How to Build Utility Functions for Becker's Economics-of-Crime Theory: Allingham-Sandmo's Example of Tax Evasion and Non-Compliance Extended

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Abstract: Allingham and Sandmo (J. Public Econ., 1972) analyse by the example of tax evasion and non-compliance the intended underreporting of taxpayers via concave and twice differentiable utility functions within Becker's economics-of-crime theory (J. Political Econ., 1968) on behavioural aspects of illicit activities. This work is concerned with how to build feasible utility functions applicable for experiments, theoretical investigations and / or numerical simulations of any kind of such illicit activities. It turns out that feasible utility functions form a set of Allingham-Sandmo-Functions applicable for Risk Averse and Neutral Taxpayers ($\mathbb{ASFRANT}$) which is a non-commutative semiring with left-annihilating zero and unity.

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1 Introduction

Hokamp et al. (2018) make use of five criteras to compare the two standard neoclassical expected utility models for tax evasion and non-compliance, i.e. Allingham and Sandmo (1972) and Srinivasan (1973) – who applied the economics-of-crime approach by Becker (1968, 1993): "(i) mathematical modeling, (ii) taxpayer's optimal choice, (iii) comparative statics, (iv) framework extensions, and (v) model critique" (p. 5 ibid.), e.g. the famous critique by Yitzhaki (1974) on ambiguous income and substitution effects. Hokamp and Cuervo Diaz (2018) show by means of computational agent-based modelling that in an Allingham-Sandmo setting the tax rate has positive whereas the fine rate has negative effects on the overall extent of tax evasion. Alm et. al (2020) enlight the ambitious effects of the audit rate.

Tax experiments, theoretical approaches to tax evasion and non-compliance as well as computerised numerical agent-based tax compliance modelling (sometimes in combination with public goods provision) make use of payment and/or utility functions often with origins in the Allingham-Sandmo approach (see Zelmer, 2003; Hokamp, 2013; Hokamp et al., 2018; Robbins and Kiser, 2018, and Alm and Malezieux, 2020, for meta-analyses and literature reviews). Note that Rizzi (2017) presents indices and profiles for tax evasion, but the author does not discuss utility functions. Hence, the leading open questions for this work are: (i) which properties should utility functions have to be in line with the Allingham-Sandmo approach for tax evasion and non-compliance and (ii) how to build such Allingham-Sandmo-Functions (ASFs). Note that such novel utility functions can then be used for experiments, theoretical investigations and / or numerical simulations of any kind of illicit activities in line with the economics-of-crime theory by Becker (1968, 1993).

Recognise that the set of Alligham-Sandmo-Functions is extended by two adjoint neutral elements to form the set of ASFs applicable for Risk Averse and Neutral Taxpayers ($\mathbb{ASFRANT}$), which has properties similar to the set of natural numbers. In fact, two binary operations, namely addition and composition, are defined, which are related to addition and multiplication operating on the set of natural numbers, respectively. Hence, this work provides a cookbook how to build novel utility functions via such binary operations and which are feasible for the Allingham-Sandmo approach and, therefore, for Becker's economics-of-crime theory. Moreover, individual behaviour under extreme conditions and large economic losses is modelled via such utility functions, in particular via intertemporal utility functions (e.g. see Hokamp and Pickhardt, 2010).

The work is organised as follows. The next section introduces the formalism of the tax evasion framework of Allingham and Sandmo (1972) based on the economics-of-crime theory by Becker (1968, 1993) together with a definition of Allingham-Sandmo-Functions (ASFs). Section 3 presents novel insights on the algebraic structure as well as the binary operations which are feasible within the set of ASFs applicable for Risk Neutral and Averse Taxpayers ($\mathbb{ASFRANT}$) to build novel utility functions, e.g for numerical computations. Section 4 provides some examples how to build feasible utility funcions for Becker's economics-of-crime-approach. The final section summarises, discusses the results and broadens the applicability of this work beyond tax evasion and non-compliance.

