A high-order compact finite volume element method for the dual-phase lag (DPL) heat conduction equations with the interface

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Abstract

In this paper, a high-order compact finite volume element method is presented for one-dimensional dual-phase lag equations with the interface. The resulting coefficient matrix is five-diagonal. This high-order method is helpful to analyze and study nano heat conduction with this equation in relative coarse grid. We apply the discrete energy method to give the error estimate in the L^2 norm with the convergence order $O(\Delta t^2 + h^{3.5})$. Finally, numerical examples are provided to show the effectiveness and feasibility of this method.

Key words: dual-phase lag equation, finite volume element method, interface.

AMS(2000) subject classifications: 65M15 65M60

1 Introduction

The well-known dual-phase lag (DPL) heat conduction model proposed by Tzou [18, 19], has attracted a considerable interest in a wide variety of scientific and engineering fields. It is utilized to simulate heat transfer in micro- or nano-structures [7], interpret the non-Fourier heat conduction phenomena in processed meats [1] and so on. For the DPL heat conduction equation, its well-posedness and solution structure as well as stability have been analyzed [20, 16], and many numerical methods have been applied [8, 4, 2]. In this paper, we discuss the DPL equation with the interface, which describe the heat transfer in multi-layer material or composite materials.

This type of problems are referred to as interface problems. Generally, the interface problems have the discontinuities in the coefficients and singular source terms due to the presence of interfaces. There are two different approaches to the design of a method for interface problems. The first is to construct the methods on the body fitted grids [21]. However, it is time consuming to generate this kind of body fitted grids. The second is on a fixed grid, such as the immersed interface method (IIM) [9], the immersed finite element method (IFEM) [6, 11, 13] and so on. It needs to modify the classical numerical schemes on fitted grids.

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By introducing non-dimensional parameters [4], the 1-D DPL equation with the interface point is expressed as follows [12],

$$\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial t^2}=\frac{K^2}{3}\left(\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^3 u}{\partial t\partial x^2}\right)+s(x,t)+w\delta(x-l),\ x\in(0,1),\ t\in[0,T],\ l\in(0,1),\quad (1.1)$$

where δ is the Dirac delta function, $s(x,t)\in C([0,T],L^2(0,1))$, the coefficients K, $B\in C([0,T],C^1(0,l)\cup C^1(l,1))$, that is, they are discontinuous at $x=l$. We set

$$s=\begin{cases}s_1,\ x\in[0,l],\\ s_2,\ x\in(l,1],\end{cases}\qquad K=\begin{cases}K_1,\ x\in[0,l],\\ K_2,\ x\in(l,1],\end{cases}\qquad B=\begin{cases}B_1,\ x\in[0,l],\\ B_2,\ x\in(l,1].\end{cases}$$

Then $u\in C^2([0,T],C(0,1))$, namely, u is continuous. By integrating (1.1) from $x=l^-$ to $x=l^+$, we get

$$\frac{K_1^2}{3}\left(\frac{\partial u(l-0,t)}{\partial x}+B_1\frac{\partial^2 u(l-0,t)}{\partial t\partial x}\right)-\frac{K_2^2}{3}\left(\frac{\partial u(l+0,t)}{\partial x}+B_2\frac{\partial^2 u(l+0,t)}{\partial t\partial x}\right)=w,\ t\in[0,T],$$

in addition to

$$u(l-0,t)=u(l+0,t),\ t\in[0,T].$$

In this paper, we consider $w=0$. Then an alterative way to state the problem is as follows,

$$\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial t^2}=\frac{K_1^2}{3}\left(\frac{\partial^2 u}{\partial x^2}+B_1\frac{\partial^3 u}{\partial t\partial x^2}\right)+s_1(x,t),\ x\in(0,l),\ t\in[0,T],\quad (1.2)$$

$$\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial t^2}=\frac{K_2^2}{3}\left(\frac{\partial^2 u}{\partial x^2}+B_2\frac{\partial^3 u}{\partial t\partial x^2}\right)+s_2(x,t),\ x\in(l,1),\ t\in[0,T].\quad (1.3)$$

The initial conditions and boundary conditions are given as

$$u(x,0)=\phi_1(x),\ \frac{\partial u(x,0)}{\partial t}=\phi_2(x),\ x\in[0,1].\quad (1.4)$$

$$-\alpha_1K_1\frac{\partial u(0,t)}{\partial x}+u(0,t)=\varphi_1(t),\ t\in[0,T],\quad (1.5)$$

$$\alpha_2K_2\frac{\partial u(1,t)}{\partial x}+u(1,t)=\varphi_2(t),\ t\in[0,T].\quad (1.6)$$

At the interface point $x=l$, we prescribe the jump conditions for the solution,

$$u(l-0,t)=u(l+0,t),\ t\in[0,T].\quad (1.7)$$

$$K_1^2\left(\frac{\partial u(l-0,t)}{\partial x}+B_1\frac{\partial^2 u(l-0,t)}{\partial t\partial x}\right)=K_2^2\left(\frac{\partial u(l+0,t)}{\partial x}+B_2\frac{\partial^2 u(l+0,t)}{\partial t\partial x}\right),\ t\in[0,T].\quad (1.8)$$

The coefficients K_1, K_2, B_1, B_2, α_1, α_2 are positive constants. In fact, K_1 or K_2 is Knudsen number. If $K_1\approx 1.0$ (or K_2) or greater than 1.0, the case is typically at nano-scale, see as [4]. Here, we consider only one interface point. Certainly, our method can be easily extended to more than one interface point.

Recently, the compact finite difference method are attracting attention for its simplicity and high accuracy [15, 17]. The finite element method and the finite volume element method are very good to deal with the differential problem with the second or third boundary condition. The finite volume element method has the simplicity of finite difference method and the accuracy of finite element method [10], and keep local conservation. Our studies are motivated by the importance of the compact finite volume element method [5, 14], which has high order and the tri- or five-diagonal symmetrical coefficient matrix. In this paper, we treat the interface point as a node point and give a uniform discretization around it. This produces two space steps, which is more similar to the body-fitted grids, so the coefficient matrix is five-diagonal, but not symmetrical. However, the numerical schemes which we get are simple and effective.

The remainder of paper is organized as follows. In section 2, the high-order compact finite volume element method is obtained. In section 3, we apply the discrete energy method to give the error estimate in L^2 norm with the convergence order $O(\Delta t^2+h^{3.5})$. In section 4, examples are used to illustrate the effectiveness and feasibility of our proposed method.

2 The high-order compact finite volume element method

We introduce a new variable $v = u + u_t$ and substitute it into (1.2), (1.3). This gives the following system:

$$v_t - \frac{K_1^2}{3}((1-B_1)u_{xx} + B_1 v_{xx}) = s_1(x,t),\ x \in (0,l),\ t \in [0,T], \tag{2.1}$$

$$v_t - \frac{K_2^2}{3}((1-B_2)u_{xx} + B_2 v_{xx}) = s_2(x,t),\ x \in (l,1),\ t \in [0,T]. \tag{2.2}$$

For the new variable v, the corresponding initial and boundary conditions are given as

$$v(x,0) = \rho(x) = \phi_1(x) + \phi_2(x),\ x \in [0,1], \tag{2.3}$$

$$-\alpha_1 K_1 \frac{\partial v(0,t)}{\partial x} + v(0,t) = \eta_1(t) = \varphi_1(t) + \varphi_{1,t}(t),\ t \in [0,T], \tag{2.4}$$

$$\alpha_2 K_2 \frac{\partial v(1,t)}{\partial x} + v(1,t) = \eta_2(t) = \varphi_2(t) + \varphi_{2,t}(t),\ t \in [0,T]. \tag{2.5}$$

First we consider a set of uniform grid in the interval $[0,l]$ with grid points $x_i = ih_1$, $i = 0,1,\cdots,k$, where $h_1 = l/k$. And another set of uniform grid in the interval $[l,1]$ with grid points $x_i = ih_2$, $i = k, k+1, \cdots, m$, where $h_2 = (1-l)/(m-k)$. This implies that the grid point x_k is the interface point l. The other grid points can be called as regular points. Then we place a dual grid $0 = x_0 < x_{1/2} < x_{3/2} < \cdots < x_{k-1/2} < x_{k+1/2} < \cdots < x_m$, where $x_{i-1/2} = (x_{i-1} + x_i)/2$. Write $I_0^* = [x_0, x_{1/2}]$, $I_i^* = [x_{i-1/2}, x_{i+1/2}]$ $(i = 1,\cdots,m-1)$, $I_m^* = [x_{m-1/2}, x_m]$. These dual elements I_i^* $(i = 0,1,\cdots,m)$ lead to a dual grid. I_i^* is usually called as the control volume which is related to the node x_i.

For the interior regular points x_i, $i = 1,\cdots,m-1$, $i \neq k$, we integrate (2.1) or (2.2) on a dual element $I_i^* = [x_{i-1/2}, x_{i+1/2}]$, and apply the formula of integral by parts to get,

$$\begin{aligned}\int_{x_{i-1/2}}^{x_{i+1/2}} v_t dx - \frac{K^2}{3}(1-B)(u_x(x_{i+1/2},t) - u_x(x_{i-1/2},t)) \\ -\frac{K^2}{3}B(v_x(x_{i+1/2},t) - v_x(x_{i-1/2},t)) = \int_{x_{i-1/2}}^{x_{i+1/2}} s(x,t)dx,\end{aligned} \tag{2.6}$$

where $K = K_1$ or K_2, $B = B_1$ or B_2, $s = s_1$ or s_2.

Expanding $u(x_{i-1},t)$ at the point $(x_{i-1/2},t)$ in Taylor formula with integral remainder term yields the following result:

$$\begin{aligned}u(x_{i-1},t) = u(x_{i-1/2},t) - \frac{h}{2}u_x(x_{i-1/2},t) + \frac{1}{2}\left(\frac{h}{2}\right)^2 u_{xx}(x_{i-1/2},t) \\ -\frac{1}{6}\left(\frac{h}{2}\right)^3 u_{xxx}(x_{i-1/2},t) + \frac{1}{24}\left(\frac{h}{2}\right)^4 u_x^{(4)}(x_{i-1/2},t) + \frac{1}{24}\int_{x_{i-1/2}}^{x_{i-1}} u_x^{(5)}(x_{i-1}-x)^4 dx,\end{aligned} \tag{2.7}$$

Similar expansions for $u(x_i,t)$ at the same point $(x_{i-1/2},t)$. Then

$$u_x(x_{i-1/2},t) = \frac{u(x_i,t) - u(x_{i-1},t)}{h} - \frac{h^2}{24}u_{xxx}(x_{i-1/2},t) + R_{u(x_{i-1/2},t)}, \tag{2.8}$$

where

$$R_{u(x_{i-1/2},t)} = -\frac{1}{24h}\left(\int_{x_{i-1/2}}^{x_i} u_x^{(5)}(x,t)(x_i-x)^4 dx + \int_{x_{i-1}}^{x_{i-1/2}} u_x^{(5)}(x,t)(x_{i-1}-x)^4 dx\right).$$

We can also get $v_x(x_{i-1/2},t)$ and $R_{v(x_{i-1/2},t)}$, similarly.

Noting that, using (2.1) or (2.2) and first-order central difference formula, we deduce that

$$\begin{aligned}\frac{K^2}{3}(1-B)u_x^{(3)}(x_{i-1/2},t) + \frac{K^2}{3}Bv_x^{(3)}(x_{i-1/2},t) = (v_t)_x(x_{i-1/2},t) - s_x(x_{i-1/2},t) \\ = \frac{v_t(x_i,t) - v_t(x_{i-1},t)}{h} - s_x(x_{i-1/2},t) + R_{v_t(x_{i-1/2},t)},\end{aligned} \tag{2.9}$$

where

$$R_{v_t(x_{i-1/2},t)} = -\frac{1}{2h}\left(\int_{x_{i-1/2}}^{x_i} (v_t)_x^{(3)}(x,t)(x_i-x)^2dx + \int_{x_{i-1}}^{x_{i-1/2}} (v_t)_x^{(3)}(x,t)(x_{i-1}-x)^2dx\right). \quad (2.10)$$

Hence, by (2.8) and (2.9), we have

$$\frac{K^2}{3}(1-B)u_x(x_{i-1/2},t) + \frac{K^2}{3}Bv_x(x_{i-1/2},t) \triangleq D(u,v)_{(x_{i-1/2},t)} + R_{i-1/2,x}, \quad (2.11)$$

where

$$D(u,v)_{(x_{i-1/2},t)} = \frac{K^2}{3}(1-B)\frac{u(x_i,t)-u(x_{i-1},t)}{h} + \frac{K^2}{3}B\frac{v(x_i,t)-v(x_{i-1},t)}{h}$$
$$-\frac{h^2}{24}\left(\frac{v_t(x_i,t)-v_t(x_{i-1},t)}{h} - s_x(x_{i-1/2},t)\right),$$

and

$$R_{i-1/2,x} = \frac{K^2}{3}(1-B)R_{u(x_{i-1/2},t)} + \frac{K^2}{3}BR_{v(x_{i-1/2},t)} - \frac{h^2}{24}R_{v_t(x_{i-1/2},t)}. \quad (2.12)$$

