The approximate method for solving Poincaré problem of nonlinear elliptic equations of second order

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Abstract. This article deals with the approximate method used to solve the Poincaré boundary value problem for nonlinear elliptic equations of second order in unbounded multiply connected domains. This type of boundary value problems are known to have applications in many fields such as mechanics and physics. We first present a formulation of the boundary value problem and the corresponding modified well-posedness. Then we obtain the representation theorem and a priori estimates of solutions for the modified problem. Finally by the estimates of solutions and the continuity method, we obtain the solvability results and error estimates of approximate solutions of the modified problem for the nonlinear elliptic equations of second order.

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1. Formulation of the Poincaré boundary value problem

Let D be an $(N+1)$-connected domain including the infinite point with the boundary $\Gamma = \cup_{j=0}^{N}\Gamma_j$ in $\mathbb{C}$, where $\Gamma \in C^2_\mu\,(0<\mu<1)$. Without loss of generality, we assume that D is a circular domain in $|z|>1$, where the boundary consists of $N+1$ circles $\Gamma_0=\Gamma_{N+1}=\{|z|=1\}$, $\Gamma_j=\{|z-z_j|=r_j\}, j=1,...,N$ and $z=\infty\in D$. In this article, the notations are the same as in references [1–8]. We consider the second order equation in the complex form

$$\begin{cases} u_{z\bar z}=F(z,u,u_z,u_{zz}),\ F=\mathrm{Re}\,[Qu_{zz}+A_1u_z]+A_2u+A_3, \\ Q=Q(z,u,u_z,u_{zz}),\ A_j=A_j(z,u,u_z),\ j=1,2,3, \end{cases} \tag{1.1}$$

satisfying the following Condition C.

Condition C (1) $Q(z,u,w,U), A_j(z,u,w)(j=1,2,3)$ are continuous in $u\in\mathbb{R}$, $w\in\mathbb{C}$ for almost every point $z\in D$, $U\in\mathbb{C}$, and $Q=0$, $A_j=0\,(j=1,2,3)$ for $z\notin D$.

(2) The functions $Q(z,u,w,U), A_j(z,u,w)(j=1,2,3)$ are measurable in $z\in D$ for all continuous functions $u(z), w(z)$ in $\overline{D}$, and $A_j(z,u,w)(j=1,2,3)$ satisfy

$$L_{p,2}[A_1(z,u,w),\overline{D}]\le k_0, L_{p,2}[A_2(z,u,w),\overline{D}]\le \varepsilon k_0,\ L_{p,2}[A_3(z,u,w),\overline{D}]\le k_1, \tag{1.2}$$

in which p,ε,k_0,k_1 are non-negative constants.

(3) The function F in equation (1.1) satisfies the uniform ellipticity condition, namely for any number $u\in\mathbb{R}$ and $w,U_1,U_2\in\mathbb{C}$, the inequality

$$|F(z,u,w,U_1)-F(z,u,w,U_2)|\le q_0|U_1-U_2|, \tag{1.3}$$

holds for almost every point $z\in D$, where $q_0(<1)$ is a non-negative constant.

We formulate the Poincaré boundary value problem as follows.

Problem P In the domain D, find a solution $u(z)$ of equation (1.1), such that it is continuously differentiable in $\overline{D}$, and satisfies the boundary condition

$$\frac{1}{2}\frac{\partial u}{\partial \nu}+c_1(z)u=c_2(z),\ \text{i.e. } \mathrm{Re}[\overline{\lambda(z)}u_z]+c_1(z)u=c_2(z),\ z\in\Gamma, \tag{1.4}$$

in which ν is any unit vector at every point on $\Gamma=\partial D$, $\lambda(z)=\cos(\nu,x)-\mathrm{i}\cos(\nu,y)$, $c_1(z)$ and $c_2(z)$ are known functions satisfying the conditions

$$C_\alpha[\lambda,\Gamma]\le k_0,\ C_\alpha[c_1,\Gamma]\le \varepsilon k_0,\ C_\alpha[c_2,\Gamma]\le k_2, \tag{1.5}$$

where $\varepsilon\,(>0)$, $\alpha\,(1/2<\alpha<1)$, k_0, k_2 are non-negative constants.

Since the directional derivative can be arbitrary, (1.4) indicates a very general boundary condition. If $\cos(\nu,n)=0$ and $c_1=0$ on Γ, where n is the outward normal vector on Γ, then Problem P is the Dirichlet boundary value problem (Problem D). If $\cos(\nu,n)=1$ and $c_1=0$ on Γ, then Problem P is the Neumann boundary value problem (Problem N). And if $\cos(\nu,n)>0$, and $c_1\ge 0$ on Γ, then Problem P is the regular oblique derivative problem, i.e. the third boundary value problem (Problem III or O).