2 Allingham-Sandmo-Functions

Following the description given in Hokamp et al. (2018, pp. 5 – 6), Allingham and Sandmo (1972) examine – within the neoclassical economics-of-crime approach by Becker (1968, 1993) – individuals, i.e. taxpayers, who are faced by a decision-making problem on how much income reflected by the decision variable X of their true income T to be stated by filing tax returns for authorities given an audit probability p, a fine rate f and a tax rate t. In addition, taxpayers are assumed to show risk aversion behaviour, so that their marginal utility $\mathcal{U}'$ is strictly decreasing, i.e. their respective utility function $\mathcal{U}$ is concave. To solve the decision-making problem, taxpayers conduct an expected maximisation procedure with respect to their individual utility

$$\mathcal{EU}[X] = (1-p)\mathcal{U}[T - tX] + p\mathcal{U}[(1-f)T + (f-t)X] \tag{1}$$

which reveals the necessary condition for a maximum

$$(1-p)(-t)\mathcal{U}'[T - tX] + p(f-t)\mathcal{U}'[(1-f)T + (f-t)X] = 0 \tag{2}$$

Then, taxpayers are equipped with an incentive towards tax evasion if their marginal expected utility is positive for full tax evasion, i.e. $X = 0$, and negative for total compliance, i.e. $X = T$. Thus, the first derivative of their expected utility with respect to filing a tax return with declared income X is forced to show a sign change, and, in addition, the second derivative needs to be negative. The latter condition is satisfied since the concavity of utility functions has been assumed and the former condition leads to

$$\frac{\partial \mathcal{EU}[X]}{\partial X}|_{X=0} = (1-p)(-t)\mathcal{U}'[T] + p(f-t)\mathcal{U}'[(1-f)T] > 0 \tag{3}$$

and

$$\frac{\partial \mathcal{EU}[X]}{\partial X}|_{X=T} = (1-p)(-t)\mathcal{U}'[(1-t)T] + p(f-t)\mathcal{U}'[(1-t)]T < 0 \tag{4}$$

Realigning Eqs. (3) and (4) results in the condition to guarantee an interior solution for the income decision-making problem

$$t > pf > t(p + (1-p)\frac{\mathcal{U}'[T]}{\mathcal{U}'[(1-f)T]}) \tag{5}$$

When the tax rate t is changed, then a fixed fine rate f on undeclared income $T - X$ could lead to a conflict by two effects on the optimal income declared X^*, that is, income versus substitution effect. Yitzhaki (1974) figured out that modelling a fine on the evaded tax (instead of a fine on undeclared income) via a sanction rate $s = f/t > 1$ terminates such a conflict.

In the following Definitions 2.1 and 2.2 sum up the properties of utility functions feasible for the Allingham-Sandmo approach, but request slightly more than the theoretical model described in Eqs. (1) to (5) by Allingham and Sandmo (1972).

Definition 2.1 (Allingham-Sandmo-Functions, ASFs)**.** Utility functions are said to be Allingham-Sandmo-Functions (ASFs) on a set $\mathbb{S} \subseteq \mathbb{R}$ when they allow to be employed for the economics-of-crime approach by Becker (1968, 1993) to model tax evasion and non-compliance according to the approach by Allingham and Sandmo (1972). Further, ASFs depend on net income $N \in \mathbb{S}$ and, in addition, possibly on a vector of other variables summarised by $\bar{N}$. Hence, ASFs are said to have the following properties, whereby all other variables $\bar{N}$ than net income N are hold fixed:
(i) Utility functions are differentiable at each net income $a \in \mathbb{S}$, i.e.

$$\forall a \in \mathbb{S} : \mathcal{U} \text{ is differentiable } \Leftrightarrow \forall a \in \mathbb{S} : \mathcal{U}'[a, \bar{N}] = lim_{h \to 0} \frac{\mathcal{U}[a+h, \bar{N}] - \mathcal{U}[a, \bar{N}]}{h} \tag{6}$$

(ii) Taxpayers are risk averse, that is
a) strictly increasing utility, i.e.

$$\frac{\partial \mathcal{U}[N, \bar{N}]}{\partial N} > 0 \tag{7}$$

and b) strictly decreasing marginal utility, i.e.

$$\frac{\partial^2 \, \mathcal{U}[N, \bar{N}]}{\partial N^2} < 0 \tag{8}$$

In particular, Definition 2.2 reflects the notion that nothing, i.e. zero net income, should lead to an utility of zero.

