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Coefficient Inequality for Certain New Sub-classes of Analytic Bi-univalent Functions

B. Srutha Keerthi[1] and Bhuvaneswari Raja[2]

Abstract

In this paper, we investigate two new subclasses of the function Σ of bi-univalent functions defined in the open unit disc. In this work, we obtain bounds for the coefficients $|a_2|$ and $|a_3|$ for functions in these new sub-classes.

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Keywords: Analytic and univalent functions; bi-univalent functions; λ-convex functions; co-efficient bounds

1 Introduction and Definitions

Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

[1] Department of Applied Mathematics, Sri Venkateswara College of Engineering, Sriperumbudur, Chennai - 602105, India, e-mail: sruthilaya06@yahoo.co.in
[2] Department of Mathematics, Aalim Muhammed Salegh College of Engineering, "Nizara Educational Campus", Muthapudupet, Avadi IAF, Chennai - 600 055, e-mail: shri_bhuva@yahoo.co.in

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which are analytic in the open unit disc $U = \{z : |z| < 1\}$. Further, by δ we shall denote the class of all functions in A which are univalent in U. It is well known that every function $f \in S$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in U)$$

$$\text{and} \qquad f(f^{-1}(w)) = w\{|w| < r_0(f); r_0(f) \geq 1/4\}$$

where $f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots.$

A function $f(z) \in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U [see [5]].

Let Σ denote the class of bi-univalent functions in U given by (1) Brannan and Taha [6] (see also [7]) introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses. $\delta^*(\alpha)$ and $K(\alpha)$ of starlike and convex functions of order α $(0 < \alpha < 1)$ respectively (see [8]). Thus, following Brannan and Taha [6] (see also [7]), a function $f(z) \in U$ is the class $\delta^*_\Sigma(\alpha)$ of strongly bi-starlike functions of order α $(0 < \alpha < 1)$ if each of the following conditions is satisfied:

$$f \in \Sigma, \left| arg\left(\frac{z^2f''(z) + zf'(z)}{zf'(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, z \in U)$$

and

$$\left| arg\left(\frac{w^2g''(w) + wg'(w)}{wg'(w)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, w \in U)$$

where g is the extension of f^{-1} to U. The classes $\delta^*_\Sigma(\alpha)$ and $K_\Sigma(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding to the function classes $\delta^*(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $\delta^*_\Sigma(\alpha)$ and $K^*_\Sigma(\alpha)$, they found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details, see [6, 7]).

The object of the present paper is to introduce two new subclasses of the function class Σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new sub-classes of the function class Σ employing the techniques used earlier by Srivastava et al. [5] (see also [6, 7, 8]).

In order to derive our main results, we have to recall here the following lemma [12].

Lemma 1.1. *If $h \in P$, then $|c_k| \leq 2$, for each k, where P is the family of all functions h analytic in U for which $Re(h(z)) > 0$.*

$$h(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots \text{ for } z \in U$$

2 Co-efficient Bounds for the Function Class $B_\Sigma(\alpha, \lambda, \mu)$

Definition 2.1. *A function $f(z)$ given by (1) is said to be in the class $B_\Sigma(\alpha, \lambda, \mu)$ if the following conditions are satisfied:*

$$f \in \Sigma : \left| arg\left[\frac{\lambda\mu z^3 f'''(z) + (2\lambda\mu + \lambda - \mu)z^2 f''(z) + zf'(z)}{\lambda\mu z^2 f''(z) + (\lambda - \mu)zf'(z) + (1 - \lambda + \mu)f(z)}\right]\right| < \frac{\alpha\pi}{2}, \quad (2)$$

$$(0 \leq \alpha \leq 1, 0 \leq \mu \leq \lambda \leq 1, z \in U)$$

and

$$\left| arg\left[\frac{\lambda\mu w^3 g'''(w) + (2\lambda\mu + \lambda - \mu)w^2 g''(w) + wg'(w)}{\lambda\mu w^2 g''(w) + (\lambda - \mu)wg'(w) + (1 - \lambda + \mu)g(w)}\right]\right| < \frac{\alpha\pi}{2}, \quad (3)$$

$$(0 \leq \alpha \leq 1, 0 \leq \mu \leq \lambda \leq 1, w \in U)$$

where the function g is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots \quad (4)$$

We note that for $\mu = 0$, we get the class $B_\Sigma(\alpha, \lambda)$ which is defined as follows:

$$f \in \Sigma, \left| arg\left[\frac{\lambda z^2 f''(z) + zf'(z)}{\lambda zf'(z) + (1 - \lambda)f(z)}\right]\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, z \in U) \quad (2.1.2)$$

and

$$f \in \Sigma, \left| arg\left[\frac{\lambda w^2 g''(w) + wg'(w)}{\lambda wg'(w) + (1 - \lambda)g(w)}\right]\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, w \in U) \quad (2.1.3)$$

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $B_\Sigma(\alpha, \lambda, \mu)$.