We call the integer

$$K=\frac{1}{2\pi}\Delta_\Gamma \arg\lambda(z)$$

the index of Problem P. Note that the Dirichlet boundary value problem, Neumann boundary value problem and regular oblique derivative boundary value problem are the special cases of Problem P, whose indexes are equal to $K=N-1$. In general, the index of Problem P can be any negative or non-negative integer, hence the boundary condition of Problem P is very general. And the index is directly related to the existence and uniqueness of the solution. When the index $K<0$, Problem P may not be solvable, and when $K\ge 0$, the solution of Problem P is not necessarily unique. Hence we consider the well-posedness of Problem P with modified boundary conditions as follows.

Problem Q Find a continuous solution $[w(z),u(z)]$ of the complex equation

$$w_{\bar z}=F(z,u,w,w_z),\ F(z,u,w,w_z)=\mathrm{Re}\,[Qw_z+A_1w]+A_2u+A_3, \tag{1.6}$$

satisfying the boundary condition

$$\mathrm{Re}[\overline{\lambda(z)}w(z)]+c_1(z)u=c_2(z)+h(z),\ z\in\varGamma, \tag{1.7}$$

and the relation

$$u(z)=-2\mathrm{Re}\int_{a_0}^{z}[\frac{w(z)}{z^2}-\sum_{j=a_0}^{N}\frac{id_jz_j}{z(z-z_j)}\mathrm{d}z]+b_0, \tag{1.8}$$

where $a_0=1$, $d_j(j=1,...,N)$ are appropriate real constants such that the function determined by the integral in (1.8) is single-valued in D, and the undetermined function $h(z)$ is as stated in

$$h(z)=\begin{cases} 0,\ z\in\varGamma, & K\ge N,\\ \left.\begin{array}{l} h_j,\ z\in\varGamma_j,\ j=1,...,N-K,\\ 0,\ z\in\varGamma_j, j=N-K+1,...,N+1 \end{array}\right\} & 0\le K<N,\\ \left.\begin{array}{l} h_j,\ z\in\varGamma_j,\ j=1,...,N,\\ h_0+\mathrm{Re}\displaystyle\sum_{m=1}^{-K-1}(h_m^+ +\mathrm{i}h_m^-)z^m,\ z\in\varGamma_0 \end{array}\right\} & K<0, \end{cases}$$

in which $h_j\,(j=0,1,...,N)$, $h_m^\pm\,(m=1,...,-K-1,K<0)$ are unknown real constants to be determined appropriately. In addition, for $K\ge 0$ the solution $w(z)$ is assumed to satisfy the point conditions

$$\mathrm{Im}[\overline{\lambda(a_j)}w(a_j)]=b_j,\ j\in J=\begin{cases}1,...,2K-N+1, & K\geq N,\\ N-K+1,...,N+1, & 0\leq K<N,\end{cases} \tag{1.9}$$

where $a_j\in \Gamma_j\,(j=1,...,N)$, $a_j\in\Gamma_0\,(j=N+1,...,2K-N+1, K\geq N)$ are distinct points, and $b_j(j\in J\cup\{0\})$ are all real constants satisfying the conditions

$$|b_j|\leq k_3,\ \ j\in J\cup\{0\}, \tag{1.10}$$

for a non-negative constant k_3. The condition $0<K<N$, a singular case which only occurs in the case of multiply connected domains, can not be easily handled.

2. Estimates of solutions for the Poincaré boundary value problem

First of all, we give a prior estimates of solutions of Problem Q for (1.6).

Theorem 2.1 *Suppose that Condition* C *holds and* $\varepsilon=0$ *in* (1.2) *and* (1.5). *Then any solution* $[w(z),u(z)]$ *of Problem* Q *for* (1.6) *satisfies the estimates*

$$C_\beta[w(z),\overline{D}]+C_\beta[u(z),\overline{D}]\leq M_1k^*, \tag{2.1}$$

$$L_{p_0,2}[|w_{\bar z}|+|w_z|,\overline{D}]\leq M_2k^*, \tag{2.2}$$

in which $\beta=\min(\alpha,1-2/p_0)$ *with* $2<p_0\leq p$, $M_j=M_j(q_0,p_0,k_0,\beta,K,D)$, $j=1,2$, $k^*=k_1+k_2+k_3$.