Definition 2.2 (Allingham-Sandmo-Function with Fixpoint at Zero, $ASF_{F=0}$). Utility functions are said to be Allingham-Sandmo-Functions with fixpoint at zero ($ASF_{F=0}$) on a set $\mathbb{S} \subseteq \mathbb{R}$ when they fulfill Definition 2.1 for ASFs and have a fixpoint at zero net income, $N = 0$, that is,

$$\mathcal{U}[0, \bar{N}] = 0 \tag{9}$$

Table 1 provides a summary of the mathematical syntax used for the Allingham-Sandmo approach to tax evasion and non-compliance.

Mathematical Syntax	Meaning
f	Fine Rate
p	Audit Probability
s	Sanction Rate
t	Tax Rate
N	Net Income
$\bar{N}$	All Variables, except Net Income
T	True Income
X	Income Declaration
X^*	Optimal Income Declaration
$\mathcal{U}$	Utility
$\mathcal{U}'$	Marginal Utility
$\mathcal{EU}$	Expected Utility

Table 1: Mathematical Syntax for the Allingham-Sandmo Approach to Tax Evasion and Non-Compliance adjusted and adopted from Hokamp et al. (2018)

The next section sheds light on how to build novel utility functions for the Allingham-Sandmo approach to tax evasion and non-compliance.

3 Algebra: How to Build Utility Functions not only for Tax Evasion and Non-Compliance

Binary operations are the key to build novel utility functions not only for the Allingham-Sandmo approach to tax evasion and non-compliance but also in general for the economics-of-crime approach by Becker (1968, 1993). To put it differently, to define a set based on Allingham-Sandmo-Functions (ASFs) there is the need to find feasible binary operations and related neutral elements. First, which binary operations are able to combine two ASFs to get another ASF on a set $\mathbb{S} \subseteq \mathbb{R}$? Candidates for binary operations are addition $(+)$, subtraction $(-)$, multiplication $(\cdot)$, division $(:)$ and composition $(\circ)$.

Theorem 3.1 (Feasible Binary Operations)**.** Addition $(+)$ and composition $(\circ)$ are feasible binary operations on Allingham-Sandmo-Functions (ASFs) on a set $\mathbb{S} \subseteq \mathbb{R}$ according to Definition 2.1.

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be ASFs. Then, it has to be shown that $\mathcal{C} := \mathcal{A} + \mathcal{B}$ and $\mathcal{D} := \mathcal{A} \circ \mathcal{B}$ are ASFs. Therefore, Definition 2.1 has to be checked for $\mathcal{C}$ and $\mathcal{D}$:

(i) Utility functions are differentiable at each $a \in \mathbb{S}$, i.e.

$\forall a \in \mathbb{S} : \mathcal{A}$ and $\mathcal{B}$ are differentiable

$\Leftrightarrow \forall a \in \mathbb{S} : \mathcal{A}'[a, \bar{N}] = lim_{h \to 0} \frac{\mathcal{A}[a+h,\bar{N}] - \mathcal{A}[a,\bar{N}]}{h} \wedge \mathcal{B}'[a, \bar{N}] = lim_{h \to 0} \frac{\mathcal{B}[a+h,\bar{N}] - \mathcal{B}[a,\bar{N}]}{h}$

$\Leftrightarrow \forall a \in \mathbb{S} : (\mathcal{A} + \mathcal{B})'[a, \bar{N}] = \mathcal{A}'[a, \bar{N}] + \mathcal{B}'[a, \bar{N}] = lim_{h \to 0} \frac{(\mathcal{A}+\mathcal{B})[a+h,\bar{N}] - (\mathcal{A}+\mathcal{B})[a,\bar{N}]}{h}$

$\Leftrightarrow \forall a \in \mathbb{S} : \mathcal{A} + \mathcal{B} = \mathcal{C}$ is differentiable

$\forall a \in \mathbb{S} : \mathcal{A}$ and $\mathcal{B}$ are differentiable

$\Leftrightarrow \forall a \in \mathbb{S} : \mathcal{A}'[a] = lim_{h \to 0} \frac{\mathcal{A}[a+h] - \mathcal{A}[a]}{h} \wedge \mathcal{B}'[a, \bar{N}] = lim_{h \to 0} \frac{\mathcal{B}[a+h,\bar{N}] - \mathcal{B}[a,\bar{N}]}{h}$