Theorem 2.2. *Let $f(z)$ given by (1) be in the class $B_\Sigma(\alpha, \lambda, \mu)$, $0 \leq \alpha \leq 1$ and $0 \leq \mu \leq \lambda \leq 1$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{4\alpha(1 + 2\lambda - 2\mu + 6\lambda\mu) + (1 - 3\alpha)(1 + \lambda - \mu + 2\lambda\mu)^2}} \quad (5)$$

and

$$|a_3| \leq \frac{4\alpha^2}{(1 + \lambda - \mu + 2\lambda\mu)^2} + \frac{\alpha}{(1 + 2\lambda - 2\mu + 6\lambda\mu)}. \quad (6)$$

Proof. We can write the argument inequalities in (2) and (3) equivalently as follows:

$$\frac{\lambda\mu z^3 f'''(z) + (2\lambda\mu + \lambda - \mu)z^2 f''(z) + zf'(z)}{\lambda\mu z^2 f''(z) + (\lambda - \mu)zf'(z) + (1 - \lambda + \mu)f(z)} = [p(z)]^\alpha \tag{7}$$

and

$$\frac{\lambda\mu w^3 g'''(w) + (2\lambda\mu + \lambda - \mu)w^2 g''(w) + wg'(w)}{\lambda\mu w^2 g''(w) + (\lambda - \mu)wg'(w) + (1 - \lambda + \mu)g(w)} = [q(w)]^\alpha. \tag{8}$$

Respectively, where $p(z)$ and $q(w)$ satisfy the following inequalities $Re(p(z)) > 0$, $(z \in U)$ and $Re(q(w)) > 0$, $(w \in U)$.

Furthermore, the functions $p(z)$ and $q(w)$ have the forms,

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{9}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots \tag{10}$$

Now, equating the coefficients in (7) and (8) we get,

$$(1 + \lambda - \mu + 2\lambda\mu)a_2 = p_1\alpha \tag{11}$$

$$(2 + 4\lambda - 4\mu + 12\lambda\mu)a_3 = p_2\alpha + \frac{\alpha(\alpha-1)}{2}p_1^2 + \alpha^2 p_1^2 \tag{12}$$

and

$$-(1 + \lambda - \mu + 2\lambda\mu)a_2 = q_1(\alpha) \tag{13}$$

$$(2 + 4\lambda - 4\mu + 12\lambda\mu)(2a_2^2 - a_3) = q_2\alpha + \frac{\alpha(\alpha-1)}{2}q_1^2 + \alpha^2 q_1^2 \tag{14}$$

from (11) and (13) we get

$$p_1 = -q_1 \tag{15}$$

and

$$2a_2^2(1 + \lambda - \mu + 2\lambda\mu)^2 = \alpha^2(p_1^2 + q_1^2). \tag{16}$$

Now from (12), (14) and (16) we obtain

$$4a_2^2(1 + \lambda - \mu + 2\lambda\mu)^2 = \alpha(p_2 + q_2) + \frac{\alpha(\alpha-1)}{2}(p_1^2 + q_1^2) + \alpha^2(p_1^2 + q_1^2). \tag{17}$$

Therefore, we have

$$a_2^2 = \frac{\alpha^2(p_2 + q_2)}{4\alpha(1 + 2\lambda - 2\mu + 6\lambda\mu) + (1 - 3\alpha)(1 + \lambda - \mu + 2\lambda\mu)^2}.$$

Applying Lemma 1.1 to the coefficients p_2 and q_2 we have

$$|a_2| \leq \frac{2\alpha}{\sqrt{4\alpha(1+2\lambda-2\mu+6\lambda\mu)+(1-3\alpha)(1+\lambda-\mu+2\lambda\mu)^2}}.$$

