VALUATION ON A FILTERED MODULE

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Abstract

In this paper we show if R is a filtered ring and M a filtered R_module then we can define a valuation on a module for M. Then we show that we can find an skeleton of valuation on M, and we prove some properties such that derived form it for a filtered module.

Key Words: Filtered module, Filtered ring, Valuation on module, skeleton of valuation.

1 Introduction

In algebra valuation module and filtered R_module are two most important structures. We know that filtered R_module is the most important structure since filtered module is a base for graded module especially associated graded module and completion and some similar results([1],[2],[3],[7],[8]). So, as these important structures, the relation between these structure is useful for finding some new structures, and if M is a valuation module then M has many properties that have many usage for example, Rees valuations and asymptotic primes of rational powers in Noetherian rings and lattices([4],[10]).

In this article we investigate the relation between filtered R_module and valuation module. We prove that if we have filtered R_module then we can find a valuation R_module on it. For this we define $\nu : M \to \mathbb{Z}$ such that for every $t \in M$, and by lemma(3.1), lemma(3.2), lemma(3.3), lemma(3.4) and theorem(3.1) we show ν has all properties of valuation on R-module M. Also we show if M is a filtered R_module then it has a skeleton of valuation, continuously we prove some properties for M that derived from skeleton of valuation([6],[9]).

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2 Preliminaries

Definition 2.1. *A filtered ring R is a ring together with a family $\{R_n\}_{n\geq 0}$ of additive subgroups of R satisfying in the following conditions:*

i) $R_0 = R$;

ii) $R_{n+1} \subseteq R_n$ *for all* $n \geq 0$;

iii) $R_n R_m \subseteq R_{n+m}$ *for all* $n, m \geq 0$.

Definition 2.2. *Let R be a ring together with a family $\{R_n\}_{n\geq 0}$ of additive subgroups of R satisfying the following conditions:*

i) $R_0 = R$;

ii) $R_{n+1} \subseteq R_n$ *for all* $n \geq 0$;

iii) $R_n R_m = R_{n+m}$ *for all* $n, m \geq 0$,

Then we say R has a strong filtration.

Definition 2.3. *Let R be a filtered ring with filtration $\{R_n\}_{n\geq 0}$ and M be a R_module with family $\{M_n\}_{n\geq 0}$ of subgroups of M satisfying the following conditions:*

i) $M_0 = M$;

ii) $M_{n+1} \subseteq M_n$ *for all* $n \geq 0$;

iii) $R_n M_m \subseteq M_{n+m}$ *for all* $n, m \geq 0$,

Then M is called filtered R_module.

Definition 2.4. *Let R be a filtered ring with filtration $\{R_n\}_{n\geq 0}$ and M be a R_module together with a family $\{M_n\}_{n\geq 0}$ of subgroups of M satisfying the following conditions:*

i) $M_0 = R$;

ii) $M_{n+1} \subseteq M_n$ *for all* $n \geq 0$;

iii) $R_n M_m = M_{n+m}$ *for all* $n, m \geq 0$,

Then we say M has a strong filtration.

Definition 2.5. *Let M be an R_module where R is a ring, and Δ an ordered set with maximum element ∞ and $\Delta \neq \{\infty\}$. A mapping υ of M onto Δ is called a* valuation *on M, if the following conditions are satisfied:*

i) For any $x, y \in M$, $\upsilon(x + y) \geq min\{\upsilon(x), \upsilon(y)\}$;

ii) *If* $\upsilon(x) \leq \upsilon(y)$, $x, y \in M$, *then* $\upsilon(ax) \leq \upsilon(ay)$ *for all* $a \in R$;

iii) *Put* $\upsilon^{-1} := \{x \in M | \upsilon(x) = \infty\}$. *If* $\upsilon(az) \leq \upsilon(bz)$, *where* $a, b \in R$, *and* $z \in M \setminus \upsilon^{-1}(\infty)$, *then* $\upsilon(ax) \leq \upsilon(ay)$ *for all* $x \in M$

iv) *For every* $a \in R \setminus (\upsilon^{-1}(\infty) : M)$, *there is an* $a^{'} \in R$ *such that* $\upsilon((a^{'}a)x) = \upsilon(x)$ *for all* $x \in M$

Definition 2.6. *Let* M *be an* R*_module where* R *is a ring, and let* ν *be a valuation on* M. *A representation system of the equivalence relation* $\sim_\nu$ *is called a skeleton of* ν.