$\Leftrightarrow \forall a \in \mathbb{S} : (\mathcal{A} \circ \mathcal{B})'[a, \bar{N}] = \mathcal{A}'[\mathcal{B}[a, \bar{N}]] \cdot \mathcal{B}'[a, \bar{N}] = lim_{h \to 0} \frac{\mathcal{A}[\mathcal{B}[a,\bar{N}]+h] - \mathcal{A}[\mathcal{B}[a,\bar{N}]]}{h} \cdot lim_{h \to 0} \frac{\mathcal{B}[a+h,\bar{N}] - \mathcal{B}[a,\bar{N}]}{h}$

$\Leftrightarrow \forall a \in \mathbb{S} : \mathcal{A} \circ \mathcal{B} = \mathcal{D}$ is differentiable

(ii) Taxpayers are risk averse, that is

a) strictly increasing utility, i.e.

$\frac{\partial \mathcal{A}[N,\bar{N}]}{\partial N} > 0 \wedge \frac{\partial \mathcal{B}[N,\bar{N}]}{\partial N} > 0 \Rightarrow \frac{\partial \mathcal{C}[N,\bar{N}]}{\partial N} = \frac{\partial (\mathcal{A}+\mathcal{B})[N,\bar{N}]}{\partial N} = \frac{\partial \mathcal{A}[N,\bar{N}]}{\partial N} + \frac{\partial \mathcal{B}[N,\bar{N}]}{\partial N} > 0$

$\frac{\partial \mathcal{A}[N]}{\partial N} > 0 \wedge \frac{\partial \mathcal{B}[N,\bar{N}]}{\partial N} > 0 \Rightarrow \frac{\partial \mathcal{D}[N,\bar{N}]}{\partial N} = \frac{\partial (\mathcal{A} \circ \mathcal{B})[N,\bar{N}]}{\partial N} = \frac{\partial \mathcal{A}[\mathcal{B}[N,\bar{N}]]}{\partial N} \cdot \frac{\partial \mathcal{B}[N,\bar{N}]}{\partial N} > 0$

and b) strictly decreasing marginal utility, i.e.

$\frac{\partial^2 \ \mathcal{A}[N,\bar{N}]}{\partial N^2} < 0 \wedge \frac{\partial^2 \ \mathcal{B}[N,\bar{N}]}{\partial N^2} < 0 \Rightarrow \frac{\partial^2 \ \mathcal{C}[N,\bar{N}]}{\partial N^2} = \frac{\partial^2 \ (\mathcal{A}+\mathcal{B})[N,\bar{N}]}{\partial N^2} = \frac{\partial^2 \ \mathcal{A}[N,\bar{N}]}{\partial N^2} + \frac{\partial^2 \ \mathcal{B}[N,\bar{N}]}{\partial N^2} < 0$

$\frac{\partial^2 \ \mathcal{A}[N]}{\partial N^2} < 0 \wedge \frac{\partial^2 \ \mathcal{B}[N,\bar{N}]}{\partial N^2} < 0 \Rightarrow \frac{\partial^2 \ \mathcal{D}[N,\bar{N}]}{\partial N^2} = \frac{\partial^2 \ (\mathcal{A} \circ \mathcal{B})[N,\bar{N}]}{\partial N^2} = \frac{\partial^2 \ \mathcal{A}[\mathcal{B}[N,\bar{N}]]}{\partial N^2} \cdot \frac{\partial \mathcal{B}[N,\bar{N}]}{\partial N} + \frac{\partial \mathcal{A}[\mathcal{B}[N,\bar{N}]]}{\partial N} \cdot \frac{\partial^2 \ \mathcal{B}[N,\bar{N}]}{\partial N^2} < 0$ □

Theorem 3.2 (Non-Feasible Operations)**.** Subtraction $(-)$, multiplication $(\cdot)$ and division $(:)$ are non-feasible operations on Allingham-Sandmo-Functions (ASFs) according to Definition 2.1.