Next, in order to find the bound on $|a_3|$, by subtracting (12) from (14), we get

$$(2a_3-2a_2^2)(2+4\lambda-4\mu+12\lambda\mu)=\alpha(p_2-q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2-q_1^2)+\alpha^2(p_1^2-q_1^2)$$

upon substituting the value of a_2^2 from (16) and observing that $p_1^2=q_1^2$, it follows that

$$a_3 = a_2^2 + \frac{\alpha(p_2-q_2)}{2(2+4\lambda-4\mu+12\lambda\mu)}$$
$$a_3 = \frac{\alpha^2(p_1^2+q_1^2)}{2(1+\lambda-\mu+2\lambda\mu)^2} + \frac{\alpha(p_2-q_2)}{2(2+4\lambda-4\mu+12\lambda\mu)}.$$

Applying Lemma (1.1) once again for coefficients p_1, p_2, q_1 and q_2 we get,

$$|a_3| \leq \frac{4\alpha^2}{(1+\lambda-\mu+2\lambda\mu)^2} + \frac{\alpha}{(1+2\lambda-2\mu+6\lambda\mu)}.$$

This completes the proof of Theorem 2.2. □

Now, putting $\mu = 0$ in Theorem 2.2 we have.

Corollary 2.3. *Let $f(z)$ given by (1) be in the class $B_\Sigma(\alpha,\lambda)$ then,*

$$|a_2| \leq \frac{2\alpha}{\sqrt{4\alpha(1+2\lambda)+(1-3\alpha)(1+\lambda)^2}} \tag{18}$$

and

$$|a_3| \leq \frac{4\alpha^2}{(1+\lambda)^2} + \frac{\alpha}{(1+2\lambda)}. \tag{19}$$

3 Coefficient Bounds for the Function Class $N_\Sigma(\beta, \lambda, \mu)$

Definition 3.1. *A function $f(z)$ given by (1) is said to be in the class $N_\Sigma(\beta, \lambda, \mu)$ if the following conditions are satisfied,*

$$f \in \Sigma, Re\left[\frac{\lambda\mu z^3 f'''(z) + (2\lambda\mu + \lambda - \mu)z^2 f''(z) + zf'(z)}{\lambda\mu z^2 f''(z) + (\lambda - \mu)zf'(z) + (1 - \lambda + \mu)f(z)}\right] > \beta,$$
$$(0 \leq \beta \leq 1, 0 \leq \mu \leq \lambda \leq 1, z \in U) \tag{20}$$

and

$$Re\left[\frac{\lambda\mu w^3 g'''(w) + (2\lambda\mu + \lambda - \mu)w^2 g''(w) + wg'(w)}{\lambda\mu w^2 g''(w) + (\lambda - \mu)wg'(w) + (1 - \lambda + \mu)g(w)}\right] > \beta,$$
$$(0 \leq \beta \leq 1, 0 \leq \mu \leq \lambda \leq 1, w \in U) \tag{21}$$

where the function $g(w)$ is defined by (4).

Definition 3.2. *We note that for $\mu = 0$, the class $N_\Sigma(\beta, \lambda, \mu)$ reduces to $N_\Sigma(\beta, \lambda)$ which satisfies the following conditions*

$$f \in \Sigma, Re\left[\frac{\lambda z^2 f''(z) + zf'(z)}{\lambda zf'(z) + (1 - \lambda)f(z)}\right] > \beta, \quad (0 \leq \beta \leq 1, 0 \leq \lambda \leq 1, z \in U) \tag{3.1.1}$$

and

$$Re\left[\frac{\lambda w^2 g''(w) + wg'(w)}{\lambda wg'(w) + (1 - \lambda)g(w)}\right] > \beta, \quad (0 \leq \beta \leq 1, 0 \leq \lambda \leq 1, w \in U) \tag{3.1.2}$$

Theorem 3.3. *Let $f(z)$ given by (1) be in the class $N_\Sigma(\beta, \lambda, \mu)$, $0 \leq \beta \leq 1$ and $0 \leq \mu \leq \lambda \leq 1$, then*

$$|a_2| \leq \sqrt{\frac{2(1 - \beta)}{(2 + 4\lambda - 4\mu + 12\lambda\mu) - (1 + \lambda - \mu + 2\lambda\mu)^2}} \tag{2}$$

and

$$|a_3| \leq \frac{4(1 - \beta)^2}{(1 + \lambda - \mu + 2\lambda\mu)^2} + \frac{(1 - \beta)}{(1 + 2\lambda - 2\mu + 6\lambda\mu)}. \tag{3}$$

Proof. It follows from (20) and (21) that there exists $p(z)$ and $q(w)$ such that

$$\frac{\lambda\mu z^3 f'''(z) + (2\lambda\mu + \lambda - \mu)z^2 f''(z) + zf'(z)}{\lambda\mu z^2 f''(z) + (\lambda - \mu)zf'(z) + (1 - \lambda + \mu)f(z)} = \beta + (1-\beta)p(z) \tag{4}$$

and

$$\frac{\lambda\mu w^3 g'''(w) + (2\lambda\mu + \lambda - \mu)w^2 g''(w) + wg'(w)}{\lambda\mu w^2 g''(w) + (\lambda - \mu)wg'(w) + (1 - \lambda + \mu)g(w)} = \beta + (1-\beta)q(w) \tag{5}$$

where $p(z)$ and $q(w)$ have the forms (9), (10) respectively.