The integral term about v_t in (2.6) can be computed as follows. First we construct second-order Lagrange interpolation for v_t as follows,

$$\Pi v_t = \frac{1}{2}\xi(\xi-1)v_t(x_{i-1},t)) - (\xi^2-1)v_t(x_i,t)) + \frac{1}{2}\xi(\xi+1)v_t(x_{i+1},t), \quad (2.13)$$

where $\xi = \frac{x-x_i}{h}$. Then integrating (2.13) gives

$$\int_{x_{i-1/2}}^{x_{i+1/2}} v_t dx \approx Sv_t(x_i,t) \triangleq \frac{h}{24}(v_t(x_{i-1},t) + 22v_t(x_i,t) + v_t(x_{i+1},t)). \quad (2.14)$$

Substituting (2.11), (2.14) into (2.6) and dropping the truncation errors, a semi-discrete high order finite volume element scheme is given as follows,

$$Sv_t(x_i,t) + D(u,v)_{(x_{i-1/2},t)} - D(u,v)_{(x_{i+1/2},t)} = \int_{x_{i-1/2}}^{x_{i+1/2}} s(x,t)dx. \quad (2.15)$$

That is,

$$\frac{h}{12}(v_t(x_{i-1},t) + 10v_t(x_i,t) + v_t(x_{i+1},t)) \quad (2.16)$$
$$-\frac{K^2}{3}(1-B)\frac{u(x_{i-1},t)-2u(x_i,t)+u(x_{i+1},t)}{h} - \frac{K^2}{3}B\frac{v(x_{i-1},t)-2v(x_i,t)+v(x_{i+1},t)}{h}$$
$$= \int_{x_{i-1/2}}^{x_{i+1/2}} s(x,t)dx + \frac{h^2}{24}(s_x(x_{i+1/2},t) - s_x(x_{i-1/2},t)).$$

For the left boundary point x_0, we integrate (2.1) on the dual element $I_0^* = [x_0, x_{1/2}]$ and get

$$\int_{x_0}^{x_{1/2}} v_t dx - \frac{K_1^2}{3}(1-B_1)(u_x(x_{1/2},t) - u_x(x_0,t)) - \frac{K_1^2}{3}B_1(v_x(x_{1/2},t) - v_x(x_0,t)) \quad (2.17)$$
$$= \int_{x_0}^{x_{1/2}} s_1(x,t)dx.$$

By (1.5) and (2.4),

$$u_x(x_0,t) = \frac{u(x_0,t)-\varphi_1(t)}{\alpha_1 K_1},\ v_x(x_0,t) = \frac{v(x_0,t)-\eta_1(t)}{\alpha_1 K_1}. \quad (2.18)$$

For the integral term about v_t, we apply the quadratic Hermite interpolation to obtain

$$\Pi v_t = (1-\xi^2)v_t(x_0,t) + \xi^2 v_t(x_1,t) + h_1\xi(1-\xi)v_{tx}(x_0,t),$$

where

$$(v_t)_x(x_0,t) = \frac{v_t(x_0,t)-\eta_{1,t}(t)}{\alpha_1 K_1},\ \xi = \frac{x-x_0}{h_1}.$$

Then integrating it gets

$$\int_{x_0}^{x_{1/2}} v_t dx \approx Sv_t(x_0,t) \triangleq \frac{11h_1}{24}v_t(x_0,t) + \frac{h_1}{24}v_t(x_1,t) + \frac{h_1^2}{12}(v_t)_x(x_0,t) \tag{2.19}$$

$$= \frac{h_1}{24}v_t(x_1,t) + \left(\frac{11}{24}h_1 + \frac{h_1^2}{12\alpha_1 K_1}\right)v_t(x_0,t) - \frac{h_1^2}{12\alpha_1 K_1}\eta_{1,t}(x_0,t).$$

Substituting (2.11), (2.18) and (2.19) into (2.17) and dropping the truncation errors, we have the semi-discrete numerical scheme for the left boundary point x_0, that is,

$$\begin{aligned}&\frac{h_1}{12}v_t(x_1,t) + \left(\frac{5}{12}h_1 + \frac{h_1^2}{12\alpha_1 K_1}\right)v_t(x_0,t) \\ &-\frac{K_1^2}{3}(1-B_1)\left(\frac{1}{h_1}u(x_1,t) - \left(\frac{1}{\alpha_1 K_1} + \frac{1}{h_1}\right)u(x_0,t)\right) - \frac{K_1^2}{3}B_1\left(\frac{1}{h_1}v(x_1,t) - \left(\frac{1}{\alpha_1 K_1} + \frac{1}{h_1}\right)v(x_0,t)\right) \\ &= \int_{x_0}^{x_{1/2}} s_1(x,t)dx + \frac{h_1^2}{12\alpha_1 K_1}\eta_{1,t}(x_0,t) + \frac{K_1}{3\alpha_1}(1-B_1)\varphi_1(x_0,t) + \frac{K_1}{3\alpha_1}B_1\eta_1(x_0,t) + \frac{h_1^2}{24}s_{1,x}(x_{1/2},t).\end{aligned} \tag{2.20}$$

Similarly, for the right boundary point x_m, we integrate (2.2) on the dual element $I_m^* = [x_{m-1/2}, x_m]$, using (1.6) and (2.5) and applying the quadratic Hermite interpolation for the integral term about v_t, we have

$$\begin{aligned}&\frac{h_2}{12}v_t(x_{m-1},t) + \left(\frac{5}{12}h_2 + \frac{h_2^2}{12\alpha_2 K_2}\right)v_t(x_m,t) \\ &-\frac{K_2^2}{3}(1-B_2)\left(\frac{1}{h_2}u(x_{m-1},t) - \left(\frac{1}{\alpha_2 K_2} + \frac{1}{h_2}\right)u(x_m,t)\right) \\ &-\frac{K_2^2}{3}B_2\left(\frac{1}{h_2}v(x_{m-1},t) - \left(\frac{1}{\alpha_2 K_2} + \frac{1}{h_2}\right)v(x_m,t)\right) \\ &= \int_{x_{m-1/2}}^{x_m} s_2(x,t)dx + \frac{h_2^2}{12\alpha_2 K_2}\eta_{2,t}(t) + \frac{K_2}{3\alpha_2}(1-B_2)\varphi_2(t) + \frac{K_2}{3\alpha_2}B_2\eta_2(t) - \frac{h_2^2}{24}s_{2,x}(x_{m-1/2},t).\end{aligned} \tag{2.21}$$

For the interface point $x_k = l$, we integrate (2.1) on the dual element $I_k^{1,*} = [x_{k-1/2}, x_k]$ and (2.2) on the dual element $I_k^{2,*} = [x_k, x_{k+1/2}]$, respectively. In fact, $I_k^* = I_k^{1,*} \cup I_k^{2,*}$. We also use the formula of integral by parts to the two equations and sum them together,

$$\begin{aligned}&\int_{x_{k-1/2}}^{x_{k+1/2}} v_t dx - \frac{K_1^2}{3}(1-B_1)(u_x(x_k^-,t) - u_x(x_{k-1/2},t)) \\ &-\frac{K_2^2}{3}(1-B_2)(u_x(x_{k+1/2},t) - u_x(x_k^+,t)) - \frac{K_1^2}{3}B_1(v_x(x_k^-,t) - v_x(x_{k-1/2},t)) \\ &-\frac{K_2^2}{3}B_2(v_x(x_{k+1/2},t) - v_x(x_k^+,t)) = \int_{x_{k-1/2}}^{x_{k+1/2}} s(x,t)dx.\end{aligned} \tag{2.22}$$

By the jump condition (1.8) and the relation $v = u + u_t$, we have

$$\frac{K_1^2}{3}(1-B_1)u_x(x_k^-,t) + \frac{K_1^2}{3}B_1 v_x(x_k^-,t) = \frac{K_2^2}{3}(1-B_2)u_x(x_k^+,t) + \frac{K_2^2}{3}B_2 v_x(x_k^+,t). \tag{2.23}$$

For the integral term about v_t, using second-order Lagrange interpolation in $I_k^{1,*}$ and $I_k^{2,*}$, respectively, namely,

$$\begin{aligned}&\int_{x_{k-1/2}}^{x_{k+1/2}} v_t dx = \int_{x_{k-1/2}}^{x_k} v_t dx + \int_{x_k}^{x_{k+1/2}} v_t dx \\ &\approx Sv_t(x_k,t) \triangleq \frac{h_1}{24}(-v_t(x_{k-2},t) + 5v_t(x_{k-1},t) + 8v_t(x_k,t)) + \frac{h_2}{24}(8v_t(x_k,t) + 5v_t(x_{k+1},t) - v_t(x_{k+2},t)).\end{aligned} \tag{2.24}$$

Substituting (2.11), (2.23) and (2.24) into (2.22), dropping the truncation errors, we have the semi-discrete numerical scheme for the interface point x_k,

$$\frac{7}{24}(h_1+h_2)v_t(x_k,t)+\frac{h_1}{4}v_t(x_{k-1},t)-\frac{h_1}{24}v_t(x_{k-2},t)+\frac{h_2}{4}v_t(x_{k+1},t)-\frac{h_2}{24}v_t(x_{k+2},t) \tag{2.25}$$
$$+\frac{K_1^2}{3}(1-B_1)\frac{u(x_k,t)-u(x_{k-1},t)}{h_1}+\frac{K_1^2}{3}B_1\frac{v(x_k,t)-v(x_{k-1},t)}{h_1}$$
$$-\frac{K_2^2}{3}(1-B_2)\frac{u(x_{k+1},t)-u(x_k,t)}{h_2}-\frac{K_2^2}{3}B_2\frac{v(x_{k+1},t)-v(x_k,t)}{h_2}$$
$$=\int_{x_{k-1/2}}^{x_{k+1/2}}s(x,t)dx-\frac{h_1^2}{24}s_{1,x}(x_{k-1/2},t)+\frac{h_2^2}{24}s_{2,x}(x_{k+1/2},t).$$

Finally, we apply the Crank-Nicolson method to (2.16), (2.20), (2.21), (2.25) at the time $t^{n-1/2}$ and have the following fully-discrete numerical schemes,

$$\frac{h}{12}\left(\delta_t v_{i-1}^{n-1/2}+10\delta_t v_i^{n-1/2}+\delta_t v_{i+1}^{n-1/2}\right) \tag{2.26}$$
$$-\frac{K^2}{3}(1-B)\frac{u_{i-1}^{n-1/2}-2u_i^{n-1/2}+u_{i+1}^{n-1/2}}{h}-\frac{K^2}{3}B\frac{v_{i-1}^{n-1/2}-2v_i^{n-1/2}+v_{i+1}^{n-1/2}}{h}$$
$$=\int_{x_{i-1/2}}^{x_{i+1/2}}s(x,t^{n-1/2})dx+\frac{h^2}{24}(s_x(x_{i+1/2},t^{n-1/2})-s_x(x_{i-1/2},t^{n-1/2})),$$
$$(i=1,2,\cdots,m-1,\ i\neq k,\ h=h_1 \textit{ or } h_2,\ K=K_1 \textit{ or } K_2,\ B=B_1 \textit{ or } B_2,\ s=s_1 \textit{ or } s_2),$$

$$\frac{h_1}{12}\delta_t v_1^{n-1/2}+\left(\frac{5}{12}h_1+\frac{h_1^2}{12\alpha_1K_1}\right)\delta_t v_0^{n-1/2} \tag{2.27}$$
$$-\frac{K_1^2}{3}(1-B_1)\left(\frac{1}{h_1}u_1^{n-1/2}-\left(\frac{1}{\alpha_1K_1}+\frac{1}{h_1}\right)u_0^{n-1/2}\right)-\frac{K_1^2}{3}B_1\left(\frac{1}{h_1}v_1^{n-1/2}-\left(\frac{1}{\alpha_1K_1}+\frac{1}{h_1}\right)v_0^{n-1/2}\right)$$
$$=\int_{x_0}^{x_{1/2}}s_1(x,t^{n-1/2})dx+\frac{h_1^2}{12\alpha_1K_1}\eta_{1,t}(t^{n-1/2})+\frac{K_1}{3\alpha_1}(1-B_1)\varphi_1(t^{n-1/2})+\frac{K_1}{3\alpha_1}B_1\eta_1(t^{n-1/2})$$
$$+\frac{h_1^2}{24}s_{1,x}(x_{1/2},t^{n-1/2}),\ (i=0),$$

$$\frac{h_2}{12}\delta_t v_{m-1}^{n-1/2}+\left(\frac{5}{12}h_2+\frac{h_2^2}{12\alpha_2K_2}\right)\delta_t v_m^{n-1/2} \tag{2.28}$$
$$-\frac{K_2^2}{3}(1-B_2)\left(\frac{1}{h_2}u_{m-1}^{n-1/2}-\left(\frac{1}{\alpha_2K_2}+\frac{1}{h_2}\right)u_m^{n-1/2}\right)-\frac{K_2^2}{3}B_2\left(\frac{1}{h_2}v_{m-1}^{n-1/2}-\left(\frac{1}{\alpha_2K_2}+\frac{1}{h_2}\right)v_m^{n-1/2}\right)$$
$$=\int_{x_{m-1/2}}^{x_m}s_2(x,t^{n-1/2})dx+\frac{h_2^2}{12\alpha_2K_2}\eta_{2,t}(t^{n-1/2})+\frac{K_2}{3\alpha_2}(1-B_2)\varphi_2(t^{n-1/2})+\frac{K_2}{3\alpha_2}B_2\eta_2(t^{n-1/2})$$
$$-\frac{h_2^2}{24}s_{2,x}(x_{m-1/2},t^{n-1/2}),\ (i=m),$$

$$\frac{7}{24}(h_1+h_2)\delta_t v_k^{n-1/2}+\frac{h_1}{4}\delta_t v_{k-1}^{n-1/2}-\frac{h_1}{24}\delta_t v_{k-2}^{n-1/2}+\frac{h_2}{4}\delta_t v_{k+1}^{n-1/2}-\frac{h_2}{24}\delta_t v_{k+2}^{n-1/2} \tag{2.29}$$
$$+\frac{K_1^2}{3}(1-B_1)\frac{u_k^{n-1/2}-u_{k-1}^{n-1/2}}{h_1}+\frac{K_1^2}{3}B_1\frac{v_k^{n-1/2}-v_{k-1}^{n-1/2}}{h_1}$$
$$-\frac{K_2^2}{3}(1-B_2)\frac{u_{k+1}^{n-1/2}-u_k^{n-1/2}}{h_2}-\frac{K_2^2}{3}B_2\frac{v_{k+1}^{n-1/2}-v_k^{n-1/2}}{h_2}$$
$$=\int_{x_{k-1/2}}^{x_{k+1/2}}s(x,t^{n-1/2})dx-\frac{h_1^2}{24}s_{1,x}(x_{k-1/2},t^{n-1/2})+\frac{h_2^2}{24}s_{2,x}(x_{k+1/2},t^{n-1/2}),\ (i=k),$$

where $n=1,\cdots,T/\Delta t$, $\delta_t v_i^{n-1/2}=\frac{v_i^n-v_i^{n-1}}{\Delta t}$, $u_i^{n-1/2}=\frac{u_i^n+u_i^{n-1}}{2}$, $v_i^{n-1/2}=\frac{v_i^n+v_i^{n-1}}{2}$, $t^{n-1/2}=(n-1/2)\Delta t$. u_i^n is the approximation of $u(x_i,n\Delta t)$, similarly to v_i^n. $\int_{x_{k-1/2}}^{x_{k+1/2}}s(x,t^{n-1/2})dx=$

$\int_{x_{k-1/2}}^{x_k} s_1(x,t^{n-1/2})dx + \int_{x_k}^{x_{k+1/2}} s_2(x,t^{n-1/2})dx$, since x_k is the interface point.

