111}, s_{111}, s_{11}, s_1\}$, then the same sequence $\{s_1, s_2, s_3, s_4, s_5, \ldots\}$ cannot be divergent. A sequence of positive odd integers generated by the same function (same rule) cannot form two different states (both convergent and divergent). Thus, if the Collatz operations are applied to any positive odd integer, the resulting sequence must be convergent, and it converges to 1. Therefore, there cannot be any positive odd integer that is not a Collatz number; all positive odd integers are Collatz numbers. All positive integers are also Collatz numbers (Remark 2.2).

4 Conclusion

The Collatz Conjecture was proved using the Collatz inverse operation method. It was shown that all positive integers reach 1 as stated in the Collatz Conjecture. With the methods described in this study for $3n + 1$, it can be found whether numbers such as $5n + 1$, $7n + 1$, $9n + 1$, $\ldots$ also reach 1.

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