Equating the coefficients in (4) and (5) yields

$$(1 + \lambda - \mu + 2\lambda\mu)a_2 = (1-\beta)p_1 \tag{6}$$

$$(2 + 4\lambda - 4\mu + 12\lambda\mu)a_3 = (1-\beta)p_2 + (1-\beta)^2 p_1^2 \tag{7}$$

and

$$-(1 + \lambda - \mu + 2\lambda\mu)a_2 = (1-\beta)q_1 \tag{8}$$

$$(2 + 4\lambda - 4\mu + 12\lambda\mu)(2a_2^2 - a_3) = (1-\beta)q_2 + (1-\beta)^2 q_1^2 \tag{9}$$

from (6) and (8) we get

$$p_1 = -q_1 \tag{10}$$

and

$$2a_2^2(1 + \lambda - \mu + 2\lambda\mu)^2 = (1-\beta)^2(p_1^2 + q_1^2). \tag{11}$$

Now from (7), (9) and (11) we obtain,

$$\begin{aligned} 4(1 + 2\lambda - 2\mu + 6\lambda\mu)a_2^2 &= (1-\beta)(p_2 + q_2) + (1-\beta)^2(p_1^2 + q_1^2) \\ &= (1-\beta)(p_2 + q_2) + 2a_2^2(1 + \lambda - \mu + 2\lambda\mu)^2. \end{aligned}$$

Therefore we have,

$$a_2^2 = \frac{(1-\beta)(p_2 + q_2)}{2[(2 + 4\lambda - 4\mu + 12\lambda\mu) - (1 + \lambda - \mu + 2\lambda\mu)^2]}$$

Applying Lemma 1.1 for the coefficients p_2 and q_2 we have,

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{(2 + 4\lambda - 4\mu + 12\lambda\mu) - (1 + \lambda - \mu + 2\lambda\mu)^2}}$$

Next, in order to find the bound on $|a_3|$, by (7) from (9) we get, and from (11),

$$(2 + 4\lambda - 4\mu + 12\lambda\mu)(2a_3 - 2a_2^2) = (1-\beta)(p_2 - q_2) + (1-\beta)^2(p_1^2 - q_1^2).$$

Since $p_1^2 = q_1^2$ it follows that,

$$\begin{aligned}(2+4\lambda-4\mu+12\lambda\mu)(2a_3-2a_2^2) &= (1-\beta)(p_2-q_2)\\ 2a_3(2+4\lambda-4\mu+12\lambda\mu) &= 2a_2^2(2+4\lambda-4\mu+12\lambda\mu)\\ &\quad+(1-\beta)(p_2-q_2)\\ 2a_3(2+4\lambda-4\mu+12\lambda\mu) &= \frac{(1-\beta)^2(p_1^2+q_1^2)}{(1+\lambda-\mu+2\lambda\mu)^2}\cdot\\ &\quad\cdot(2+4\lambda-4\mu+12\lambda\mu)+(1-\beta)(p_2-q_2).\end{aligned}$$

Once again for the coefficients p_1, q_1, p_2 and q_2 applying Lemma 1.1, we get,

$$|a_3| \leq \frac{4(1-\beta)^2}{(1+\lambda-\mu+2\lambda\mu)^2} + \frac{(1-\beta)}{(1+2\lambda-2\mu+6\lambda\mu)}.$$

This completes the proof of Theorem 3.3. □

Now, putting $\mu = 0$ in Theorem 3.3, we have the following corollary.

Corollary 3.4. *Let $f(z)$ given by (1) be in the class $N_\Sigma(\beta, \lambda)$, $(0 \leq \beta \leq 1)$ and $(0 \leq \lambda \leq 1)$ then,*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{2(1+2\lambda)-(1+\lambda)^2}} \qquad (12)$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{(1+\lambda)^2} + \frac{(1-\beta)}{1+2\lambda}. \qquad (13)$$

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