Definition 2.7. *A subset* S *of* M *is said to be* ν*-independent if* $S \cap \nu^{-1}(\infty) = \phi$, *and* $\nu(x) \notin \nu(Ry)$ *for any pair of distinct elements* $x, y \in S$. *Here, we adopt the convention that the empty subset* ϕ *is* ν*-independent.*

Proposition 2.1. *Let* M *be an* R*_module where* R *is a ring, and let* $\upsilon : M \to \Delta$ *be a valuation on* M. *Then the following statements are true:*

i) *If* $\nu(x) = \nu(y)$ *for* $x, y \in M$, *then* $\nu(ax) = \nu(ay)$ *for all* $a \in R$;

ii) $\nu(-x) = \nu(x)$ *for all* $x \in M$;

iii) *If* $\nu(x) \neq \nu(y)$, *then* $\nu(x + y) = min\{\nu(x), \nu(y)\}$;

iv) *If* $\nu(az) = \nu(bz)$ *for some* $a, b \in R$ *and* $z \in M \setminus \nu^{-1}(\infty)$,*then* $\nu(ax) = \nu(bx)$ *for all* $x \in M$;

v) *If* $\nu(az) < \nu(bz)$ *for some* $a, b \in R$ *and* $z \in M$, *then* $\nu(ax) < \nu(bx)$ *for all* $x \in M \setminus \nu^{-1}(\infty)$;

vi) *The core* ν^{-1} *of* ν *is prime submodule of* M;

vii) *The following subsets constitute a valuation pair of* R *with core* $(M : \nu^{-1}(\infty))$:

$$A_\nu = \{a \in A | \nu(ax) \geq \nu(x) \ for \ all \ x \in M\},$$

$$P_\nu = \{a \in A | \nu(ax) \geq \nu(x) \ for \ all \ x \in M \setminus \nu^{-1}(\infty)\}$$

Proof. see proposition 1.1 [6] □

Definition 2.8. *The pair* (A_ν, P_ν) *as in Proposition (2.1) is called the valuation pair of* R *induced by* ν *or the induced valuation pair of* ν.

3 Valuation derived from filtered module

In this section we use the four following lemmas for showing the existence of valuation on filtered module. Let R be a ring with unit and R a filtered ring with filtration $\{R_n\}_{n>0}$ and M be filtered R_module with filtration $\{M_n\}_{n>0}$.

Lemma 3.1. *Let M be filtered R_module with filtration $\{M_n\}_{n>0}$. Now we define $\nu : M \to \mathbb{Z}$ such that for every $t \in M$ and $\nu(t) = \min\{i\,|t \in M_i \backslash M_{h+1}\}$. Then for all $x, y \in M$ we have $\nu(x + y) \geq min\{\nu(x), \nu(u)\}$.*

Proof. For any $x, y \in M$ such that $\nu(x) = i$ also $\nu(y) = j$, and $\nu(x+y) = h$, so we have $x + y \in M_k \backslash M_{k+1}$. Without losing the generality, let $i < j$ so $M_j \subset M_i$ hence $y \in R_i$. Now if $k < i$, then $k + 1 \leq i$ and $M_i \subset M_{k+1}$ so $x + y \in M_i \subset M_{k+1}$ it is contradiction. Hence $k \geq i$ and so we have $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$.