Applying the Crank-Nicolson method to the relation $v = u + u_t$ and dropping the truncation errors give

$$\frac{v_i^n + v_i^{n-1}}{2} = \frac{u_i^n + u_i^{n-1}}{2} + \frac{u_i^n - u_i^{n-1}}{\Delta t}. \tag{2.30}$$

We may solve u_i^{n+1} from (2.30) and substitute it into the numerical schemes (2.26)-(2.29). As a result, a linear system for v_i^{n+1} can be obtained. In the computing, the integral term about s can be solved by using the Gaussian quadrature with enough degree of precision, and s_x can be approximated by the central difference formula.

3 Error analysis and estimates

For any integer $m \geq 0$, let

$$\tilde{H}^m(\Omega) = \{q |\ q|_{I_i} \in H^m(I_i),\ i = 1,2\},$$

where $I_1 = [0,l]$, $I_2 = [l,1]$, then the domain $I = [0,1] = I_1 \cup I_2$. We define the norm of $\tilde{H}^m$ to be with

$$\|q\|_{m,I} = \sqrt{\|q\|_{m,I_1}^2 + \|q\|_{m,I_2}^2},$$

and semi-norms of $\tilde{H}^m$ are defined accordingly.

For any grid function e, let us define the following discrete zero norm, semi-norm, and full-norm:

$$\|e\|_{0,h} = \left\{\|e\|_{0,h_1}^2 + \|e\|_{0,h_2}^2\right\}^{\frac{1}{2}} = \left\{\frac{h_1}{2}e_0^2 + \sum_{i=1}^{k-1} e_i^2 h_1 + \frac{h_1}{2}e_k^2 + \frac{h_2}{2}e_k^2 + \sum_{i=k+1}^{m-1} e_i^2 h_2 + \frac{h_2}{2}e_m^2\right\}^{\frac{1}{2}},$$

$$|e|_{1,h} = \left\{|e|_{1,h_1}^2 + |e|_{1,h_2}^2\right\}^{\frac{1}{2}} = \left\{\sum_{i=1}^{k}\left(\frac{e_i - e_{i-1}}{h_1}\right)^2 h_1 + \sum_{i=k+1}^{m}\left(\frac{e_i - e_{i-1}}{h_2}\right)^2 h_2\right\}^{\frac{1}{2}},$$

$$\|e\|_{1,h} = \left(\|e\|_{0,h}^2 + |e|_{1,h}^2\right)^{\frac{1}{2}}.$$

Lemma 3.1. *When $u \in \tilde{H}^1(I) \cap \tilde{H}^4(I)$, we set $x_{-1/2} = x_0$, $x_{m+1/2} = x_m$ and have*

$$\left|Su(x_i) - \int_{x_{i-1/2}}^{x_{i+1/2}} u dx\right| \leq \begin{cases} ch_1^{\frac{7}{2}}|u|_{3,[x_0,x_1]},\ i = 0, \\ ch^{\frac{9}{2}}|u|_{4,[x_{i-1},x_{i+1}]},\ i = 1,2,\cdots,m-1,\ i \neq k,\ h = h_1 \text{ or } h_2, \\ ch_1^{\frac{9}{2}}|u|_{3,[x_{k-1/2},x_k]} + ch_2^{\frac{9}{2}}|u|_{3,[x_k,x_{k+1/2}]},\ i = k, \\ ch_2^{\frac{7}{2}}|u|_{3,[x_{m-1},x_m]},\ i = m. \end{cases}$$

Proof. In $[x_0,x_1]$, let $\xi = \frac{x - x_0}{h_1}$, $\hat{u}(\xi) = u(\frac{x - x_0}{h_1})$, then

$$Su(x_0) - \int_{x_0}^{x_{1/2}} u dx = \frac{h_1}{24}\left[11\hat{u}(0) + \hat{u}(1) + 2\hat{u}'(0)\right] - h_1\int_0^{1/2}\hat{u}(\xi)d\xi$$
$$= h_1\left[\frac{11}{24}\hat{u}(0) + \frac{1}{24}\hat{u}(1) + \frac{1}{12}\hat{u}'(0) - \int_0^{1/2}\hat{u}(\xi)d\xi\right] \triangleq h_1 F(\hat{u}).$$

As a linear functional of $\hat{u}$, $|F(\hat{u})| \leq C\|\hat{u}\|_{1,\infty,[0,1]}$.

Since $H^3 \hookrightarrow C^1$, then $|F(\hat{u})| \leq \|\hat{u}\|_{3,[0,1]}$. A straightforward calculation shows $F(\hat{u}) = 0$ for $\hat{u} = 1,\ \xi,\ \xi^2$. According to Bramble-Hilbert Lemma [3], $|F(\hat{u})| \leq C|\hat{u}|_{3,[0,1]}$. By the coordinate transformation, we have $|F(\hat{u})| \le Ch_1^{5/2}|u|_{3,[x_0,x_1]}$. Thus, $\left|Su(x_i) - \int_{x_{i-1/2}}^{x_{i+1/2}} udx\right| \le ch_1^{\frac{7}{2}}|u|_{3,[x_0,x_1]}$. The other conclusions can be proofed analogously. □

Lemma 3.2. *Define a matrix* $A \in R^{(n+1)\times(n+1)}$,

$$A = \frac{h}{12}\begin{pmatrix} 5 & 1 & & & \\ 1 & 10 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 10 & 1 \\ & & & 1 & 5 \end{pmatrix}.$$

For any $(n+1)$*-dimensional vectors* $e^k = (e_0^k, e_1^k, \cdots, e_n^k)^T$ *and* $e^{k-1} = (e_0^{k-1}, e_1^{k-1}, \cdots, e_n^{k-1})^T$, *we have*

(1) $(e^k)^T Ae^{k-1} = (e^{k-1})^T Ae^k$,

(2) $(e^k)^T Ae^k \ge \frac{2}{3}\|e^k\|_{0,h}^2$,

(3) $|(e^k)^T Ae^{k-1}| \le \left[(e^k)^T Ae^k\right]^{1/2}\left[(e^{k-1})^T Ae^{k-1}\right]^{1/2} \le h\left[(e^k)^T e^k\right]^{1/2}\left[(e^{k-1})^T e^{k-1}\right]^{1/2}$.

Proof. Since the matrix A is positive definite and symmetric, and its greatest eigenvalue is h, then (1) and (3) are true obviously. For (2), because

$$\begin{aligned}
&(e^k)^T Ae^k = \frac{h}{12}[5(e_0^k)^2 + 2e_0^ke_1^k + 10(e_1^k)^2 + 2e_1^ke_2^k + \cdots + 2e_{n-1}^ke_n^k + 5(e_n^k)^2] \\
&\ge \frac{h}{12}[5(e_0^k)^2 - (e_0^k)^2 - (e_1^k)^2 + 10(e_1^k)^2 - (e_1^k)^2 - (e_2^k)^2 + \cdots - (e_{n-1}^k)^2 - (e_n^k)^2 + 5(e_n^k)^2] \\
&= \frac{h}{12}[4(e_0^k)^2 + 8(e_1^k)^2 + \cdots + 8(e_{n-1}^k)^2 + 4(e_n^k)^2] \\
&= \frac{2}{3}\|e^k\|_{0,h}^2,
\end{aligned}$$

the proof is completed. □

Lemma 3.3. $\|e\|_\infty \le C\|e\|_{1,h}$.

Proof. Let $|e_{j_0}| = \min_{0\le i\le m}|e_i|$, then $|e_{j_0}| \le \sum_{i=0}^m |e_i|h$. If $j > j_0$, then

$$e_j = e_j - e_{j-1} + e_{j-1} - e_{j-2} + \cdots + e_{j_0+1} - e_{j_0} + e_{j_0} = e_{j_0} + \sum_{i=j_0+1}^{j} \delta_{\overline{x}}e_ih.$$

When $j \le j_0$, we can get the similar result. So

$$|e_j| \le \sum_{i=0}^m |e_i|h + \sum_{i=1}^m |\delta_{\overline{x}}e_i|h.$$

By the Cauchy inequality, we have $|e_j| \le (C_1\|e\|_{0,h}^2 + C_2|e|_{1,h}^2)^{1/2}$, namely, $\|e\|_\infty \le C\|e\|_{1,h}$. □

In the following, we derive error estimates. First we consider the truncation errors at the time direction. Expanding $u(x_i,t^n)$ and $u(x_i,t^{n-1})$ at $t^{n-1/2}$ in Taylor formula with integral remainder term gets,

$$u_t(x_i,t^{n-1/2}) = \frac{u(x_i,t^n) - u(x_i,t^{n-1})}{\Delta t} + R_{i,t}^{(1)}(u),$$

where

$$R_{i,t}^{(1)}(u) = -\frac{1}{\Delta t}\left(\int_{t^{n-1/2}}^{t^n} u_t^{(3)}(x_i,\eta)(t^n-\eta)^2d\eta + \int_{t^{n-1}}^{t^{n-1/2}} u_t^{(3)}(x_i,\eta)(t^{n-1}-\eta)^2d\eta\right).$$

And

$$u(x_i, t^{n-1/2}) = \frac{u(x_i, t^n) + u(x_i, t^{n-1})}{2} + R^{(2)}_{i,t}(u),$$

where

$$R^{(2)}_{i,t}(u) = -\frac{1}{2}\left(\int_{t^{n-1/2}}^{t^n} u^{(2)}_t(x_i,\eta)(t^n-\eta)d\eta - \int_{t^{n-1}}^{t^{n-1/2}} u^{(2)}_t(x_i,\eta)(t^{n-1}-\eta)d\eta\right).$$

Remark 3.4. *It follows from the Cauchy-Schwarz inequality that,*

$$(R^{(1)}_{i,t}(u))^2 \le C\Delta t^3 \int_{t^{n-1}}^{t^n} (u^{(3)}_t(x_i,t))^2 dt, \tag{3.1}$$

$$(R^{(2)}_{i,t}(u))^2 \le C\Delta t^3 \int_{t^{n-1}}^{t^n} (u^{(2)}_t(x_i,t))^2 dt. \tag{3.2}$$

Hence, the numerical schemes (2.26)-(2.29) have the following truncation errors at the time direction,

$$\tilde{R}^{n-1/2}_{i,t} = \frac{h}{12}R^{(1)}_{i-1,t}(v) + \frac{10h}{12}R^{(1)}_{i,t}(v) + \frac{h}{12}R^{(1)}_{i+1,t}(v),$$
$$(i = 1,2,\cdots,m-1,\ i \ne k,\ h = h_1 \ or\ h_2),$$
$$\tilde{R}^{n-1/2}_{0,t} = \frac{h_1}{12}R^{(1)}_{1,t}(v) + \frac{5h_1}{12}R^{(1)}_{0,t}(v) + \frac{h_1^2}{12\alpha_1 K_1}R^{(1)}_{0,t}(v),\ (i=0),$$
$$\tilde{R}^{n-1/2}_{m,t} = \frac{h_2}{12}R^{(1)}_{m-1,t}(v) + \frac{5h_2}{12}R^{(1)}_{m,t}(v) + \frac{h_2^2}{12\alpha_2 K_2}R^{(1)}_{m,t}(v),\ (i=m),$$
$$\tilde{R}^{n-1/2}_{k,t} = \frac{7}{24}(h_1+h_2)R^{(1)}_{k,t}(v) + \frac{h_1}{4}R^{(1)}_{k-1,t}(v) - \frac{h_1}{24}R^{(1)}_{k-2,t}(v) + \frac{h_2}{4}R^{(1)}_{k+1,t}(v) - \frac{h_2}{24}R^{(1)}_{k+2,t}(v),\ (i=k).$$