Proof Note that the solution $[w(z),u(z)]$ of Problem Q satisfies the equation and boundary conditions

$$w_{\bar z}-\mathrm{Re}[Qw_z+A_1w]=A_3\ \text{ in }\ D, \tag{2.3}$$

$$\mathrm{Re}[\overline{\lambda(z)}w]=c_2(z)+h(z)\ \text{ on }\ \Gamma, \tag{2.4}$$

$$\mathrm{Im}[\overline{\lambda(a_j)}w(a_j)]=b_j,\ j\in J,\ u(1)=b_0. \tag{2.5}$$

According to the method in the proof of Theorem 3.1, Chapter V, [3] or Theorem 2.2.1, [5], we can derive that the solution $w(z)$ satisfies the estimates

$$C_\beta[w(z),\overline{D}]\leq M_3k^*, \tag{2.6}$$

$$L_{p_0,2}[|w_{\bar z}|+|w_z|,\overline{D}]\leq M_4k^*, \tag{2.7}$$

where $M_j=M_j(q_0,p_0,k_0,\beta,K,D), j=3,4$ and $k^*=k_1+k_2+k_3$. From (1.8), it follows that

$$C_\beta[u(z),\overline{D}]\leq M_5C_\beta[w(z),\overline{D}]+k_3, \tag{2.8}$$

$$L_{p_0,2}[|u_{\bar z}|+|u_z|,\overline{D}]\leq M_5C_\beta[w(z),\overline{D}]+k_3, \tag{2.9}$$

in which $M_5=M_5(p_0,D)$ is a non-negative constant. Combining (2.6)–(2.9), we see that the estimates (2.1) and (2.2) are obtained.

Theorem 2.2 *Let the equation* (1.6) *satisfy Condition* C *and* ε *in* (1.2), (1.5) *be small enough. Then any solution* $[w(z),u(z)]$ *of Problem* Q *for* (1.6) *satisfies the estimates*

$$C_\beta[w(z),\overline{D}]+C_\beta[u(z),\overline{D}]\leq M_6k^*, \tag{2.10}$$

$$L_{p_0,2}[|w_{\bar z}|+|w_z|,\overline{D}]+L_{p_0,2}[u_z,\overline{D}]\leq M_7k^*, \tag{2.11}$$

where β,k^* *are as stated in Theorem* 2.1, $M_j=M_j(q_0,p_0,k_0,\beta,K,D), j=6,7$.

Proof It is easy to see that $[w(z),u(z)]$ satisfies the equation and boundary conditions

$$w_{\bar z}-\mathrm{Re}[Qw_z]+A_1w]=A_2u+A_3,\ z\in D, \tag{2.12}$$

$$\mathrm{Re}[\overline{\lambda(z)}w(z)]=-c_1u+c_2(z)+h(z),\ z\in\Gamma, \tag{2.13}$$

$$\mathrm{Im}[\overline{\lambda(a_j)}w(a_j)]=b_j,\ j\in J,\ u(1)=b_0. \tag{2.14}$$

Similarly to the derivations of (2.6) and (2.7), we can obtain

$$\begin{cases} C_\beta[w(z),\overline{D}] \le M_3\{k^* + \varepsilon k_0 C_\beta[u,\overline{D}]\}, \\ L_{p_0,2}[|w_{\bar z}| + |w_z|,\overline{D}] \le M_4\{k^* + \varepsilon k_0 C_\beta[u,\overline{D}]\}. \end{cases} \tag{2.15}$$

Then from (2.8), it follows that

$$\begin{cases} C_\beta[w(z),\overline{D}] \le M_3\{k^* + \varepsilon k_0[M_5 C_\beta[w(z),\overline{D}] + k_3]\}, \\ L_{p_0,2}[|w_{\bar z}| + |w_z|,\overline{D}] \le M_4\{k^* + \varepsilon k_0[M_5 C_\beta[w(z),\overline{D}] + k_3]\}. \end{cases} \tag{2.16}$$

If the positive constant ε is small enough such that $1-\varepsilon k_0 M_3 M_5 \ge 1/2$, then the first inequality in (2.16) implies that

$$C_\beta[w(z),\overline{D}] \le \frac{(1+\varepsilon k_0)M_3}{1-\varepsilon k_0 M_3 M_5} k^* \le 2(1+\varepsilon k_0)M_3 k^* = M_8 k^*. \tag{2.17}$$

Combining (2.8) and (2.17), we obtain

$$C_\beta[w(z),\overline{D}] + C_\beta[u(z),\overline{D}] \le [1+(1+M_5)M_8]k^* = M_6 k^*, \tag{2.18}$$

which is the estimate (2.10). The estimates in (2.11) can be easily derived from (2.9) and the second inequality in (2.16), i.e.

$$\begin{cases} L_{p_0,2}[|w_{\bar z}| + |w_z|,\overline{D}] + L_{p_0,2}[u_z,\overline{D}] \\ \le M_4\{k^* + \varepsilon k_0[M_5 C_\beta[w(z),\overline{D}] + k_3]\} + M_5 C_\beta[w(z),\overline{D}] + k_3 \\ \le [1 + M_4(1+\varepsilon k_0) + M_5 M_8(1+\varepsilon k_0 M_4)]k^* = M_7 k^*. \end{cases} \tag{2.19}$$

In the following, we prove the uniqueness theorem of solutions of Problem Q for equation (1.1) as follows.