Proof. It has to be shown by contradiction that Allingham-Sandmo-Functions linked by subtraction $(-)$, multiplication $(\cdot)$ and/or division $(:)$ do not generate necessarily another ASF:

(Subtraction) Assume $\mathcal{A}$ and $\mathcal{B}$ are ASFs with $\frac{\partial \mathcal{B}[N,\bar{N}]}{\partial N} > \frac{\partial \mathcal{A}[N,\bar{N}]}{\partial N} > 0$. Then $\mathcal{C} := \mathcal{A} - \mathcal{B}$ is no ASF, since in Definition 2.1 the condition (ii) a) of strictly increasing utility is violated according to Eq. (7) $\frac{\partial \mathcal{C}[N,\bar{N}]}{\partial N} = \frac{\partial (\mathcal{A}-\mathcal{B})[N,\bar{N}]}{\partial N} = \frac{\partial \mathcal{A}[N,\bar{N}]}{\partial N} - \frac{\partial \mathcal{B}[N,\bar{N}]}{\partial N} < 0$.

(Multiplication) Assume $\mathcal{A}$ is an ASF with $\mathcal{A}[0,\bar{N}]=0$. Then $\mathcal{A}\cdot\mathcal{A}$ is no ASF, because in Definition 2.1 the condition (ii) a) of strictly increasing utility is violated at zero according to Eq. (7) $\frac{\partial(\mathcal{A}\cdot\mathcal{A})[0,\bar{N}]}{\partial N}=2\cdot\mathcal{A}[0,\bar{N}]\cdot\frac{\partial\mathcal{A}[0,\bar{N}]}{\partial N}=0$.
(Division) Assume $\mathcal{A}$ is an ASF. Then $\mathcal{A}:\mathcal{A}\equiv 1$ is no ASF, because in Definition 2.1 the condition (ii) a) of strictly increasing utility is violated according to Eq. (7) $\frac{\partial(\mathcal{A}:\mathcal{A})[N,\bar{N}]}{\partial N}\equiv\frac{\partial 1}{\partial N}=0$. □

Second, how do neutral elements look like for the two feasible binary operations addition and composition? Possible candidates are the utility functions which reflect risk neutral taxpayers, i.e. the identity function $id[N,\bar{N}]=(N,\bar{N})$ as well as the constant function $\mathcal{O}[N,\bar{N}]\equiv 0$.

Theorem 3.3 (Neutral Elements)**.** For Allingham-Sandmo-Functions (ASFs) in line with Definition 2.1 the identity function $id[N,\bar{N}]=(N,\bar{N})$ with $(N,0)=N$ is the neutral element with respect to the binary operation composition ($\circ$) and the constant function $\mathcal{O}[N,\bar{N}]\equiv 0$ is the neutral element with respect to the binary operation addition (+).

Proof. Assume $\mathcal{A}$ is an ASF. Then $\mathcal{A}\circ id=id\circ\mathcal{A}=\mathcal{A}$ as well as $\mathcal{A}+\mathcal{O}=\mathcal{O}+\mathcal{A}=\mathcal{A}$ are ASFs. □

Theorems 3.1 to 3.3 also work for $ASF_{F=0}$, ASFs with fixpoint at zero, according to Definition 2.2. Note that these neutral elements for ASFs and $ASF_{F=0}$ take special roles like unity and zero, respectivily, for the set of natural numbers. Recognise that it depends on the definition whether zero is a natural number or not. Transferred to ASFs this means that Definition 2.1 could be changed to allow also for utility functions modelling risk neutral taxpayers. Nonetheless, the set of ASFs applicable for risk averse and neutral taxpayers can be defined as follows by adjoining the function constantly set to zero and the identity function.

Definition 3.1 (Set of Allingham-Sandmo-Functions applicable for Risk Averse and Neutral Taxpayers, $\mathbb{ASFRANT}$)**.** The set of Allingham-Sandmo-Functions applicable for risk averse and neutral taxpayers is defined as
$\mathbb{ASFRANT}:=\{\mathcal{U}\mid\mathcal{U}$ fulfills Definition 2.1 for Allingham-Sandmo-Functions$\}$
$\cup\ \{id\}\cup\{\mathcal{O}\}$.