In addition, we have another kind of truncation errors at the time direction as follows,

$$\hat{R}^{n-1/2}_{i,t} = -\frac{K^2(1-B)}{3h}\left(R^{(2)}_{i-1,t}(u) - 2R^{(2)}_{i,t}(u) + R^{(2)}_{i+1,t}(u)\right)$$
$$-\frac{K^2 B}{3h}\left(R^{(2)}_{i-1,t}(v) - 2R^{(2)}_{i,t}(v) + R^{(2)}_{i+1,t}(v)\right),$$
$$(i = 1,2,\cdots,m-1,\ i\ne k,\ h = h_1\ or\ \ h_2,\ K = K_1\ or\ K_2,\ B = B_1\ or\ B_2),$$
$$\hat{R}^{n-1/2}_{0,t} = -\frac{K_1^2(1-B_1)}{3h_1}\left(R^{(2)}_{1,t}(u) - R^{(2)}_{0,t}(u)\right) + \frac{K_1(1-B_1)}{3\alpha_1}R^{(2)}_{0,t}(u)$$
$$-\frac{K_1^2 B_1}{3h_1}\left(R^{(2)}_{1,t}(v) - R^{(2)}_{0,t}(v)\right) + \frac{K_1 B_1}{3\alpha_1}R^{(2)}_{0,t}(v),\ (i=0),$$
$$\hat{R}^{n-1/2}_{m,t} = -\frac{K_2^2(1-B_2)}{3h_2}\left(R^{(2)}_{m-1,t}(u) - R^{(2)}_{m,t}(u)\right) + \frac{K_2(1-B_2)}{3\alpha_2}R^{(2)}_{m,t}(u)$$
$$-\frac{K_2^2 B_2}{3h_2}\left(R^{(2)}_{m-1,t}(v) - R^{(2)}_{m,t}(v)\right) + \frac{K_2 B_2}{3\alpha_2}R^{(2)}_{m,t}(v),\ (i=m),$$
$$\hat{R}^{n-1/2}_{k,t} = \frac{K_1^2(1-B_1)}{3h_1}\left(R^{(2)}_{k,t}(u) - R^{(2)}_{k-1,t}(u)\right) + \frac{K_1^2 B_1}{3h_1}\left(R^{(2)}_{k,t}(v) - R^{(2)}_{k-1,t}(v)\right)$$
$$-\frac{K_2^2(1-B_2)}{3h_2}\left(R^{(2)}_{k+1,t}(u) - R^{(2)}_{k,t}(u)\right) - \frac{K_2^2 B_2}{3h_2}\left(R^{(2)}_{k+1,t}(v) - R^{(2)}_{k,t}(v)\right),\ (i=k).$$

At the space direction, the numerical schemes (2.26)-(2.29) have the truncation errors at the fixed time $t^{n-1/2}$,

$$\overline{R}^{n-1/2}_{i,x} = \left(R_{i-1/2,x} - R_{i+1/2,x}\right) + \left(S(v_t)_i - \int_{x_{i-1/2}}^{x_{i+1/2}} v_t dx\right) \triangleq R^{n-1/2}_{i(1),x} + R^{n-1/2}_{i(2),x},\ (i = 1,2,\cdots,m-1),$$
$$\overline{R}^{n-1/2}_{0,x} = \left(-R_{1/2,x}\right) + \left(S(v_t)_0 - \int_{x_0}^{x_{1/2}} v_t dx\right) \triangleq R^{n-1/2}_{0(1),x} + R^{n-1/2}_{0(2),x},\ (i=0),$$

$$\overline{R}_{m,x}^{n-1/2} = \left(R_{m-1/2,x}\right) + \left(S(v_t)_m - \int_{x_{m-1/2}}^{x_m} v_t dx\right) \triangleq R_{m(1),x}^{n-1/2} + R_{m(2),x}^{n-1/2}, \ (i=m).$$

Let $e_{u,i}^n = u_i^n - u(x_i, t^n)$, $e_{v,i}^n = v_i^n - v(x_i, t^n)$, $\delta_x e_i = \dfrac{e_{i+1} - e_i}{h}$, $\delta_{\overline{x}} e_i = \dfrac{e_i - e_{i-1}}{h}$, $\delta_x^2 e_i = \dfrac{1}{h}(\delta_x e_i - \delta_{\overline{x}} e_i)$. The error equations for the numerical schemes (2.26)-(2.29) are given as follows,

$$\begin{aligned}
&\frac{h}{12}\left(\frac{e_{v,i-1}^n - e_{v,i-1}^{n-1}}{\Delta t} + 10\frac{e_{v,i}^n - e_{v,i}^{n-1}}{\Delta t} + \frac{e_{v,i+1}^n - e_{v,i+1}^{n-1}}{\Delta t}\right) - \frac{K^2Bh}{3}\delta_x^2\frac{e_{v,i}^n + e_{v,i}^{n-1}}{2} \qquad (3.3)\\
&= \frac{K^2(1-B)h}{3}\delta_x^2\frac{e_{u,i}^n + e_{u,i}^{n-1}}{2} + \overline{R}_{i,x}^{n-1/2} + \tilde{R}_{i,t}^{n-1/2} + \hat{R}_{i,t}^{n-1/2},\\
&(i = 1, 2, \cdots, m-1,\ i \neq k,\ h = h_1 \ or\ h_2,\ K = K_1\ or\ K_2,\ B = B_1\ or\ B_2),\\
&\frac{h_1}{12}\left(5\frac{e_{v,0}^n - e_{v,0}^{n-1}}{\Delta t} + \frac{e_{v,1}^n - e_{v,1}^{n-1}}{\Delta t}\right) - \frac{K_1^2B_1}{3}\delta_x\frac{e_{v,0}^n + e_{v,0}^{n-1}}{2} + \frac{h_1^2}{12\alpha_1K_1}\frac{e_{v,0}^n - e_{v,0}^{n-1}}{\Delta t} \qquad (3.4)\\
&+\frac{K_1B_1}{3\alpha_1}\frac{e_{v,0}^n + e_{v,0}^{n-1}}{2} = \frac{K_1^2(1-B_1)}{3}\delta_x\frac{e_{u,0}^n + e_{u,0}^{n-1}}{2} - \frac{K_1(1-B_1)}{3\alpha_1}\frac{e_{u,0}^n + e_{u,0}^{n-1}}{2}\\
&+\overline{R}_{0,x}^{n-1/2} + \tilde{R}_{0,t}^{n-1/2} + \hat{R}_{0,t}^{n-1/2}, (i=0),\\
&\frac{h_2}{12}\left(\frac{e_{v,m-1}^n - e_{v,m}^{n-1}}{\Delta t} + 5\frac{e_{v,m}^n - e_{v,m}^{n-1}}{\Delta t}\right) + \frac{K_2^2B_2}{3}\delta_{\overline{x}}\frac{e_{v,m}^n + e_{v,m}^{n-1}}{2} + \frac{h_2^2}{12\alpha_2K_2}\frac{e_{v,m}^n - e_{v,m}^{n-1}}{\Delta t} \qquad (3.5)\\
&+\frac{K_2B_2}{3\alpha_2}\frac{e_{v,m}^n + e_{v,m}^{n-1}}{2} = -\frac{K_2^2(1-B_2)}{3}\delta_{\overline{x}}\frac{e_{u,m}^n + e_{u,m}^{n-1}}{2} - \frac{K_2(1-B_2)}{3\alpha_2}\frac{e_{u,m}^n + e_{u,m}^{n-1}}{2}\\
&+\overline{R}_{m,x}^{n-1/2} + \tilde{R}_{m,t}^{n-1/2} + \hat{R}_{m,t}^{n-1/2},\ (i=m),\\
&\frac{7}{24}(h_1+h_2)\frac{e_{v,k}^n - e_{v,k}^{n-1}}{\Delta t} + \frac{h_1}{4}\frac{e_{v,k-1}^n - e_{v,k-1}^{n-1}}{\Delta t} - \frac{h_1}{24}\frac{e_{v,k-2}^n - e_{v,k-2}^{n-1}}{\Delta t} \qquad (3.6)\\
&+\frac{h_2}{4}\frac{e_{v,k+1}^n - e_{v,k+1}^{n-1}}{\Delta t} - \frac{h_2}{24}\frac{e_{v,k+2}^n - e_{v,k+2}^{n-1}}{\Delta t} + \frac{K_1^2B_1}{3}\delta_{\overline{x}}\frac{e_{v,k}^n + e_{v,k}^{n-1}}{2} - \frac{K_2^2B_2}{3}\delta_x\frac{e_{v,k}^n + e_{v,k}^{n-1}}{2}\\
&= \frac{K_2^2(1-B_2)}{3}\delta_x\frac{e_{u,k}^n + e_{u,k}^{n-1}}{2} - \frac{K_1^2(1-B_1)}{3}\delta_{\overline{x}}\frac{e_{u,k}^n + e_{u,k}^{n-1}}{2} + \overline{R}_{k,x}^{n-1/2} + \tilde{R}_{k,t}^{n-1/2} + \hat{R}_{k,t}^{n-1/2},\ (i=k).
\end{aligned}$$

We multiply the error equations (3.3)-(3.6) by $\dfrac{e_{v,i}^n + e_{v,i}^{n-1}}{2}$ and sum them together for i from 0 to m to get a "big" error equation. Firstly, we consider the left-hand side of the "big" equation term by term. Denote these terms by $Y_1^{n-1/2}$, $Y_2^{n-1/2}$, $Y_3^{n-1/2}$, successively.

$$\begin{aligned}
Y_1^{n-1/2} &= \frac{h_1}{12}\left(5\frac{e_{v,0}^n - e_{v,0}^{n-1}}{\Delta t} + \frac{e_{v,1}^n - e_{v,1}^{n-1}}{\Delta t}\right)\frac{e_{v,0}^n + e_{v,0}^{n-1}}{2}\\
&+\sum_{i=1}^{k-1}\frac{h_1}{12}\left(\frac{e_{v,i-1}^n - e_{v,i-1}^{n-1}}{\Delta t} + 10\frac{e_{v,i}^n - e_{v,i}^{n-1}}{\Delta t} + \frac{e_{v,i+1}^n - e_{v,i+1}^{n-1}}{\Delta t}\right)\frac{e_{v,i}^n + e_{v,i}^{n-1}}{2}\\
&+\frac{h_1}{12}\left(\frac{e_{v,k-1}^n - e_{v,k-1}^{n-1}}{\Delta t} + 5\frac{e_{v,k}^n - e_{v,k}^{n-1}}{\Delta t}\right)\frac{e_{v,k}^n + e_{v,k}^{n-1}}{2}\\
&+\frac{h_2}{12}\left(5\frac{e_{v,k}^n - e_{v,k}^{n-1}}{\Delta t} + \frac{e_{v,k+1}^n - e_{v,k+1}^{n-1}}{\Delta t}\right)\frac{e_{v,k}^n + e_{v,k}^{n-1}}{2}\\
&+\sum_{i=k+1}^{m-1}\frac{h_2}{12}\left(\frac{e_{v,i-1}^n - e_{v,i-1}^{n-1}}{\Delta t} + 10\frac{e_{v,i}^n - e_{v,i}^{n-1}}{\Delta t} + \frac{e_{v,i+1}^n - e_{v,i+1}^{n-1}}{\Delta t}\right)\frac{e_{v,i}^n + e_{v,i}^{n-1}}{2}\\
&+\frac{h_2}{12}\left(\frac{e_{v,m-1}^n - e_{v,m-1}^{n-1}}{\Delta t} + 5\frac{e_{v,m}^n - e_{v,m}^{n-1}}{\Delta t}\right)\frac{e_{v,m}^n + e_{v,m}^{n-1}}{2} + M^{n-1/2}
\end{aligned}$$

$$= \frac{1}{2\Delta t}\left(e^n_{v[1]}+e^{n-1}_{v[1]}\right)^T A_{k+1}\left(e^n_{v[1]}-e^{n-1}_{v[1]}\right)+\frac{1}{2\Delta t}\left(e^n_{v[2]}+e^{n-1}_{v[2]}\right)^T A_{m-k+1}\left(e^n_{v[2]}-e^{n-1}_{v[2]}\right)+M^{n-1/2}.$$

By Lemma 3.2 (1), we have

$$Y_1^{n-1/2} = \frac{1}{2\Delta t}\left((e^n_{v[1]})^T A_{k+1} e^n_{v[1]} - (e^{n-1}_{v[1]})^T A_{k+1} e^{n-1}_{v[1]}\right) + \frac{1}{2\Delta t}\left((e^n_{v[2]})^T A_{m-k+1} e^n_{v[2]} - (e^{n-1}_{v[2]})^T A_{m-k+1} e^{n-1}_{v[2]}\right) + M^{n-1/2},$$

where $e^n_{v[1]} = \left[e^n_{v,0}, e^n_{v,1}, \cdots, e^n_{v,k}\right]^T$, $e^n_{v[2]} = \left[e^n_{v,k}, e^n_{v,k+1}, \cdots, e^n_{v,m}\right]^T$,

$$M^{n-1/2} = \frac{h_1}{12}\left(-\frac{3}{2}\frac{e^n_{v,k}-e^{n-1}_{v,k}}{\Delta t} + 2\frac{e^n_{v,k-1}-e^{n-1}_{v,k-1}}{\Delta t} - \frac{1}{2}\frac{e^n_{v,k-2}-e^{n-1}_{v,k-2}}{\Delta t}\right)\frac{e^n_{v,k}+e^{n-1}_{v,k}}{2}$$
$$+\frac{h_2}{12}\left(-\frac{3}{2}\frac{e^n_{v,k}-e^{n-1}_{v,k}}{\Delta t} + 2\frac{e^n_{v,k+1}-e^{n-1}_{v,k+1}}{\Delta t} - \frac{1}{2}\frac{e^n_{v,k+2}-e^{n-1}_{v,k+2}}{\Delta t}\right)\frac{e^n_{v,k}+e^{n-1}_{v,k}}{2}.$$