Theorem 2.3 *Suppose that equation* (1.6) *satisfies Condition* C. *Also suppose that for any real functions* $u_j(z)$, $w_j(z)$ $(u_j(z), w_j(z) \in C(\overline{D}), j=1,2)$, $V(z) \in L_{p,2}(\overline{D})$, *the equality*

$$F(z,u_1,w_1,V) - F(z,u_2,w_2,V) = \mathrm{Re}[\tilde{Q}V + \tilde{A}_1(w_1-w_2)] + \tilde{A}_2(u_1-u_2) \tag{2.20}$$

holds, where $|\tilde{Q}| \le q_0 (<1)$, $L_{p,2}[\tilde{A}_1,\overline{D}] \le k_0$, $L_{p,2}[\tilde{A}_2,\overline{D}] \le \varepsilon k_0$ *with the sufficiently small positive constant* ε. *Then Problem* Q *for equation* (1.1) *has at most one solution.*

Proof Denote by $[w_j(z), u_j(z)](j=1,2)$ the two solutions of Problem Q for (1.6), and substitute them into (1.6)-(1.9), we see that $[w(z),u(z)] = [w_1(z) - w_2(z), u_1(z)] - u_2(z)]$ is a solution of the following homogeneous boundary value problem

$$\begin{aligned} & w_{\bar z} = \mathrm{Re}[\tilde{Q}w_z + \tilde{A}_1 w] + \tilde{A}_2 u,\ z \in D, \\ & \mathrm{Re}[\overline{\lambda(z)}w(z) + c_1(z)u(z)] = h(z),\ z \in \Gamma^*, \\ & \mathrm{Im}[\overline{\lambda(a_j)}w(z)] = 0,\ j \in J, \\ & u(z) = \textstyle\int_{a_0}^{z}[w(z)dz + \sum_{m=1}^{N} \frac{id_j}{z-z_j}dz] \text{ in } D, \end{aligned}$$

the coefficients of which satisfy the same conditions of (1.2),(1.3),(1.5) and (1.10) with $k_1 = k_2 = k_3 = 0$. On the basis of Theorem 2.2, provided that ε is sufficiently small, we can derive that $w(z) = u(z) = 0$ in $\overline{D}$, i.e. $w_1(z) = w_2(z)$, $u_1(z) = u_2(z)$ in $\overline{D}$.

3. The approximate method of solving Poincaré boundary value problem

In this section, we shall prove the solvability of Poincaré boundary value problem by the continuity method.

Theorem 3.1 *Suppose that the nonlinear elliptic equation* (1.1) *satisfies Condition* C *and* (2.20) *and* ε *in* (1.2) *and* (1.5) *is small enough. Then there exists a solution* $u(z)$ *of Problem* Q *for* (1.6) *and* $u(z) \in B = C^1_\beta(\overline{D}) \cap W^2_{p_0,2}(D)$, *where* β *and* p_0 *are positive constants as before.*

Proof We introduce the nonlinear elliptic equation with the parameter $t \in [0,1]$:

$$u_{z\bar{z}} = tF(z,u,u_z,u_{zz}) + A(z), \tag{3.1}$$

where $A(z)$ is any measurable function in D and $A(z) \in L_{p_0,2}(\overline{D})$ for $2 < p_0 \le p$. Let E be a subset of $0 \le t \le 1$ such that Problem Q is solvable for (3.1) with any $t \in E$ and any $A(z) \in L_{p_0,2}(\overline{D})$. We can prove that when $t = 0$, Problem Q has the unique solution

$$u(z)=U(z)+\Psi(z),\ \Psi(z)=HA=\frac{2}{\pi}\iint_{D_0}\frac{A(1/\zeta)}{|\zeta|^4}\ln\left|\frac{1-\zeta z}{\zeta}\right|d\sigma_\zeta, \tag{3.2}$$

where $\Psi(z)$, $U(z)$ are the solutions of

$$\Psi_{z\bar{z}} = A(z),\ U_{z\bar{z}} = 0 \text{ in } D \tag{3.3}$$

satisfying the boundary conditions

$$\begin{aligned}&\mathrm{Re}[\overline{\lambda(z)}U_z] + c_1(z)U(z) = R(z) + h(z),\\ &R(z) = c_2(z) - c_1\Psi(z) - \mathrm{Re}[\overline{\lambda(z)}\Psi_z],\ z \in \Gamma,\end{aligned} \tag{3.4}$$

$$\mathrm{Re}[\overline{\lambda(z)}(U_z+\Psi_z)]|_{z=a_j}=b_j,\ j\in J,\ U(1)+\Psi(1)=b_0, \tag{3.5}$$

respectively(see Chapter VI in [3]). According to Theorem 2.3, we know that Problem Q for the above equation has a unique solution. This shows that the point set E is not empty. If we can prove that E is both open and closed in $0 \le t \le 1$, then E is $0 \le t \le 1$. Hence Problem Q for (3.1) with $t = 1$ and $A(z) = (1-t)F(z,0,0,0)$ is solvable. Thus this theorem is proved.