Definition 3.2 (Set of Allingham-Sandmo-Functions applicable for Risk Averse and Neutral Taxpayers with Fixpoint at Zero, $\mathbb{ASFRANT}_{F=0}$)**.** The set of Allingham-Sandmo-Functions applicable for risk averse and neutral taxpayers with fixpont at zero is defined as $\mathbb{ASFRANT}_{F=0}:=\{\mathcal{U}\mid\mathcal{U}$ fulfills Definition 2.2 for Allingham-Sandmo-Functions$\}\cup\{id\}\cup\{\mathcal{O}\}$.

However, which algebraic structure have $\mathbb{ASFRANT}$ and $\mathbb{ASFRANT}_{F=0}$ together with addition ($+$) and composition ($\circ$)? Because of $\mathbb{ASFRANT}_{F=0} \subset \mathbb{ASFRANT}$, Theorems 3.4 to 3.7 elaborate on this question.

Theorem 3.4 (Structure of the Algebra ($\mathbb{ASFRANT}$, $+$, $\mathcal{O}$))**.** The algebra ($\mathbb{ASFRANT}$, $+$, $\mathcal{O}$) is a commutative monoid.

Proof. According to Theorem 3.1 $\mathbb{ASFRANT}$ is equipped with the binary operation $+$. Assume three arbitrary $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{ASFRANT}$. Then, the binary operation $+$ is associative, because of $(\mathcal{A}+\mathcal{B})+\mathcal{C} = \mathcal{A}+\mathcal{B}+\mathcal{C} = \mathcal{A}+(\mathcal{B}+\mathcal{C})$. Further, according to Theorem 3.3 there exists a neutral element for each $\mathcal{A} \in \mathbb{ASFRANT}$, that is $\mathcal{O}[N, \bar{N}] \equiv 0$. Finally, the binary operation $+$ is commutative, that is, $\mathcal{A} + \mathcal{B} = \mathcal{B} + \mathcal{A} \; \forall \; \mathcal{A}, \mathcal{B} \in \mathbb{ASFRANT}$. □

To give another example, the set of natural numbers including zero, denoted as $\mathbb{N}_{\geq 0}$, together with the binary operation addition forms the algebra ($\mathbb{N}_{\geq 0}$, $+$, 0), which is also a commutative monoid. Note that there exists no inverse element since for $n \in \mathbb{N}_{>0}$, then the inverse $-n \notin \mathbb{N}_{>0}$.

Theorem 3.5 (Structure of the Algebra ($\mathbb{ASFRANT}$, $\circ$, id))**.** The algebra ($\mathbb{ASFRANT}$, $\circ$, id) is a non-commutative monoid.

Proof. According to Theorem 3.1 $\mathbb{ASFRANT}$ is equipped with the binary operation $\circ$. Assume three arbitrary $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{ASFRANT}$. Then, the binary operation $\circ$ is associative, because of $(\mathcal{A} \circ \mathcal{B}) \circ \mathcal{C} = \mathcal{A}[\mathcal{B}[\mathcal{C}]] = \mathcal{A} \circ (\mathcal{B} \circ \mathcal{C})$. Further, according to Theorem 3.3 there exist the neutral element $id[N, \bar{N}] = (N, \bar{N})$, i.e. the identity function, for each $\mathcal{A} \in \mathbb{ASFRANT}$. Finally, the binary operation $\circ$ is non-commutative, which is shown by contradiction as follows: Let $\mathcal{A}[N, \rho] := 1 - e^{-\rho N} \in \mathbb{ASFRANT}$, then $\mathcal{B} := \mathcal{A} + id \in \mathbb{ASFRANT}$ because of Theorem 3.4. However, $\mathcal{A} \circ \mathcal{B} \neq \mathcal{B} \circ \mathcal{A}$ and, therefore, the algebra ($\mathbb{ASFRANT}$, $\circ$, id) is non-commutative. □

Theorem 3.6 (Structure of the Algebra ($\mathbb{ASFRANT}$, $\circ$, id, $+$, $\mathcal{O}$))**.** The algebra ($\mathbb{ASFRANT}$, $\circ$, id, $+$, $\mathcal{O}$) is a non-commutative semiring with unity and left-annihilating zero.