We will move the term $M^{n-1/2}$ into the left-hand side of the "big" error equation, then use the Cauchy-Schwarz inequality to estimate it, that is,

$$\begin{aligned}
&|M^{n-1/2}| \qquad\qquad (3.7)\\
&\leq \frac{1}{2}h_1\left(\frac{e^n_{v,k}+e^{n-1}_{v,k}}{2}\right)^2 + Ch_1\left[\left(\frac{e^n_{v,k}-e^{n-1}_{v,k}}{\Delta t}\right)^2+\left(\frac{e^n_{v,k-1}-e^{n-1}_{v,k-1}}{\Delta t}\right)^2+\left(\frac{e^n_{v,k-2}-e^{n-1}_{v,k-2}}{\Delta t}\right)^2\right]\\
&+\frac{1}{2}h_2\left(\frac{e^n_{v,k}+e^{n-1}_{v,k}}{2}\right)^2 + Ch_2\left[\left(\frac{e^n_{v,k}-e^{n-1}_{v,k}}{\Delta t}\right)^2+\left(\frac{e^n_{v,k+1}-e^{n-1}_{v,k+1}}{\Delta t}\right)^2+\left(\frac{e^n_{v,k+2}-e^{n-1}_{v,k+2}}{\Delta t}\right)^2\right]\\
&\leq \frac{1}{2}\left\|\frac{e^n_v+e^{n-1}_v}{2}\right\|^2_{0,h} + C\left\|\frac{e^n_v-e^{n-1}_v}{\Delta t}\right\|^2_{0,h} \leq \frac{1}{2}\left\|\frac{e^n_v+e^{n-1}_v}{2}\right\|^2_{0,h} + C\Delta t^3\int_{t_{n-1}}^{t^n}\|v_{ttt}\|^2_{0,I}\,dt,
\end{aligned}$$

where the last inequality is deduced by the relation $v_t = u_t + u_{tt}$, (3.1) and the L^2 error estimate in [10].

Using the summation formula by parts, we have

$$\begin{aligned}
Y_2^{n-1/2} &= -\frac{K_1^2B_1}{3}\delta_x\frac{e^n_{v,0}+e^{n-1}_{v,0}}{2}\frac{e^n_{v,0}+e^{n-1}_{v,0}}{2} - \frac{K_1^2B_1}{3}\sum_{i=1}^{k-1}h_1\delta_x^2\frac{e^n_{v,i}+e^{n-1}_{v,i}}{2}\frac{e^n_{v,i}+e^{n-1}_{v,i}}{2}\\
&+\frac{K_1^2B_1}{3}\delta_{\bar{x}}\frac{e^n_{v,k}+e^{n-1}_{v,k}}{2}\frac{e^n_{v,k}+e^{n-1}_{v,k}}{2} - \frac{K_2^2B_2}{3}\delta_x\frac{e^n_{v,k}+e^{n-1}_{v,k}}{2}\frac{e^n_{v,k}+e^{n-1}_{v,k}}{2}\\
&-\frac{K_2^2B_2}{3}\sum_{i=k+1}^{m-1}h_2\delta_x^2\frac{e^n_{v,i}+e^{n-1}_{v,i}}{2}\frac{e^n_{v,i}+e^{n-1}_{v,i}}{2} + \frac{K_2^2B_2}{3}\delta_{\bar{x}}\frac{e^n_{v,m}+e^{n-1}_{v,m}}{2}\frac{e^n_{v,m}+e^{n-1}_{v,m}}{2}\\
&= \frac{K_1^2B_1}{3}\left|\frac{e^n_v+e^{n-1}_v}{2}\right|^2_{1,h_1} + \frac{K_2^2B_2}{3}\left|\frac{e^n_v+e^{n-1}_v}{2}\right|^2_{1,h_2}.
\end{aligned}$$

A straightforward computation shows that

$$\begin{aligned}
Y_3^{n-1/2} &= \left(\frac{h_1^2}{12\alpha_1K_1}\frac{e^n_{v,0}-e^{n-1}_{v,0}}{\Delta t} + \frac{K_1B_1}{3\alpha_1}\frac{e^n_{v,0}+e^{n-1}_{v,0}}{2}\right)\frac{e^n_{v,0}+e^{n-1}_{v,0}}{2}\\
&+\left(\frac{h_2^2}{12\alpha_2K_2}\frac{e^n_{v,m}-e^{n-1}_{v,m}}{\Delta t} + \frac{K_2B_2}{3\alpha_2}\frac{e^n_{v,m}+e^{n-1}_{v,m}}{2}\right)\frac{e^n_{v,m}+e^{n-1}_{v,m}}{2}\\
&\geq \frac{h_1^2}{24\Delta t\alpha_1K_1}\left((e^n_{v,0})^2-(e^{n-1}_{v,0})^2\right) + \frac{h_2^2}{24\Delta t\alpha_2K_2}\left((e^n_{v,m})^2-(e^{n-1}_{v,m})^2\right).
\end{aligned}$$

Hence, we get the estimate of the left-hand side of the "big" error equation as follows,

$$\begin{aligned}
&Y_1^{n-1/2}+Y_2^{n-1/2}+Y_3^{n-1/2}\\
&\geq \frac{1}{2\Delta t}\Big((e_{v[1]}^n)^TA_{k+1}e_{v[1]}^n-(e_{v[1]}^{n-1})^TA_{k+1}e_{v[1]}^{n-1}\Big)\\
&+\frac{1}{2\Delta t}\Big((e_{v[2]}^n)^TA_{m-k+1}e_{v[2]}^n-(e_{v[2]}^{n-1})^TA_{m-k+1}e_{v[2]}^{n-1}\Big)+\frac{K_1^2B_1}{3}\left|\frac{e_v^n+e_v^{n-1}}{2}\right|_{1,h_1}^2\\
&+\frac{K_2^2B_2}{3}\left|\frac{e_v^n+e_v^{n-1}}{2}\right|_{1,h_2}^2+\frac{h_1^2}{24\Delta t\alpha_1K_1}((e_{v,0}^n)^2-(e_{v,0}^{n-1})^2)+\frac{h_2^2}{24\Delta t\alpha_2K_2}((e_{v,m}^n)^2-(e_{v,m}^{n-1})^2).
\end{aligned}$$

Next we consider the right-hand side of the "big" error equation term by term.

$$\sum_{i=0}^m\overline{R}_{i,x}^{n-1/2}\frac{e_{v,i}^n+e_{v,i}^{n-1}}{2}=\sum_{i=0}^mR_{i(1),x}^{n-1/2}\frac{e_{v,i}^n+e_{v,i}^{n-1}}{2}+\sum_{i=0}^mR_{i(2),x}^{n-1/2}\frac{e_{v,i}^n+e_{v,i}^{n-1}}{2}\triangleq T_1+T_2.$$

By the summation formula by parts and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&T_1=-R_{1/2,x}\frac{e_{v,0}^n+e_{v,0}^{n-1}}{2}+\sum_{i=1}^{k-1}\big(R_{i-1/2,x}(v)-R_{i+1/2,x}\big)\frac{e_{v,i}^n+e_{v,i}^{n-1}}{2}+R_{k-1/2,x}\frac{e_{v,k}^n+e_{v,k}^{n-1}}{2}\\
&-R_{k+1/2,x}\frac{e_{v,k}^n+e_{v,k}^{n-1}}{2}+\sum_{i=k+1}^{m-1}\big(R_{i-1/2,x}-R_{i+1/2,x}\big)\frac{e_{v,i}^n+e_{v,i}^{n-1}}{2}+R_{m-1/2,x}\frac{e_{v,m}^n+e_{v,m}^{n-1}}{2}\\
&=\sum_{i=1}^kR_{i-1/2,x}\delta_{\overline{x}}\frac{e_{v,i}^n+e_{v,i}^{n-1}}{2}h_1+\sum_{i=k+1}^mR_{i-1/2,x}\delta_{\overline{x}}\frac{e_{v,i}^n+e_{v,i}^{n-1}}{2}h_2.\\
&\leq C\sum_{i=1}^k(R_{i-1/2,x})^2h_1+C\sum_{i=k+1}^m\big(R_{i-1/2,x}\big)^2h_2+\frac{1}{4}\frac{K_1^2B_1}{3}\left|\frac{e_v^n+e_v^{n-1}}{2}\right|_{1,h_1}^2+\frac{1}{4}\frac{K_2^2B_2}{3}\left|\frac{e_v^n+e_v^{n-1}}{2}\right|_{1,h_2}^2.
\end{aligned}$$

Further, by (2.12), we get

$$\begin{aligned}
&\sum_{i=1}^k(R_{i-1/2,x})^2h_1 &(3.8)\\
&=\sum_{i=1}^k\left(\frac{K^2}{3}(1-B)R_{u(x_{i-1/2},t^{n-1/2})}+\frac{K^2}{3}BR_{v(x_{i-1/2},t^{n-1/2})}-\frac{h^2}{24}R_{v_t(x_{i-1/2},t^{n-1/2})}\right)^2h_1\\
&\leq Ch_1\sum_{i=1}^k\Big(R_{u(x_{i-1/2},t^{n-1/2})}^2+R_{v(x_{i-1/2},t^{n-1/2})}^2+h_1^4R_{v_t(x_{i-1/2},t^{n-1/2})}^2\Big)\\
&\leq Ch_1^8\int_0^l\left(\Big(u_x^{(5)}(x,t^{n-1/2})\Big)^2+\Big(v_x^{(5)}(x,t^{n-1/2})\Big)^2+\Big((v_t)_x^{(3)}(x,t^{n-1/2})\Big)^2\right)dx.
\end{aligned}$$

We remark that

$$u(t^{n-1/2})=\frac{1}{\Delta t}\left[\int_{t^{n-1}}^{t^n}u(t)dt-\int_{t^{n-1}}^{t^n}g(t)dt\right],$$

where $g(t)=\displaystyle\int_{t^{n-1/2}}^tu_{tt}(s)(t-s)ds$. Using the Cauchy-Schwarz inequality we have

$$\begin{aligned}
\Big(u(t^{n-1/2})\Big)^2&\leq\frac{2}{\Delta t}\int_{t^{n-1}}^{t^n}u^2(t)dt+\frac{2}{\Delta t}\int_{t^{n-1}}^{t^n}g^2(t)dt &(3.9)\\
&\leq\frac{2}{\Delta t}\int_{t^{n-1}}^{t^n}u^2(t)dt+\frac{\Delta t^3}{24}\int_{t^{n-1}}^{t^n}u_{tt}^2(t)dt.
\end{aligned}$$

Since the estimate (3.9) is only related to the time t, we have

$$\Big(u_x^{(5)}(x,t^{n-1/2})\Big)^2\leq\frac{2}{\Delta t}\int_{t^{n-1}}^{t^n}\Big(u_x^{(5)}(x,t)\Big)^2dt+\frac{\Delta t^3}{24}\int_{t^{n-1}}^{t^n}\Big((u_{tt})_x^{(5)}(x,t)\Big)^2dt. \qquad (3.10)$$

Similarly to estimate the term $\left(v_x^{(5)}\left(x,t^{n-1/2}\right)\right)^2$.