Now, we verify that E is open in $0 \le t \le 1$. Suppose that $t_0 \in E$ and $0 \le t_0 \le 1$, namely Problem Q for (3.1) with $t = t_0$ for any $A(z) \in L_{p_0,2}(\overline{D})$ is solvable. We shall prove that there exists a neighborhood $E = \{|t-t_0| \le \delta, 0 \le t \le 1, \delta > 0\}$, so that for every $t \in E$ and any function $A(z) \in L_{p_0,2}(\overline{D})$, Problem Q for (3.1) has a unique solution. The equation (3.1) can be rewritten in the form

$$u_{z\bar{z}} - t_0F(z,u,u_z,u_{zz}) = (t-t_0)F(z,u,u_z,u_{zz}) + A(z). \tag{3.6}$$

Choosing any function $u_0(z) \in B = C^1_\beta(\overline{D}) \cap W^2_{p_0,2}(D)$, in particular $u_0(z) = 0$ in $\overline{D}$ and substituting $u_0(z)$ into the position of $u_0(z)$ in the right hand side of (3.6), we see that

$$(t-t_0)F(z,u_0,u_{0z},u_{0zz}) + A(z) \in L_{p_0,2}(\overline{D}).$$

Consequently the equation (3.6) has a unique solution $u_1(z) \in B$. By using the successive iteration, we can find out a sequence of functions: $u_n(z) \in B$, $n = 1,2,...$, which are the solutions of the boundary value problems

$$u_{n+1z\bar{z}}-t_0F(z,u_{n+1},u_{n+1z},u_{n+1zz})=(t-t_0)F(z,u_n,u_{nz},u_{nzz})+A(z), \tag{3.7}$$

$$\mathrm{Re}[\overline{\lambda(z)}u_{n+1z}] + c_1(z)u_{n+1}(z) = c_2(z) + h(z),\ z \in \Gamma, \tag{3.8}$$

$$\mathrm{Im}[\overline{\lambda(z)}u_{n+1z}]|_{z=a_j} = b_j,\ j \in J,\ u_{n+1}(1) = b_0. \tag{3.9}$$

From (3.7)–(3.9), it follows that

$$\begin{aligned}&(u_{n+1}-u_n)_{z\bar{z}} - t_0[F(z,u_{n+1},u_{n+1z},u_{n+1zz}) - F(z,u_n,u_{nz},u_{nzz})]\\ &=(t-t_0)[F(z,u_n,u_{nz},u_{nzz})-F(z,u_{n-1},u_{n-1z},u_{n-1zz})],\ n=1,2,....\end{aligned} \tag{3.10}$$

$$\mathrm{Re}[\overline{\lambda(z)}(u_{n+1}(z)-u_n(z))_z] + c_1(z)(u_{n+1}-u_n) = h(z),\ z \in \Gamma, \tag{3.11}$$

$$\mathrm{Im}[\overline{\lambda(z)}(u_{n+1z}-u_{nz})]|_{z=a_j} = 0,\ j \in J,\ u_{n+1}(1) - u_n(1) = 0. \tag{3.12}$$

By Condition C, we have

$$F(z,u_{n+1},u_{n+1z},u_{n+1zz})-F(z,u_n,u_{nz},u_{nzz})$$
$$=\mathrm{Re}[\tilde{Q}(u_{n+1}-u_n)_{zz}+\tilde{A}_1(u_{n+1}-u_n)_z]+\tilde{A}_2(u_{n+1}-u_n),$$

where $|\tilde{Q}|\le q_0<1, L_{p_0,2}[\tilde{A}_j,\overline{D}]\le k_0, j=1,2,\ n=0,1,2,....$ In addition, we can obtain the estimate

$$\begin{aligned}&S(u_{n+1}-u_n)=C^1_\beta[u_{n+1}-u_n,\overline{D}]+L_{p_0,2}[|(u_{n+1}-u_n)_{z\bar{z}}|+|(u_{n+1}-u_n)_{zz}|,\overline{D}]\\&\le M|t-t_0|L_{p_0,2}[F(z,u_n,u_{nz},u_{nzz})-F(z,u_{n-1},u_{n-1z},u_{n-1zz}),\overline{D}]\\&\le M|t-t_0|\{q_0L_{p_0,2}[|(u_n-u_{n-1})_{z\bar{z}}|+|(u_n-u_{n-1})_{zz}|,\overline{D}]\\&\quad+k_0C^1[u_n-u_{n-1},\overline{D}]\le M_9|t-t_0|(q_0+k_0)S(u_n-u_{n-1}),\end{aligned}\tag{3.13}$$