Proof. ($\mathbb{ASFRANT}$, $+$, $\mathcal{O}$) is a commutative monoid according to Theorem 3.4 and ($\mathbb{ASFRANT}$, $\circ$, id) is a non-commutative monoid according Theorem 3.5. Assume three arbitrary $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{ASFRANT}$. Compositon left and right distributes over addition, that is $\mathcal{A} \circ (\mathcal{B} + \mathcal{C}) = \mathcal{A}[\mathcal{B} + \mathcal{C}] = \mathcal{A} \circ \mathcal{B} + \mathcal{A} \circ \mathcal{C}$ and $(\mathcal{A}+\mathcal{B}) \circ \mathcal{C} = \mathcal{A}[\mathcal{C}] + \mathcal{B}[\mathcal{C}] = \mathcal{A} \circ \mathcal{C} + \mathcal{B} \circ \mathcal{C}$. Composition with $\mathcal{O}$ left-annihilates $\mathbb{ASFRANT}$, that is, $\mathcal{O} \circ \mathcal{A} = \mathcal{O} = 0 \; \forall \; \mathcal{A} \in \mathbb{ASFRANT}$. □

Theorem 3.7 (Structure of the Algebra $(\mathbb{ASFRANT}_{F=0}, \circ, id, +, \mathcal{O})$). The algebra $(\mathbb{ASFRANT}_{F=0}, \circ, id, +, \mathcal{O})$ is a non-commutative semiring with unity and annihilating zero.

Proof. Since $\mathbb{ASFRANT}_{F=0} \subset \mathbb{ASFRANT}$ it obviously follows that the algebra $(\mathbb{ASFRANT}_{F=0}, \circ, id, +, \mathcal{O})$ is a non-commutative semiring with unity. Composition with $\mathcal{O}$ annihilates $\mathbb{ASFRANT}_{F=0}$, that is, $\mathcal{O} \circ \mathcal{A} = \mathcal{O} = 0 = \mathcal{A}[0] = \mathcal{A} \circ \mathcal{O} \ \forall \ \mathcal{A} \in \mathbb{ASFRANT}_{F=0}$. □

4 Examples

To sum up, the rewards can now be raped. For instance, two examples for novel utility functions build in $\mathbb{ASFRANT}_{F=0}$ via the binary operations addition (+) and composition (∘) are $\mathcal{E}_1(N, \rho) = 1 - e^{-\rho N} + id(N) = 1 - e^{-\rho N} + N$ and $\mathcal{E}_2(N, \rho) = (1 - e^{-\rho N}) \circ (1 - e^{-\rho N}) = (1 - e^{-\rho(1-e^{-\rho N})})$, where N stands for net income and ρ for individual risk aversion. Because of $\mathcal{E}_1 \circ \mathcal{E}_2 \neq \mathcal{E}_2 \circ \mathcal{E}_1$ these functions provide an example that the moniod $(\mathbb{ASFRANT}_{F=0}, \circ, id)$ is not commutative.[1]

The next and final section summarises and broadens the applicability of the results beyond tax evasion and non-compliance.

5 Discussion and Conclusion

Utility functions are at the beating heart of many numerical computerised simulations, theoretical investigations and / or experiments dealing with tax evasion and non-compliance. This work has shed light on Allingham-Sandmo-Functions (ASFs) given by Definition 2.1 and on the set of ASFs applicable for Risk Averse and Neutral Taxpayers ($\mathbb{ASFRANT}$) in line with Definition 3.1. Based on these Definitions ASFs have been introduced with fixpoint at zero according to Definition 2.2 and the related set $\mathbb{ASFRANT}_{F=0}$ with fixpoint at zero referring to Definition 3.2. In particular, it was shown by Theorems 3.1 to 3.7 how to build novel utility functions feasible to computational numerical simulate and to investigate tax evasion and non-compliance as well as which algebraic structure prevails. To put it differently, to find novel ASFs the key is linking two ASFs by the binary operations addition (+) and / or composition (∘). The algebraic structure of $(\mathbb{ASFRANT}, \circ, id, +, \mathcal{O})$ turns out to be a non-commutative semiring with unity and left-annihilating zero. The results

[1]The set of all functions $\mathbb{F} : f(\mathbb{T}) \to \mathbb{T}$, on a set $\mathbb{T} \subset \mathbb{R}$ together with the binary operation composition ∘ provides another example for a non-commutative monoid with the identity function as neutral element.

might be transferred to ASFs with fixpoint at zero and ($\mathbb{ASFRANT}_{F=0}$, $\circ$, id, $+$, $\mathcal{O}$) is a non-commutative semiring with unity and annihilating zero.