Noting that

$$u_t(t^{n-1/2})=\frac{1}{\Delta t}\left[\int_{t^{n-1}}^{t^n}u_t(s)ds-\int_{t^{n-1/2}}^{t^n}u_{tt}(s)(t^n-s)ds-\int_{t^{n-1}}^{t^{n-1/2}}u_{tt}(s)(t^{n-1}-s)ds\right],$$

we have

$$\left(u_t(t^{n-1/2})\right)^2\le\frac{2}{\Delta t}\int_{t^{n-1}}^{t^n}u_t^2dt+\frac{\Delta t}{6}\int_{t^{n-1}}^{t^n}u_{tt}^2dt.$$

Then

$$\left((v_t)_x^{(3)}(x,t^{n-1/2})\right)^2\le\frac{2}{\Delta t}\int_{t^{n-1}}^{t^n}\left((v_t)_x^{(3)}(x,t)\right)^2dt+\frac{\Delta t}{6}\int_{t^{n-1}}^{t^n}\left((v_{tt})_x^{(3)}(x,t)\right)^2dt. \tag{3.11}$$

Substituting the above estimates (3.10) and (3.11) into (3.8), we have the following estimate,

$$\begin{aligned}T_1\le&C\frac{h_1^8}{\Delta t}\left(\int_{t^{n-1}}^{t^n}(|u|_{5,I_1}^2+|v|_{5,I_1}^2+|v_t|_{3,I_1}^2)dt+\Delta t^4\int_{t^{n-1}}^{t^n}(|u_{tt}|_{5,I_1}^2+|v_{tt}|_{5,I_1}^2)dt+\Delta t^2\int_{t^{n-1}}^{t^n}|v_{tt}|_{3,I_1}^2dt\right)\\&+C\frac{h_2^8}{\Delta t}\left(\int_{t^{n-1}}^{t^n}(|u|_{5,I_2}^2+|v|_{5,I_2}^2+|v_t|_{3,I_2}^2)dt+\Delta t^4\int_{t^{n-1}}^{t^n}(|u_{tt}|_{5,I_2}^2+|v_{tt}|_{5,I_2}^2)dt+\Delta t^2\int_{t^{n-1}}^{t^n}|v_{tt}|_{3,I_2}^2dt\right)\\&+\frac{1}{4}\frac{K_1^2B_1}{3}\left|\frac{e_v^n+e_v^{n-1}}{2}\right|_{1,h_1}^2+\frac{1}{4}\frac{K_2^2B_2}{3}\left|\frac{e_v^n+e_v^{n-1}}{2}\right|_{1,h_2}^2.\end{aligned}$$

By Lemma 3.1 and the estimate (3.11), we have

$$\begin{aligned}T_2\le&\left[\frac{2}{h_1}(R_{0(2),x}^{n-1/2})^2+\frac{1}{h_1}\sum_{i=1}^{k-1}(R_{i(2),x}^{n-1/2})^2+\frac{2}{h_1}(R_{k(2),x}^{n-1/2})^2\right]^{1/2}\left\|\frac{e_v^n+e_v^{n-1}}{2}\right\|_{0,h_1}^2\\&+\left[\frac{2}{h_2}(R_{k(2),x}^{n-1/2})^2+\frac{1}{h_2}\sum_{i=k+1}^{m-1}(R_{i(2),x}^{n-1/2})^2+\frac{2}{h_2}(R_{m(2),x}^{n-1/2})^2\right]^{1/2}\left\|\frac{e_v^n+e_v^{n-1}}{2}\right\|_{0,h_2}^2\\\le&C\left[\frac{2}{h_1}(R_{0(2),x}^{n-1/2})^2+\frac{1}{h_1}\sum_{i=1}^{k-1}(R_{i(2),x}^{n-1/2})^2+\frac{2}{h_1}(R_{k(2),x}^{n-1/2})^2\right]\\&+C\left[\frac{2}{h_2}(R_{k(2),x}^{n-1/2})^2+\frac{1}{h_2}\sum_{i=k+1}^{m-1}(R_{i(2),x}^{n-1/2})^2+\frac{2}{h_2}(R_{m(2),x}^{n-1/2})^2\right]+\frac{1}{2}\left\|\frac{e_v^n+e_v^{n-1}}{2}\right\|_{0,h}^2\\\le&C\frac{h_1^8}{\Delta t}\left(\int_{t^{n-1}}^{t^n}|v_t|_{4,I_1}^2dt+\Delta t^2\int_{t^{n-1}}^{t^n}|v_{tt}|_{4,I_1}^2dt\right)\\&+C\frac{h_2^8}{\Delta t}\left(\int_{t^{n-1}}^{t^n}|v_t|_{4,I_2}^2dt+\Delta t^2\int_{t^{n-1}}^{t^n}|v_{tt}|_{4,I_2}^2dt\right)\\&+C\frac{h_1^6}{\Delta t}\left(\int_{t^{n-1}}^{t^n}|v_t|_{3,[x_0,x_1]}^2dt+\Delta t^2\int_{t^{n-1}}^{t^n}|v_{tt}|_{3,[x_0,x_1]}^2dt\right)\\&+C\frac{h_2^6}{\Delta t}\left(\int_{t^{n-1}}^{t^n}|v_t|_{3,[x_{m-1},x_m]}^2dt+\Delta t^2\int_{t^{n-1}}^{t^n}|v_{tt}|_{3,[x_{m-1},x_m]}^2dt\right)+\frac{1}{2}\left\|\frac{e_v^n+e_v^{n-1}}{2}\right\|_{0,h}^2.\end{aligned}$$

Therefore, we get the estimate about the term $\sum_{i=0}^{m}\overline{R}_{i,x}^{n-1/2}\frac{e_{v,i}^n+e_{v,i}^{n-1}}{2}$.

By Lemma 3.2 (3), Lemma 3.3, the Cauchy-Schwarz inequality and the ε inequality, we give

$$
\begin{aligned}
&\sum_{i=0}^{m}\tilde{R}_{i,t}^{n-1/2}\frac{e_{v,i}^{n}+e_{v,i}^{n-1}}{2}\\
&=\left(\sqrt{2}R_{0,t}^{(1)}(v),R_{1,t}^{(1)}(v),\cdots,\sqrt{2}R_{k,t}^{(1)}(v)\right)A_{k+1}\left(\frac{e_{v,0}^{n}+e_{v,0}^{n-1}}{2\sqrt{2}},\frac{e_{v,1}^{n}+e_{v,1}^{n-1}}{2},\cdots,\frac{e_{v,k}^{n}+e_{v,k}^{n-1}}{2\sqrt{2}}\right)^{T}\\
&+\left(\sqrt{2}R_{k,t}^{(1)}(v),R_{k+1,t}^{(1)}(v),\cdots,\sqrt{2}R_{m,t}^{(1)}(v)\right)A_{m-k+1}\left(\frac{e_{v,k}^{n}+e_{v,k}^{n-1}}{2\sqrt{2}},\frac{e_{v,k+1}^{n}+e_{v,k+1}^{n-1}}{2},\cdots,\frac{e_{v,m}^{n}+e_{v,m}^{n-1}}{2\sqrt{2}}\right)^{T}\\
&+\left(-\frac{3}{24}(h_1+h_2)R_{k,t}^{(1)}(v)+\frac{h_1}{6}R_{k-1,t}^{(1)}(v)+\frac{h_2}{6}R_{k+1,t}^{(1)}(v)-\frac{h_1}{24}R_{k-2,t}^{(1)}(v)-\frac{h_2}{24}R_{k+2,t}^{(1)}(v)\right)\frac{e_{v,k}^{n}+e_{v,k}^{n-1}}{2}\\
&+\frac{h_1^2}{12\alpha_1K_1}R_{0,t}^{(1)}(v)\frac{e_{v,0}^{n}+e_{v,0}^{n-1}}{2}+\frac{h_2^2}{12\alpha_2K_2}R_{m,t}^{(1)}(v)\frac{e_{v,m}^{n}+e_{v,m}^{n-1}}{2}\\
&\leq\left[(R_{0,t}^{(1)}(v))^2 2h_1+\sum_{i=1}^{k-1}(R_{i,t}^{(1)}(v))^2h_1+(R_{k,t}^{(1)}(v))^2 2h_1\right]^{1/2}\left\|\frac{e_v^n+e_v^{n-1}}{2}\right\|_{0,h_1}\\
&+\left[(R_{k,t}^{(1)}(v))^2 2h_2+\sum_{i=k+1}^{m-1}(R_{i,t}^{(1)}(v))^2h_2+(R_{m,t}^{(1)}(v))^2 2h_2\right]^{1/2}\left\|\frac{e_v^n+e_v^{n-1}}{2}\right\|_{0,h_2}\\
&+M_{\varepsilon_1}\left((h_1+h_2)(R_{k,t}^{(1)}(v))^2+h_1(R_{k-1,t}^{(1)}(v))^2+h_1(R_{k-2,t}^{(1)}(v))^2+h_2(R_{k+1,t}^{(1)}(v))^2+h_2(R_{k+2,t}^{(1)}(v))^2\right)\\
&+\varepsilon_1(h_1+h_2)\left(\frac{e_{v,k}^{n}+e_{v,k}^{n-1}}{2}\right)^2+M_{\varepsilon_2}h_1^4\Delta t^3\int_{t^{n-1}}^{t^n}v_{ttt}^2(0,t)dt+M_{\varepsilon_3}h_2^4\Delta t^3\int_{t^{n-1}}^{t^n}v_{ttt}^2(1,t)dt\\
&+\varepsilon_2\left(\frac{e_{v,0}^{n}+e_{v,0}^{n-1}}{2}\right)^2+\varepsilon_3\left(\frac{e_{v,m}^{n}+e_{v,m}^{n-1}}{2}\right)^2\\
&\leq C\Delta t^3\int_{t^{n-1}}^{t^n}\left[v_{ttt}^2(0,t)\frac{h_1}{2}+\sum_{i=1}^{k-1}v_{ttt}^2(x_i,t)h_1+v_{ttt}^2(x_k,t)\frac{h_1}{2}\right]dt\\
&+C\Delta t^3\int_{t^{n-1}}^{t^n}\left[v_{ttt}^2(x_k,t)\frac{h_2}{2}+\sum_{i=k+1}^{m-1}v_{ttt}^2(x_i,t)h_2+v_{ttt}^2(x_m,t)\frac{h_2}{2}\right]dt+\frac{1}{2}\left\|\frac{e_v^n+e_v^{n-1}}{2}\right\|_{0,h}^2\\
&+M_{\varepsilon_2}h_1^4\Delta t^3\int_{t^{n-1}}^{t^n}v_{ttt}^2(0,t)dt+M_{\varepsilon_3}h_2^4\Delta t^3\int_{t^{n-1}}^{t^n}v_{ttt}^2(1,t)dt+\varepsilon\left\|\frac{e_v^n+e_v^{n-1}}{2}\right\|_{1,h}^2\\
&\leq C\Delta t^3\int_{t^{n-1}}^{t^n}\|v_{ttt}\|_{0,I}^2\,dt+M_{\varepsilon_2}h_1^4\Delta t^3\int_{t^{n-1}}^{t^n}v_{ttt}^2(0,t)dt+M_{\varepsilon_3}h_2^4\Delta t^3\int_{t^{n-1}}^{t^n}v_{ttt}^2(1,t)dt\\
&+\frac{1}{2}\left\|\frac{e_v^n+e_v^{n-1}}{2}\right\|_{0,h}^2+\varepsilon\left\|\frac{e_v^n+e_v^{n-1}}{2}\right\|_{1,h}^2,
\end{aligned}
$$

where the last inequality is explained as follows: since

$$\lim_{h\to 0}\left|(v_{ttt}(x_0,t))^2\frac{h_1}{2}+\sum_{i=1}^{k-1}(v_{ttt}(x_i,t))^2\frac{h_1}{2}+(v_{ttt}(x_k,t))^2\frac{h_1}{2}-\int_0^l(v_{ttt}(x,t))^2dx\right|=0,$$

then for $\varepsilon=\int_0^l(v_{ttt}(x,t))^2dx$, there exists a $h_0>0$, such that

$$v_{ttt}(x_0,t))^2\frac{h_1}{2}+\sum_{i=1}^{k-1}(v_{ttt}(x_i,t))^2\frac{h_1}{2}+(v_{ttt}(x_k,t))^2\frac{h_1}{2}\leq 2\int_0^l(v_{ttt}(x,t))^2dx=2\|v_{ttt}\|_{0,I_1}^2,$$

whenever $h<h_0$. The similar conclusion can be obtained at $I_2=[l,1]$.

Using precisely the same arguments as above, we have

$$\sum_{i=0}^{m} \hat{R}_{i,t}^{n-1/2} \frac{e_{v,i}^n + e_{v,i}^{n-1}}{2}$$

$$= \frac{K_1^2}{3}(1-B_1) \sum_{i=1}^{k} \left(R_{i,t}^{(2)}(u) - R_{i-1,t}^{(2)}(u) \right) \delta_{\overline{x}} \frac{e_{v,i}^n + e_{v,i}^{n-1}}{2} + \frac{K_1^2}{3} B_1 \sum_{i=1}^{k} \left(R_{i,t}^{(2)}(v) - R_{i-1,t}^{(2)}(v) \right) \delta_{\overline{x}} \frac{e_{v,i}^n + e_{v,i}^{n-1}}{2}$$

$$+ \frac{K_2^2}{3}(1-B_2) \sum_{i=k+1}^{m} \left(R_{i,t}^{(2)}(u) - R_{i-1,t}^{(2)}(u) \right) \delta_{\overline{x}} \frac{e_{v,i}^n + e_{v,i}^{n-1}}{2} + \frac{K_2^2}{3} B_2 \sum_{i=k+1}^{m} \left(R_{i,t}^{(2)}(v) - R_{i-1,t}^{(2)}(v) \right) \delta_{\overline{x}} \frac{e_{v,i}^n + e_{v,i}^{n-1}}{2}$$

$$+ \frac{K_1(1-B_1)}{3\alpha_1} R_{0,t}^{(2)}(u) \frac{e_{v,0}^n + e_{v,0}^{n-1}}{2} + \frac{K_1 B_1}{3\alpha_1} R_{0,t}^{(2)}(v) \frac{e_{v,0}^n + e_{v,0}^{n-1}}{2}$$

$$+ \frac{K_2(1-B_2)}{3\alpha_2} R_{m,t}^{(2)}(u) \frac{e_{v,m}^n + e_{v,m}^{n-1}}{2} + \frac{K_2 B_2}{3\alpha_2} R_{m,t}^{(2)}(v) \frac{e_{v,m}^n + e_{v,m}^{n-1}}{2}$$

$$\leq C \left(\sum_{i=1}^{k} \left(\frac{R_{i,t}^{(2)}(u) - R_{i-1,t}^{(2)}(u)}{h_1} \right)^2 h_1 \right)^{1/2} \left| \frac{e_v^n + e_v^{n-1}}{2} \right|_{1,h_1}$$

$$+ C \left(\sum_{i=k+1}^{m} \left(\frac{R_{i,t}^{(2)}(u) - R_{i-1,t}^{(2)}(u)}{h_2} \right)^2 h_2 \right)^{1/2} \left| \frac{e_v^n + e_v^{n-1}}{2} \right|_{1,h_2}$$