in which $M_9=M_9(q_0,p_0,k_0,\alpha,K,D)$ is a positive constant. Choosing δ to be small enough so that $\eta=\delta M_9(q_0+k_0)<1$, we can derive

$$S(u_{n+1}-u_n)\le\eta^nS(u_1-u_0)\le\eta^n[C^1_\beta(u,\overline{D}]+L_{p_0,2}[|u_{1z\bar{z}}|+|u_{1zz}|,\overline{D}]\tag{3.14}$$

for every $t\in E$, and

$$\begin{aligned}S(u_n-u_m)&\le(\eta^{n-1}+\eta^{n-2}+\cdots+\eta^m)S(u_1)\\&\le\eta^{N+1}\frac{1-\eta^{n-m}}{1-\eta}S(u_1)\le\frac{\eta^{N+1}}{1-\eta}S(u_1)\end{aligned}\tag{3.15}$$

for $n\ge m>N$, where N is a positive integer. Thus $S(w_n-w_m)\to0$ as $n,m\to\infty$. Since $B=C^1_\beta(\overline{D})\cap W^2_{p_0,2}(D)$ is a Banach space, there exists a solution of Problem Q for (3.6), i.e. (3.1) with $t\in E$. This shows that E is an open set in $0\le t\le1$.

Finally, we verify that E is closed in $0\le t\le1$. Choosing an arbitrary sequence $t_n\in E(n=1,2,...)$ and $t_n\to t_0$ as $n\to\infty$, it is sufficient to prove Problem Q for (3.1) with $t=t_0$ is solvable. In fact, substituting the above solutions $u_n(z),u_m(z)$ into the equations (3.1) with $t=t_n,t=t_m$ respectively, we obtain the following difference between the two equations:

$$(u_n-u_m)_{z\bar{z}}-t_n[F(z,u_n,u_{nz},u_{nzz})-F(z,u_m,u_{mz},u_{mzz})]$$
$$=A_{nm}(z),\ A_{nm}(z)=(t_n-t_m)f(z,u_m,u_{mz},u_{mzz}),$$

where

$$f(z,u_m,u_{mz},u_{mzz})=F(z,u_m,u_{mz},u_{mzz})-F(z,0,0,0),$$

and

$$\begin{aligned}&F(z,u_n,u_{nz},u_{nzz})-F(z,u_m,u_{mz},u_{mzz})\\&=F(z,u_n,u_{nz},u_{nzz})-F(z,u_n,u_{nz},u_{mzz})\\&\quad+F(z,u_n,u_{nz},u_{mzz})-F(z,u_m,u_{mz},u_{mzz})\\&=\mathrm{Re}[\tilde{Q}(u_n-u_m)_{zz}+\tilde{A}_1(u_n-u_m)]+\tilde{A}_2(u_n-u_m),\end{aligned}\tag{3.16}$$

with

$$|\tilde{Q}|\le q_0<1,\ L_{p_0,2}[\tilde{A}_j,\overline{D}]\le k_0,\ j=1,2,$$

and

$$L_{p_0,2}[A_{nm},\overline{D}]\le|t_n-t_m|[q_0L_{p_0,2}(u_{mzz},\overline{D})+k_0C^1(u_m,\overline{D})]$$
$$\le|t_n-t_m|(q_0+k_0)S(u_m)\le|t_n-t_m|(q_0+k_0)k,$$

in which

$$S(u_m)=C^1_\beta[u_m,\overline{D}]+L_{p_0,2}[u_{mz\bar{z}}|+|u_{mzz}|,\overline{D}]$$
$$\le M_1(k_1+k_2+k_3)=k,\ M_1=M_1(q_0,p_0,k_0,\alpha,K,D),$$

for

$$k_1 = L_{p_0,2}[F(z,0,0,0),\overline{D}],\ k_2 = C_\alpha[c_2(z),\overline{D}],\ k_3 = \max_{1\le j\le N_0} |b_j|. \tag{3.17}$$

Moreover, taking into account that $u_n(z) - u_m(z)$ satisfies the boundary conditions

$$\mathrm{Re}[\overline{\lambda(z)}(u_n(z) - u_m(z))_z] + c_1(z)(u_n - u_m) = h(z),\ z \in \Gamma, \tag{3.18}$$

$$\mathrm{Im}[\overline{\lambda(z)}(u_{nz} - u_{mz})]|_{z=a_j} = 0,\ j \in J,\ u_n(1) - u_m(1) = 0, \tag{3.19}$$

we have

$$\begin{aligned} S(u_n-u_m) &= C^1_\beta[u_n-u_m,\overline{D}]+L_{p_0,2}[|(u_n-u_m)_{z\bar z}|+|(u_n-u_m)_{zz}|,\overline{D}] \\ &\le M_1L_{p_0,2}[A_{nm},\overline{D}] \le |t_n - t_m|M_1(q_0 + k_0)k. \end{aligned} \tag{3.20}$$

Since $|t_n - t_m| \to 0$ as $n, m \to \infty$, it is easy to see that $S(u_n - u_m) \to 0$ as $n, m \to \infty$. And then there exists a function $u_0(z) \in B$, such that $S(u_n - u_0) \to 0$ as $n \to \infty$. From this we can derive that $u_0(z)$ is a solution of Problem Q for (3.1) with $t = t_0$. Hence E is a closed point set in $0 \le t \le 1$.