However, results are not restricted to tax evasion and non-compliance because of the possibility to broaden it up. In particular, intertemporal utility functions allow to incorporate the deterrent effect of large economic losses. Each problem works which allow for investigation via Becker's economics-of-crime approach due to Becker (1968, 1993). For example Westmattelmann et al. (2014) and Westmattelmann et al. (2020) successfully transferred Hokamp and Pickhardt (2010) to examine via agent-based modelling the pecuniary incentives to dope or not to dope in professional sport competitions. Thus, the transfer of this work to other topics beyond tax evasion and non-compliance delineates a rich research agenda for the future.

A Appendix

FOR REVIEW: This Appendix provides a brief mathematical background with respect to algebra based on Droste et. al (2009) and Karpfinger and Meyberg (2021) introducing the Definitions A.1 to A.5 with respect to binary operations, commutativity (and non-commutativity), monoids, semirings and annihilators.

Definition A.1 (Binary Operation)**.** A binary operation $*$ on a set $\mathbb{A}$ is a function that relates two elements a and b from $\mathbb{A}$ to another element c of $\mathbb{A}$ denoted as $* : \mathbb{A} \times \mathbb{A}$, $(a, b) \mapsto a * b = c$.[2]

Definition A.2 (Commutative and Non-Commutative)**.** A binary operation $*$ on a set $\mathbb{A}$ is said to be commutative if $\forall\, a, b \in \mathbb{A} : a * b = b * a$. A binary operation $*$ on a set $\mathbb{A}$ is said to be non-commutative if $\exists\, a, b \in \mathbb{A} : a * b \neq b * a$.[3]

Definition A.3 (Monoid)**.** An algebra ($\mathbb{M}$, $*$) is said to be a monoid if
(i) the binary operation $*$ is associative, i.e. $\forall\, a, b, c, \in \mathbb{M} : (a*b)*c = a*(b*c)$, and
(ii) there exists a neutral element ne, i.e. $\exists\, ne \in \mathbb{M} : a * ne = ne * a = a\ \forall\, a \in \mathbb{M}$. [4]

[2]Examples for binary operations are addition (+) and composition ($\circ$), which has been shown in Theorem 3.1.

[3]An equivalent synonym for commutative is abelian (and for non-commutative non-abelian) in honour of the mathematician Nils Hendrik Abel (1802–1829). The properties commutative and/or non-commutative can be tranferred to groups and rings and, hence, to monoids and semirings.

[4]An algebra ($\mathbb{M}$, $*$) is said to be a semigroup if the binary operation $*$ is associative. Therefore, a monoid is a semigroup with a neutral element.

Definition A.4 (Semiring)**.** An algebra $(\mathbb{M}, *, ne_*, \times, ne_\times)$ is said to be a semiring if
(i) $(\mathbb{M}, *, ne_*)$ is a commutative monoid,
(ii) $(\mathbb{M}, \times, ne_\times)$ is a non-commutative monoid, and
(iii) the binary operation $\times$ distributes over the binary operation $*$, i.e. $\forall\, a, b, c \in \mathbb{M} : a \times (b * c) = a \times b * a \times c$.[5]

Definition A.5 (Annihilator)**.** Within an algebra $(\mathbb{M}, *, ne_*, \times, ne_\times)$ a neutral element ne_* is said to be an annihilator if $\forall\ a \in \mathbb{M} : a \times ne_* = ne_* \times a = ne_*$.[6]

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[5] For example, the algebra $(\mathbb{N}_0, +, 0, \cdot, 1)$ is a commutative semiring (i.e. both monoids are commutative), where multiplication $\cdot$ distributes over addition $+$.

[6] Note that annihilator is a more broader term than annihilating zero, since for the later the binary operation is interpreted as addition.

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