$$+ C \left(\sum_{i=1}^{k} \left(\frac{R_{i,t}^{(2)}(v) - R_{i-1,t}^{(2)}(v)}{h_1} \right)^2 h_1 \right)^{1/2} \left| \frac{e_v^n + e_v^{n-1}}{2} \right|_{1,h_1}$$

$$+ C \left(\sum_{i=k+1}^{m} \left(\frac{R_{i,t}^{(2)}(v) - R_{i-1,t}^{(2)}(v)}{h_2} \right)^2 h_2 \right)^{1/2} \left| \frac{e_v^n + e_v^{n-1}}{2} \right|_{1,h_2}$$

$$+ C\Delta t^3 \int_{t_{n-1}}^{t^n} \left(u_{tt}^2(0,t) + v_{tt}^2(0,t) + u_{tt}^2(1,t) + v_{tt}^2(1,t) \right) dt + \varepsilon_1 \left(\frac{e_{v,0}^n + e_{v,0}^{n-1}}{2} \right)^2 + \varepsilon_2 \left(\frac{e_{v,m}^n + e_{v,m}^{n-1}}{2} \right)^2$$

$$\leq C\Delta t^3 \int_{t^{n-1}}^{t^n} |u_{tt}|_{1,I}^2 dt + C\Delta t^3 \int_{t^{n-1}}^{t^n} |v_{tt}|_{1,I}^2 dt + \frac{1}{4} \frac{K_1^2 B_1}{3} \left| \frac{e_v^n + e_v^{n-1}}{2} \right|_{1,h_1}^2 + \frac{1}{4} \frac{K_2^2 B_2}{3} \left| \frac{e_v^n + e_v^{n-1}}{2} \right|_{1,h_2}^2$$

$$+ C\Delta t^3 \int_{t_{n-1}}^{t^n} \left(u_{tt}^2(0,t) + v_{tt}^2(0,t) + u_{tt}^2(1,t) + v_{tt}^2(1,t) \right) dt + \varepsilon \left\| \frac{e_v^n + e_v^{n-1}}{2} \right\|_{1,h}^2 .$$

For the remaining terms, we use the summation formula by parts, the ε inequality and the similar estimate of the last inequality in (3.7) to get,

$$
\begin{aligned}
&\frac{K_1^2}{3}(1-B_1)\left\{\delta_x \frac{e_{u,0}^n+e_{u,0}^{n-1}}{2}\frac{e_{v,0}^n+e_{v,0}^{n-1}}{2}+\sum_{i=1}^{k-1}h_1\delta_x^2\frac{e_{u,i}^n+e_{u,i}^{n-1}}{2}\frac{e_{v,i}^n+e_{v,i}^{n-1}}{2}-\delta_{\overline{x}}\frac{e_{u,k}^n+e_{u,k}^{n-1}}{2}\frac{e_{v,k}^n+e_{v,k}^{n-1}}{2}\right\}\\
&+\frac{K_2^2}{3}(1-B_2)\left\{\delta_x \frac{e_{u,k}^n+e_{u,k}^{n-1}}{2}\frac{e_{v,k}^n+e_{v,k}^{n-1}}{2}+\sum_{i=k+1}^{m-1}h_2\delta_x^2\frac{e_{u,i}^n+e_{u,i}^{n-1}}{2}\frac{e_{v,i}^n+e_{v,i}^{n-1}}{2}-\delta_{\overline{x}}\frac{e_{u,m}^n+e_{u,m}^{n-1}}{2}\frac{e_{v,m}^n+e_{v,m}^{n-1}}{2}\right\}\\
&-\frac{K_1(1-B_1)}{3\alpha_1}\frac{e_{u,0}^n+e_{u,0}^{n-1}}{2}\frac{e_{v,0}^n+e_{v,0}^{n-1}}{2}-\frac{K_2(1-B_2)}{3\alpha_2}\frac{e_{u,m}^n+e_{u,m}^{n-1}}{2}\frac{e_{v,m}^n+e_{v,m}^{n-1}}{2}\\
&=-\frac{K_1^2}{3}(1-B_1)\sum_{i=1}^{k}h_1\left(\delta_{\overline{x}}\frac{e_{u,i}^n+e_{u,i}^{n-1}}{2}\right)\left(\delta_{\overline{x}}\frac{e_{v,i}^n+e_{v,i}^{n-1}}{2}\right)\\
&-\frac{K_2^2}{3}(1-B_2)\sum_{i=k+1}^{m}h_2\left(\delta_{\overline{x}}\frac{e_{u,i}^n+e_{u,i}^{n-1}}{2}\right)\left(\delta_{\overline{x}}\frac{e_{v,i}^n+e_{v,i}^{n-1}}{2}\right)\\
&-\frac{K_1(1-B_1)}{3\alpha_1}\frac{e_{u,0}^n+e_{u,0}^{n-1}}{2}\frac{e_{v,0}^n+e_{v,0}^{n-1}}{2}-\frac{K_2(1-B_2)}{3\alpha_2}\frac{e_{u,m}^n+e_{u,m}^{n-1}}{2}\frac{e_{v,m}^n+e_{v,m}^{n-1}}{2}\\
&\le M_{\varepsilon_1}\sum_{i=1}^{k}h_1\left(\delta_{\overline{x}}\frac{e_{u,i}^n+e_{u,i}^{n-1}}{2}\right)^2+\varepsilon_1\left|\frac{e_v^n+e_v^{n-1}}{2}\right|_{1,h_1}^2+M_{\varepsilon_2}\sum_{i=1}^{k}h_1\left(\delta_{\overline{x}}\frac{e_{u,i}^n+e_{u,i}^{n-1}}{2}\right)^2+\varepsilon_2\left|\frac{e_v^n+e_v^{n-1}}{2}\right|_{1,h_1}^2\\
&+M_{\varepsilon_3}\left(\frac{e_{u,0}^n+e_{u,0}^{n-1}}{2}\right)^2+\varepsilon_3\left(\frac{e_{v,0}^n+e_{v,0}^{n-1}}{2}\right)^2+M_{\varepsilon_4}\left(\frac{e_{u,m}^n+e_{u,m}^{n-1}}{2}\right)^2+\varepsilon_4\left(\frac{e_{v,m}^n+e_{v,m}^{n-1}}{2}\right)^2\\
&\le M_{\varepsilon_1}\sum_{i=1}^{k}h_1\left(\delta_{\overline{x}}R_{i,t}^{(2)}(u)\right)^2+\varepsilon_1\left\|\frac{e_v^n+e_v^{n-1}}{2}\right\|_{1,h_1}^2+M_{\varepsilon_2}\sum_{i=1}^{k}h_2\left(\delta_{\overline{x}}R_{i,t}^{(2)}(u)\right)^2+\varepsilon_2\left\|\frac{e_v^n+e_v^{n-1}}{2}\right\|_{1,h_2}^2\\
&\le M_\varepsilon\Delta t^3\int_{t^{n-1}}^{t^n}|u_{tt}|_{1,I}^2dt+\varepsilon\left\|\frac{e_v^n+e_v^{n-1}}{2}\right\|_{1,h}^2.
\end{aligned}
$$

Combining the above estimates, we can get an error inequality. Then multiplying the inequality by $2\Delta t$, summing up it for n from 1 to N and making ε sufficiently small and noting that $e_v^0=0$, we have

$$
\begin{aligned}
&\left(e^N_{v[1]}\right)^T A_{k+1} e^N_{v[1]} + \left(e^N_{v[2]}\right)^T A_{m-k+1} e^N_{v[2]} \\
&\leq C h_1^8 \int_0^{t^N} \left(|u|^2_{5,I_1} + |v|^2_{5,I_1} + |v_t|^2_{3,I_1} + |v_t|^2_{4,I_1}\right) dt \\
&+ C h_1^8 \left(\Delta t^4 \int_0^{t^N} \left(|u_{tt}|^2_{5,I_1} + |v_{tt}|^2_{5,I_1}\right) dt + \Delta t^2 \int_0^{t^N} \left(|v_{tt}|^2_{3,I_1} + |v_{tt}|^2_{4,I_1}\right) dt\right) \\
&+ C h_2^8 \int_0^{t^N} \left(|u|^2_{5,I_2} + |v|^2_{5,I_2} + |v_t|^2_{3,I_2} + |v_t|^2_{4,I_2}\right) dt \\
&+ C h_2^8 \left(\Delta t^4 \int_0^{t^N} \left(|u_{tt}|^2_{5,I_2} + |v_{tt}|^2_{5,I_2}\right) dt + \Delta t^2 \int_0^{t^N} \left(|v_{tt}|^2_{3,I_2} + |v_{tt}|^2_{4,I_2}\right) dt\right) \\
&+ C h_1^6 \left(\int_0^{t^N} |v_t|^2_{3,[x_0,x_1]} dt + \Delta t^2 \int_0^{t^N} |v_{tt}|^2_{3,[x_0,x_1]} dt\right) \\
&+ C h_2^6 \left(\int_0^{t^N} |v_t|^2_{3,[x_{m-1},x_m]} dt + \Delta t^2 \int_0^{t^N} |v_{tt}|^2_{3,[x_{m-1},x_m]} dt\right) \\
&+ C \Delta t^4 \int_0^{t^N} \left(\|v_{ttt}\|^2_{0,I} + |u_{tt}|^2_{1,I} + |v_{tt}|^2_{1,I} + u^2_{tt}(0,t) + v^2_{tt}(0,t) + u^2_{tt}(1,t) + v^2_{tt}(1,t)\right) dt \\
&+ C h_1^4 \Delta t^4 \int_0^{t^N} v^2_{ttt}(0,t) dt + C h_2^4 \Delta t^4 \int_0^{t^N} v^2_{ttt}(1,t) dt + C \Delta t \sum_{n=1}^{N} \left\| \frac{e^n_v + e^{n-1}_v}{2} \right\|^2_{0,h}.
\end{aligned}
$$

Using Lemma 3.2 and applying the discrete Gronwall inequality argument, we have

$$\left\|e^N_v\right\|^2_{0,h} \leq C(\Delta t^4 + h_1^6 + h_2^6). \tag{3.12}$$

Further, if $|v_{tt}|_{3,\infty}$ exists, we have

$$|v_t|^2_{3,[x_0,x_1]} + |v_t|^2_{3,[x_{m-1},x_m]} = \int_{x_0}^{x_1} v^2_{txxx} dx + \int_{x_m}^{x_{m-1}} v^2_{txxx} dx \leq 2h_1 |v_t|^2_{3,I_1,\infty} + 2h_2 |v_t|^2_{3,I_2,\infty}.$$

Similarly, we have

$$|v_{tt}|^2_{3,[x_0,x_1]} + |v_{tt}|^2_{3,[x_{m-1},x_m]} \leq 2h_1 |v_{tt}|^2_{3,I_1,\infty} + 2h_2 |v_{tt}|^2_{3,I_2,\infty}.$$

It follows that

$$\|e^N_v\|^2_{0,h} \leq C(\Delta t^4 + h_1^7 + h_2^7). \tag{3.13}$$

Subsequently, we discuss the error of the numerical scheme (2.30) for $v = u + u_t$. The error equation is

$$\frac{e^n_{u,i} - e^{n-1}_{u,i}}{\Delta t} + \frac{e^n_{u,i} + e^{n-1}_{u,i}}{2} = \frac{e^n_{v,i} + e^{n-1}_{v,i}}{2} + R^{(1)}_{i,t}(u) + R^{(2)}_{i,t}(u) - R^{(2)}_{i,t}(v). \tag{3.14}$$