From the above theorem, the next result can be derived.

Theorem 3.2 *Under the same conditions as in Theorem* 3.1, *the following statements hold.*

(1) *When the index $K > N$, Problem* P *for* (1.1) *has N solvability conditions, and the solution of Problem* P *depends on $2K - N + 2$ arbitrary real constants.*

(2) *When $0 \le K < N$, Problem* P *for* (1.1) *is solvable, if $2N - K$ solvability conditions are satisfied, and the solution of Problem* P *depends on $K + 2$ arbitrary real constants.*

(3) *When $K < 0$, Problem* P *for* (1.1) *is solvable under $2N - 2K - 1$ conditions, and the solution of Problem* P *depends on one arbitrary real constant.*

Moreover the solvability conditions of Problem P *can be explicitly stated.*

Proof Let the solution $[w(z), u(z)]$ of Problem Q for (1.6) be substituted into the boundary condition (1.7) and the relation (1.8). If the function $h(z) = 0,\ z \in \Gamma$, i.e.

$$\begin{cases} h_j = 0,\ j = 1, ..., N - K, \quad \text{if } 0 \le K < N, \\ h_j = 0,\ j = 0, 1, ..., N,\ h^\pm_m = 0,\ m = 1, ..., -K-1, \text{ if } K < 0, \end{cases}$$

and $d_j = 0,\ j = 1, ..., N$, then we have $w(z) = u_z$ in D and the function $w(z)$ is just a solution of Problem P for (1.1). Hence the total number of above equalities is just the number of solvability conditions as stated in this theorem. Also note that the real constants b_0 in (1.8) and $b_j (j \in J)$ in (1.9) are arbitrarily chosen. This shows that the general solution of Problem P for (1.1) depends on the number of arbitrary real constants as stated in the theorem.

4. Error estimates of approximate solutions for Poincaré problem

We provide the following error estimate of the approximate solutions of the boundary value problem.

Theorem 4.1 *Under the same conditions in Theorem* 3.1, *let $u(z)$ be a solution of Problem* Q *for* (3.1) *and $u^t_n = u_n(z, t)$ be its approximation as stated in the proof of Theorem* 3.1 *with $A(z) = (1 - t)F(z, 0, 0, 0)$. Then $u(z)$ $-u^t_n(z)$ processes the estimate*

$$\begin{aligned} S(u - u^t_n) &= C^1_\beta(u - u^t_n,\overline{D}] + L_{p_0,2}[|(u - u^t_n)_{\bar z z}| + |(u - u^t_n)_{zz}|,\overline{D}] \\ &\le \gamma k[\frac{1 - \gamma^n|t - t_0|^n}{1 - \gamma|t - t_0|}(1 - t) + (\gamma|t - t_0|)^n(1 - t_0)], \end{aligned} \tag{4.1}$$

where $\gamma = M_{10}(q_0+k_0)$, $k = M_{10}(k_1+k_2+k_3)$, $M_{10} = M_6+M_7$, M_6, M_7, q_0 and $k_j (j = 0, 1, 2, 3)$ are non-negative constants as stated in Sections 1 *and* 2.

Proof From (3.1) and (3.7) with $A(z)=(1-t)F(z,0,0,0)$, we obtain

$$\begin{aligned}&(u-u^t_{n+1})_{z\bar z}=f(z,u,u_n,u_{zz})-t_0f(z,u^t_{n+1},u^t_{n+1z},u^t_{n+1zz})\\&-(t-t_0)f(z,u_n,u_{nz},u_{nzz})=t_0[f(z,u,u_z,u_{zz})-\\&-f(z,u^t_{n+1},u^t_{n+1z},u^t_{n+1zz})]+(t-t_0)[f(z,u,u_z,u_{zz})\\&-f(z,u^t_n,u^t_{nz},u^t_{nzz})]+(1-t)f(z,u,u_z,u_{zz}),\end{aligned}\tag{4.2}$$

in which $f(z,u,u_z,u_{zz})=F(z,u,u_z,u_{zz})-F(z,0,0,0)$. Similarly to (3.16), we have