□

Lemma 3.2. *Let M be filtered R_module with filtration $\{M_n\}_{n>0}$. Now we define ν as lemma(3.1). If $\upsilon(y) \leq \upsilon(x)$, $x, y \in M$, then $\upsilon(ax) \leq \upsilon(ay)$ for all $a \in R$;*

Proof. Let $\nu(x) = i$ and $\nu(y) = j$, since $\nu(x) \geq \nu(y)$ then $M_j \supseteq M_i$. Since R is filtered ring, there exists $k \in \mathbb{Z}$ such that $a \in R_k$ so

$$ax \in R_k M_i \subseteq M_{k+i}$$

$$ay \in R_k M_j \subseteq M_{k+j}$$

we have $i + k \geq j + k$ by $i \geq j$, then $\nu(ax) \geq \nu(ay)$ for all $a \in R$. □

Lemma 3.3. *Let M be filtered R_module with filtration $\{M_n\}_{n>0}$. Now we define ν as lemma(3.1). Put $\upsilon^{-1} := \{x \in M|\upsilon(x) = \infty\}$. If $\upsilon(az) \leq \upsilon(bz)$, where $a, b \in R$, and $z \in M \setminus \upsilon^{-1}(\infty)$, then $\upsilon(ax) \leq \upsilon(ay)$ for all $x \in M$.*

Proof. Since $a, b \in R$ and $z \in M$ then there exist $i, j, k \in \mathbb{Z}$ such that
$a \in R_i$, $b \in R_j$ and $z \in M_k$ hence

$$az \in R_i M_k \subseteq M_{i+k}$$

$$bz \in R_j M_k \subseteq M_{j+k}$$

Now if $\nu(az) \leq \nu(bz)$ then

$$k + i \leq k + j \Longrightarrow i \leq j \Longrightarrow R_j \subseteq R_i$$

So we have $\nu(ax) \leq \nu(bx)$ for all $x \in M$ □

Lemma 3.4. *Let M be filtered R_module with filtration $\{M_n\}_{n>0}$. Now we define ν as lemma(3.1). For every $a \in R \setminus (\upsilon^{-1}(\infty) : M)$, there is an $a^{'} \in R$ such that $\upsilon((a^{'}a)x) = \upsilon(x)$ for all $x \in M$.*

Proof. Let $x \in \nu^{-1}(\infty)$ then for all $a^{'}, a \in R$ $\upsilon((a^{'}a)x) = \upsilon(x) = \infty$.
Now let $x \notin \nu^{-1}(\infty)$ and for all $a^{'} \in R$ we have $\nu((a^{'}a)x) \neq \nu(x)$. So if $a^{'} \in R \setminus (\nu^{-1}(\infty) : M)$, then $a^{'}a \in R \setminus (\nu^{-1}(\infty) : M)$ and hence $\nu((a^{'}a)x) \neq \infty$.
Let $a \in R_k$, $a^{'} \in R_{k'}$ and $x \in M_i$, then $a^{'}a \in R_{k+k'}$ so $(a^{'}a)x \in M_{i+k+k'}$.
We may have one of following conditions:

1) $\nu((a^{'}a)x) < \nu(x)$.
2) $\nu(x) < \nu((a^{'}a)x)$

Now if we have (1) then $i + k + k^{'} < i$, it is contradiction .
Consequently $a^{'} \in R_{k'}$ and $a \in R_k$ for $k \in \mathbb{Z}$ then

$$a^{'}a \in R_{k'+k} \Longrightarrow (a^{'}a)x \in R_{k+k'}M_i \subseteq M_{i+k'+k}.$$

Since $M_{k'+k+i} \subseteq M_i$ hence $(a^{'}a)x \in M_i$. So we have $\nu((a^{'}a)x) < i$ therefore $\nu(x) > \nu((a^{'}a)x)$, it is contradiction with (2). By now we have
$\nu(x) = \nu((a^{'}a)x)$. □

Theorem 3.1. *Let R be a filtered ring with filtration $\{R_n\}_{n>0}$, and M be a filtered R_module with filtration $\{M_n\}_{n>0}$. Now we define $\nu : M \to \mathbb{Z}$ such that for every $t \in M$ and $\nu(t) = \min\{i \,|t \in M_i \backslash M_{h+1}\}$. Then ν is a valuation on M.*