First, we consider the error equation at [0,l]. Multiplying $h_1 \dfrac{e^n_{u,i} + e^{n-1}_{u,i}}{2}$ to both sides of (3.14) and summing together for i from 0 to k gives

$$\frac{1}{2\Delta t}\sum_{i=0}^{k} h_1((e_{u,i}^n)^2-(e_{u,i}^{n-1})^2)+\sum_{i=0}^{k} h_1\left(\frac{e_{u,i}^n+e_{u,i}^{n-1}}{2}\right)^2$$

$$=\sum_{i=0}^{k} h_1\frac{e_{v,i}^n+e_{v,i}^{n-1}}{2}\frac{e_{u,i}^n+e_{u,i}^{n-1}}{2}+\sum_{i=0}^{k} h_1 R_{i,t}^{(1)}(u)\frac{e_{u,i}^n+e_{u,i}^{n-1}}{2}+\sum_{i=0}^{k} h_1 R_{i,t}^{(2)}(u)\frac{e_{u,i}^n+e_{u,i}^{n-1}}{2}$$

$$-\sum_{i=0}^{k} h_1 R_{i,t}^{(2)}(v)\frac{e_{u,i}^n+e_{u,i}^{n-1}}{2}$$

$$\leq \frac{1}{2}\left(2h_1\left(\frac{e_{v,0}^n+e_{v,0}^{n-1}}{2}\right)^2+\sum_{i=1}^{k-1} h_1\left(\frac{e_{v,i}^n+e_{v,i}^{n-1}}{2}\right)^2+2h_1\left(\frac{e_{v,k}^n+e_{v,k}^{n-1}}{2}\right)^2\right)+\frac{1}{2}\left\|\frac{e_u^n+e_u^{n-1}}{2}\right\|_{0,h_1}^2$$

$$+\frac{1}{2}\left(2h_1\left(R_{0,t}^{(1)}(u)\right)^2+\sum_{i=1}^{k-1} h_1\left(R_{i,t}^{(1)}(u)\right)^2+2h_1\left(R_{k,t}^{(1)}(u)\right)^2\right)+\frac{1}{2}\left\|\frac{e_u^n+e_u^{n-1}}{2}\right\|_{0,h_1}^2$$

$$+\frac{1}{2}\left(2h_1\left(R_{0,t}^{(2)}(u)\right)^2+\sum_{i=1}^{k-1} h_1\left(R_{i,t}^{(2)}(u)\right)^2+2h_1\left(R_{k,t}^{(2)}(u)\right)^2\right)+\frac{1}{2}\left\|\frac{e_u^n+e_u^{n-1}}{2}\right\|_{0,h_1}^2$$

$$+\frac{1}{2}\left(2h_1\left(R_{0,t}^{(2)}(v)\right)^2+\sum_{i=1}^{k-1} h_1\left(R_{i,t}^{(2)}(v)\right)^2+2h_1\left(R_{k,t}^{(2)}(v)\right)^2\right)+\frac{1}{2}\left\|\frac{e_u^n+e_u^{n-1}}{2}\right\|_{0,h_1}^2$$

$$\leq C\|\frac{e_v^n+e_v^{n-1}}{2}\|_{0,h_1}^2+2\left\|\frac{e_u^n+e_u^{n-1}}{2}\right\|_{0,h_1}^2$$

$$+C\Delta t^3\int_{t^{n-1}}^{t^n}\left[u_{ttt}^2(0,t)\frac{h_1}{2}+\sum_{i=1}^{k-1}u_{ttt}^2(x_i,t)h_1+u_{ttt}^2(x_k,t)\frac{h_1}{2}\right]dt$$

$$+C\Delta t^3\int_{t^{n-1}}^{t^n}\left[u_{tt}^2(0,t)\frac{h_1}{2}+\sum_{i=1}^{k-1}u_{tt}^2(x_i,t)h_1+u_{tt}^2(x_k,t)\frac{h_1}{2}\right]dt$$

$$+C\Delta t^3\int_{t^{n-1}}^{t^n}\left[v_{tt}^2(0,t)\frac{h_1}{2}+\sum_{i=1}^{k-1}v_{tt}^2(x_i,t)h_1+v_{tt}^2(x_k,t)\frac{h_1}{2}\right]dt$$

$$\leq C\|\frac{e_v^n+e_v^{n-1}}{2}\|_{0,h_1}^2+2\left\|\frac{e_u^n+e_u^{n-1}}{2}\right\|_{0,h_1}^2+C\Delta t^3\int_{t^{n-1}}^{t^n}\left(\|u_{tt}\|_{0,I_1}^2+\|u_{ttt}\|_{0,I_1}^2+\|v_{tt}\|_{0,I_1}^2\right)$$

Multiplying $2\Delta t$, summing together for n from 1 to N, using the Gronwall inequality argument and noting that $e_u^0=0$, we have

$$\|e_u^N\|_{0,h_1}^2\leq C\Delta t\sum_{n=1}^{N}\left\|\frac{e_v^n+e_v^{n-1}}{2}\right\|_{0,h_1}^2+C\Delta t^4\int_0^{t^N}\left(\|u_{tt}\|_{0,I_1}^2+\|u_{ttt}\|_{0,I_1}^2+\|v_{tt}\|_{0,h_1}^2\right)dt.$$

We can get the similar conclusion of the error equation at $[l,1]$, namely,

$$\|e_u^N\|_{0,h_2}^2\leq C\Delta t\sum_{n=1}^{N}\left\|\frac{e_v^n+e_v^{n-1}}{2}\right\|_{0,h_2}^2+C\Delta t^4\int_0^{t^N}\left(\|u_{tt}\|_{0,I_2}^2+\|u_{ttt}\|_{0,I_2}^2+\|v_{tt}\|_{0,I_2}^2\right)dt.$$

Combining them gives

$$\|e_u^N\|_{0,h}^2\leq C\Delta t\sum_{n=1}^{N}\left\|\frac{e_v^n+e_v^{n-1}}{2}\right\|_{0,h}^2+C\Delta t^4\int_0^{t^N}\left(\|u_{tt}\|_{0,I}^2+u_{ttt}\|_{0,I}^2+v_{tt}\|_{0,I}^2\right)dt. \tag{3.15}$$

Finally, using the Gronwall inequality argument and combining (3.12), (3.15), we have,

$$\|e_u^N\|_{0,h}+\|e_v^N\|_{0,h}\leq C(\Delta t^2+h_1^3+h_2^3).$$

Further, if $|v_{tt}|_{3,\infty}$ exists, we combine (3.13) and (3.15) to get

$$\|e_u^N\|_{0,h}+\|e_v^N\|_{0,h}\leq C(\Delta t^2+h_1^{3.5}+h_2^{3.5}).$$

Table 1: Numerical results at T=1 with $\Delta t = 10^{-6}$

h	e_0	order	e_∞	order
0.1	1.181×10^{-6}	-	2.404×10^{-6}	-
0.05	6.789×10^{-8}	4.12	1.4134×10^{-7}	4.01
0.025	4.076×10^{-9}	4.05	8.5362×10^{-9}	4.05

Theorem 3.5. *Let $u(x,t)$, $v(x,t) \in H^3([0,T], \tilde{H}^5(I))$ be the solutions to the dual phase equations (1.7)-(1.6), (2.1)-(2.5) and the relation $v = u + u_t$. Let u_i^n, v_i^n be the solutions to the fully-discrete numerical schemes (2.26)-(2.30).Then existing a $h_0 > 0$, such that*

$$\left\|u(x_i,t^N) - u_i^N\right\|_{0,h} + \left\|v(x_i,t^N) - v_i^N\right\|_{0,h} \le C(\Delta t^2 + h_1^3 + h_2^3), \tag{3.16}$$

whenever $\max(h_1,h_2) < h_0$. *Further, if $|v_{tt}|_{3,\infty}$ exists, we have*

$$\left\|u(x_i,t^N) - u_i^N\right\|_{0,h} + \left\|v(x_i,t^N) - v_i^N\right\|_{0,h} \le C(\Delta t^2 + h_1^{3.5} + h_2^{3.5}). \tag{3.17}$$

4 Numerical example

In this section, the examples are provided to verify the accuracy and effectiveness of the above schemes.

Example 1. *Let the coefficients $B_1 = 1$, $K_1 = \dfrac{\sqrt{2}}{\pi} < 1$, $\alpha_1 = 0.2$, $B_2 = 4$, $K_2 = \dfrac{1}{\pi} < 1$, $\alpha_2 = 0.2$. The accurate solution is as follows,*

$$u = \begin{cases} e^{-\frac{t}{2}} \sin(\frac{\pi}{2}x), & x \in [0, 0.5], \\ e^{-\frac{t}{2}} \cos(\frac{\pi}{2}x), & x \in (0.5, 1]. \end{cases}$$

This implies that the interface point $l = 0.5$. The initial-boundary values can be determined by the accurate solution. Setting $h = \max(h_1,h_2)$. For convenience, we select $h_1 = h_2$ in computation, namely, $h = h_1$ *or* h_2. The maximal L^2 norm error and the L^∞ norm error at the final time $T = t^N = N\Delta t$ are respectively defined by

$$e_0 = \max_{0\le n\Delta t\le T}\left(h_1\sum_{i=0}^{k}|u(x_i,t^n) - u_i^n|^2 + h_2\sum_{i=k}^{m}|u(x_i,t^n) - u_i^n|^2\right)^{1/2}, \quad e_\infty = \max_{0\le i\le m}\left|u(x_i,t^N) - u_i^N\right|.$$

First, we investigate the convergence order with respect to the spatial variable x, when the time step Δt is small enough. This implies that $e(m,\Delta t) = O(h^p + \Delta t^q) \approx O(h^p)$, where m is the number of grid points and e denotes e_0 or e_∞. In our computation, we set $\Delta t = 10^{-6}$, and choose the spacial size to be $h = 0.1$, 0.05 and 0.025. The corresponding results are listed in Table 1. Table 1 shows that the convergence order p with respect to the spatial variable x is around 4.0 , i.e., $O(h^4)$, where $order = \log_2\dfrac{e(2m,\Delta t)}{e(m,\Delta t)}$.

Conversely, we discuss the convergence order with respect to the time step, when the spacial size h is very small. We set $h = 1/5000$, choose $\Delta t = 0.01$, 0.005, 0.0025 and compute $order = \log_2\dfrac{e(m,2\Delta t)}{e(m,\Delta t)}$. The results are listed in Table 2, which shows the order q is around 2.0, i.e., $O(\Delta t^2)$.

Subsequently, we consider the following cases, that is, $\Delta t = h^2$ and $\Delta t = h$. The results are listed in Table 3, 4, respectively. For example, when $\Delta t = h^2$, $e(m,\Delta t) = O(h^p + \Delta t^q) = O(h^p + h^{2q}) \approx O(h^p)$ if $p < 2q$. We find that these orders are around 4.0, 2.0, respectively, which are in accord with $O(\Delta t^2 + h^4)$.

Table 2: Numerical results at T=1 with $h = 1/5000$

Δt	e_0	order	e_∞	order
0.01	1.994×10^{-7}	-	8.004×10^{-8}	-
0.005	4.984×10^{-8}	2.00	2.001×10^{-8}	2.00
0.0025	1.246×10^{-8}	2.00	5.003×10^{-9}	2.00

Table 3: Numerical results at T=1 with $\Delta t = h^2$

h	e_0	order	e_∞	order
0.1	1.377×10^{-6}	-	2.752×10^{-6}	-
0.05	8.009×10^{-8}	4.10	1.631×10^{-7}	4.08
0.025	4.837×10^{-9}	4.05	9.894×10^{-9}	4.04

The left one of Fig. 1 shows the true solution u and the numerical solution u_h at nodes, when $h = 0.1$, $\Delta t = h^2$, $T = 1$. The right one of Fig. 1 shows the absolute error at $h = 0.1$, $\Delta t = h^2$, $T = 1$. We find that the absolute error at the interface point $x = 0.5$ is a bit bigger than the other points. But it does not affect the error orders.

Example 2. *The accurate solution is the same as Example 1. Let the coefficients* $B_1 = 1.5$, $K_1 = \frac{8}{\pi} > 1$, $\alpha_1 = 0.1$, $B_2 = 4$, $K_2 = \frac{4}{\pi} > 1$, $\alpha_2 = 0.2$. *Setting* $\Delta t = h^2$.

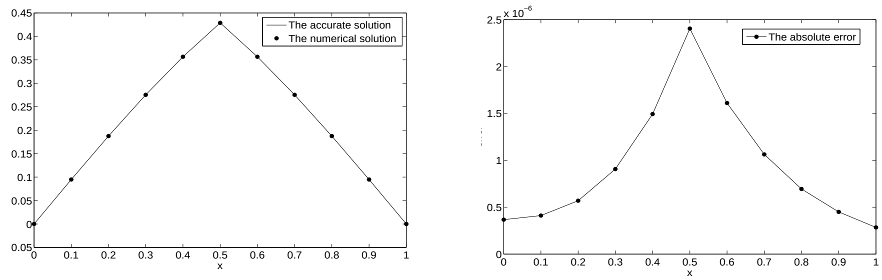
The left one of Fig. 2 illustrates the comparison of the numerical solution u with respect to x when $h = 0.1$, $T = 1$ or $T = 10$. We can see that the maximum norm errors of u are both $O(10^{-7})$. The right one of Fig. 2 shows the convergence orders with respect to the maximal L_2 norm error and the maximum norm error when $T = 1$. They are both nearly 4.

5 Conclusions

We have developed a high-order compact finite volume element method for solving 1-D dual-phase lag equation with the interface. Around the interface point, we make modification to keep the high-order accuracy of the method. The examples are given to discuss the nano-heat conduction where the Knudsen number K_1 or K_2 is around 1.0 or is grater than 1.0. Further research will focus on a convenient high-order finite volume element method to solve multi-dimensional dual-phase lag equation with the interface, for example, ADI scheme, which can simplify the computation.

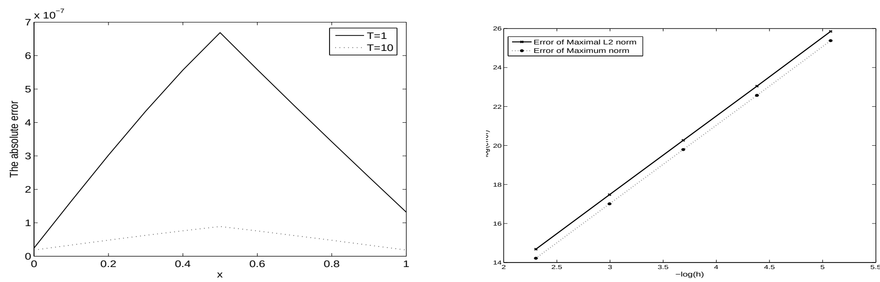
Table 4: Numerical results at T=1 with $\Delta t = h$

h	e_0	order	e_∞	order
0.1	2.142×10^{-5}	-	3.725×10^{-5}	-
0.05	5.077×10^{-6}	2.08	8.847×10^{-6}	2.07
0.025	1.252×10^{-6}	2.02	2.185×10^{-6}	2.02



(a) (b)

Figure 1: (a) The true solution u and the numerical solution u_h. (b) the absolute error $|u - u_h|$.



(a) (b)

Figure 2: (a)The numerical results of u with respect to x with $h = 0.1$, $\Delta t = h^2$ when $T = 1$ or 10. (b) The convergence orders of u.

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