$$\begin{aligned}&f(z,u,u_z,u_{zz})-f(z,u^t_n,u^t_{nz},u^t_{nzz})=\\&=\mathrm{Re}[\tilde Q(u-u^t_n)_{zz}+\tilde A_1(u-u^t_n)_z]+\tilde A_2(u-u^t_n),\ |\tilde Q|\le q_0,\\&f(z,u,u_z,u_{zz})=\mathrm{Re}[\tilde Qu_{zz}+\tilde A_1u_z]+\tilde A_2u,\ L_{p_0,2}[\tilde A_j,\overline D]\le k_0,j=1,2,\end{aligned}$$

and

$$\begin{aligned}&L_{p_0,2}[(t-t_0)(f(z,u,u_z,u_{zz})-f(z,u^t_n,u^t_{nz},u^t_{nzz}))+(1-t)f(z,u,u_z,u_{zz}),\overline D]\\&\le|t-t_0|[q_0L_{p_0,2}((u-u^t_n)_{zz},\overline D)+k_0C^1(u-u^t_n,\overline D)]+(1-t)[q_0L_{p_0,2}(u_{zz},\overline D)\\&+k_0C^1(u,\overline D)]\le(q_0+k_0)[|t-t_0|S(u-u^t_n)+(1-t)S(u)].\end{aligned}$$

Noting that the function $u(z)-u^t_{n+1}(z)$ satisfies the homogeneous boundary conditions

$$\mathrm{Re}[\overline{\lambda(z)}(u-u^t_{n+1})_z]+c_1(z)(u-u^t_{n+1})=h(z),\ z\in\Gamma,\tag{4.3}$$

$$\mathrm{Im}[\overline{\lambda(z)}(u_z-u^t_{n+1z})]|_{z=a_j}=0,\ j\in J,\ u(1)-u_{n+1}(1)=0,\tag{4.4}$$

and using Theorem 2.2, we have

$$\begin{aligned}&S(u-u^t_{n+1})\le M_9(q_0+k_0)[|t-t_0|S(u-u^t_n)+(1-t)S(u)]\\&\le\gamma^{n+1}|t-t_0|^{n+1}S(u-u^t_0)+\gamma(1-t)S(u)(1+\gamma|t-t_0|+\\&+\gamma^2|t-t_0|^2+\cdots+\gamma^n|t-t_0|^n)\le\gamma^{n+1}|t-t_0|^{n+1}S(u-u^t_0)\\&+\gamma(1-t)S(u)(1-\gamma^{n+1}|t-t_0|^{n+1})/(1-\gamma|t-t_0|),\end{aligned}\tag{4.5}$$

where $\gamma=M_{10}(q_0+k_0)$ and $u^t_0=u(z,t_0)$ is the solution of Problem Q for (3.6) with $t=t_0$ and $A(z)=(1-t_0)F(z,0,0,0)$. Since $u(z)$ is a solution of Problem Q for (3.1), and $u-u^t_0$ is a solution of the following boundary value problem

$$\begin{aligned}&(u-u^t_0)_{z\bar z}-t_0[f(z,u,u_z,u_{zz})-\\&-f(z,u^t_0,u^t_{0z},u^t_{0zz})]=(1-t_0)f(z,u,u_z,u_{zz}),\end{aligned}\tag{4.6}$$

$$\mathrm{Re}[\overline{\lambda(z)}(u-u^t_{0z})]+c_1(z)(u-u^t_0)=h(z),\ z\in\Gamma,\tag{4.7}$$

$$\mathrm{Im}[\overline{\lambda(z)}(u_z-u^t_{0z})]|_{z=a_j}=0,j\in J,\ u(1)-u_0(1)=0,\tag{4.8}$$

it can be seen that

$$S(u)\le M_9(k_1+k_2+k_3)=k,\tag{4.9}$$

$$\begin{aligned}&S(u-u^t_0)\le M_9(1-t_0)L_{p_0,2}[f(z,u,u_z,u_{zz},\overline D]\\&\le M_9(q_0+k_0)(1-t_0)S(u)\le\gamma(1-t_0)k.\end{aligned}\tag{4.10}$$

Thus from (4.5), it follows that

$$\begin{aligned}&S(u-u^t_{n+1})\le\gamma^{n+1}|t-t_0|^{n+1}\gamma(1-t_0)k+\frac{\gamma(1-t)k(1-\gamma^{n+1}|t-t_0|^{n+1})}{1-\gamma|t-t_0|}\\&=\gamma k[\frac{1-\gamma^{n+1}|t-t_0|^{n+1}}{1-\gamma|t-t_0|}(1-t)+(\gamma|t-t_0|)^{n+1}(1-t_0)],\end{aligned}$$

Hence (4.1) is true. If the positive constant δ is small enough, then when $|t-t_0| \leq \delta$, $\gamma|t-t_0| < 1$, n is sufficiently large and t is close to 1, the right hand side in (4.1) becomes very small.

Note: The opinions expressed herein are those of the authors and do not necessarily represent those of the Uniformed Services University of the Health Sciences and the Department of Defense.

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