Proof.
i) By lemma (3.1) we have For any $x, y \in M$, $\upsilon(x+y) \geq min\{\upsilon(x), \upsilon(y)\}$;
ii) We have If $\upsilon(x) \leq \upsilon(y)$, $x, y \in M$, then $\upsilon(ax) \leq \upsilon(ay)$ for all $a \in R$ by lemma(3.2);
iii) Put $\upsilon^{-1} := \{x \in M | \upsilon(x) = \infty\}$. If $\upsilon(az) \leq \upsilon(bz)$, where $a, b \in R$, and $z \in M \setminus \upsilon^{-1}(\infty)$, then then by lemma (3.3) $\upsilon(ax) \leq \upsilon(ay)$ for all $x \in M$;
iv) For every $a \in R \backslash (\upsilon^{-1}(\infty) : M)$, then by lemma(3.4) there is an $a^{'} \in R$ such that $\upsilon((a^{'}a)x) = \upsilon(x)$ for all $x \in M$.

So by definition(2.5) ν is a valuation on$M$ if has those conditions. □

Corollary 3.1. *If M be a filtered R_module, then $\nu : M \to \mathbb{Z}$ has all of properties that explained in Proposition(2.1).*

Proposition 3.1. *I R is a strongly filtered ring and M is a strongly filtered R_module and there exist valuation $\nu : M \to \mathbb{Z}$ on M, then R should be a trivial filtered R_module.*

Proof. By definition(2.5)(iv) and theorem(3.1) we have for every $a \in R \setminus (\upsilon^{-1}(\infty) : M)$, there is an $a^{'} \in R$ such that $\upsilon((a^{'}a)x) = \upsilon(x)$. Now if $\nu(a) = i$, $\nu(a\prime) = j$ and $\nu(x) = k$ then $i + j + k = k$ so $i + j = 0$, consequently $R_i = R$ for every $i > 0$. □

Proposition 3.2. *Let M be an R_module, where R is a ring. Then there is a valuation on M, if and only if there exists a prime ideal P of R such that $PM_P \neq M_P$, where M_P is the localization of M at P.*

Proof. see (Proposition 1.3 [6]) □

Corollary 3.2. *Let M be an filtered R_module, where R is a filtered ring. Then there exists a prime ideal P of R such that $PM_P \neq M_P$, where M_P is the localization of M at P.*

Proof. By theorem(3.1) there is an valuation on M, then by proposition(3.2) there exists a prime ideal P of R such that $PM_P \neq M_P$, where M_P is the localization of M at P. □

Corollary 3.3. *Let M be an filtered R_module, where R is a filtered ring. Then there is a skeleton on M.*

Proof. By theorem(3.1) there is a valuation on M,then by definition(2.6) we have there is a skeleton on M. □

Proposition 3.3. *Let M be an filtered R_module where R is a filtered ring, and ν a valuation on M. If Λ is a skeleton of ν, then the following conditions are satisfied:*

- *i) Λ is a ν-independent subset of M;*
- *ii) For every $x \in M\nu_{-1}(\infty)$, there exists a unique $\lambda \in \Lambda$ such that $\nu(x) = \nu(R\lambda)$.*

Proof. By corollary(3.3) Λ is a skeleton of ν and by proposition(1.4, [6]) we have the above conditions. □

Proposition 3.4. *Let M be an filtered R_module where R is a filtered ring, and ν a valuation on M. If Λ is a skeleton of ν. If $a_1\lambda_1 + \cdots + a_n\lambda_n = 0$ where $a_1, \cdots a_n \in R$ and $\lambda_1 \cdots \lambda_n \in \Lambda$ are mutually distinct, then $a_i \in (\nu_{-1}(\infty) : M)$,$i = 1, \cdots , n$.*

Proof. By corollary(3.3) Λ is a skeleton of ν and by proposition(1.5, [6]) we have If $a_1\lambda_1 + \cdots + a_n\lambda_n = 0$ where $a_1, \cdots a_n \in R$ and $\lambda_1 \cdots \lambda_n \in \Lambda$ are mutually distinct, then $a_i \in (\nu_{-1}(\infty) : M)$,$i = 1, \cdots , n$. □

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