ON TRIDIAGONAL CONJUGATE SECONDARY NORMAL MATRICES

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Abstract:

Con-s-normal matrices play the same role in the theory of s-unitary congruences as conventional s-normal matrices do with respect to s-unitary similarities. Naturally, the properties of both matrix classes are fairly similar up to the distinction between the congruence and similarity. However, in certain respects, con-s-normal matrices differ substantially from s-normal ones. Our goal in this paper is to indicate one of such distinctions. It is shown that none of the familiar characterizations of s-normal matrices having the irreducible tridiagonal form has a natural counterpart in the case of con-s-normal matrices.

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1. INTRODUCTION

Let C_{nxn} be the space of nxn complex matrices of order n. For $A \in C_{n\times n}$, let A^T, $\overline{A}$, A^*, A^S, $A^\theta \left(= \overline{A}^s \right)$ and A^{-1} denote the transpose, conjugate, conjugate transpose, secondary transpose, conjugate secondary transpose and inverse of matrix A respectively. The conjugate secondary transpose of A satisfies the following properties such as $\left(A^\theta\right)^\theta = A$, $\left(A+B\right)^\theta = A^\theta + B^\theta$, $\left(AB\right)^\theta = B^\theta A^\theta$. etc

Definition 1

A matrix $A \in C_{n\times n}$ is said to be normal if $AA^* = A^*A.$

Definition 2

A Matrix $A \in C_{n\times n}$ is said to be conjugate normal (con-normal) if $AA^* = \overline{A^*A}.$

Definition 3

A matrix $A \in C_{n\times n}$ is said to be secondary normal (s-normal) if $AA^\theta = A^\theta A.$

Definition 4

A matrix $A \in C_{n\times n}$ is said to be unitary if $AA^{*} = A^{*}A = I$.

Definition 5

A matrix $A \in C_{n\times n}$ is said to be *s*-unitary if $AA^{\theta} = A^{\theta}A = I$.

Definition 6 [5]

A matrix $A \in C_{n\times n}$ is said to be a conjugate secondary normal matrix (con-*s*-normal) if $AA^{\theta} = \overline{A^{\theta}A}$ where $A^{\theta} = \overline{A}^{S}$. **. . . (1)**

This matrix class plays the same role in the theory of s-unitary congruences as conventional normal matrices do with respect to s-unitary similarities. Accordingly, the properties of both matrix classes are fairly similar up to the distinction between congruence and similarity.

However, in certain respects, con-s-normal matrices substantially differ from s-normal ones. Our goal in this paper is to indicate one of such distinctions that concerns matrices having a tridiagonal form.

A tridiagonal matrix

$$A = \begin{pmatrix} \alpha_1 & \beta_2 & & & & \\ \gamma_2 & \alpha_2 & \beta_3 & & & \\ & \gamma_3 & \alpha_3 & \cdots & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & \alpha_{n-1} & \beta_n \\ & & & & \gamma_n & \alpha_n \end{pmatrix} \quad \textbf{. . . (2)}$$

is said to be irreducible if

$$\beta_2\beta_3 \ldots \beta_n \neq 0 \quad \textbf{. . . (3)}$$

and

$$\gamma_2\gamma_3 \ldots \gamma_n \neq 0. \quad \textbf{. . . (4)}$$

For a s-normal *A*, inequalities **(3)** and **(4)** are implications of each other; therefore, irreducibility can be characterized by any one of these inequalities.

There exist several descriptions of s-normal matrices having the irreducible tridiagonal form. One of these descriptions is based on a well-known characteristic property of s-normal matrices; namely, a matrix $A \in M_n(C)$ is s-normal if and only if its s-Hermitian adjoint A^{θ} is a polynomial in *A*. Moreover, in the representation $A^{\theta} = f(A)$, **. . . (5)**
one can choose *f* to be a polynomial with a degree less than *n*.

Proposition 1

Irreducible matrix **(2)** is s-normal if and only if A^{θ} is a linear polynomial in A.

The following description is an easy corollary of **Proposition 1**.

Proposition 2

Irreducible matrix **(2)** is s-normal if and only if

$$A = e^{i\phi} H + \alpha I_n, \qquad \textbf{. . . (6)}$$

Where $\phi \in \mathbb{R}$, $\alpha \in \mathbb{C}$, and H is a s-Hermitian matrix. In particular, if A is real, then A is either s-symmetric or has the for

$$A = K + \alpha I_n, \qquad \textbf{. . . (7)}$$

Where $\alpha \in \mathbb{R}$ and K is a s-skew symmetric matrix.

One more description can be derived from representation **(6)**.

Proposition 3

Irreducible matrix **(2)** is s-normal if and only if its secondary spectrum belongs to a line.

2. TRIDIAGONAL CON-s-NORMAL MATRICES

Here after, matrix **(2)** is assumed to be irreducible. Moreover, without loss of generality, we can assume $\beta_2, \ldots, \beta_n$ to be real positive scalars. Indeed, performing for matrix **(2)** the congruence transformation

$$A \to \tilde{A} = DAD$$

with the diagonal s-unitary matrix

$$D = diag\{1, d_2, \ldots, d_n\}, \qquad d_j = e^{i\delta_j}, \qquad j = 2, 3, \ldots, n,$$

we have

$$\tilde{a}_{12} = \beta_2 d_2, \qquad \tilde{a}_{j,j+1} = \beta_{j+1} d_j d_{j+1}, \qquad j = 2, 3, \ldots, n-1.$$

Setting

$$\delta_2 = -arg\,\beta_2, \qquad \delta_{j+1} = -arg\,\beta_{j+1} - \delta_j, \qquad j = 2, 3, \ldots, n-1,$$

we obtain a matrix $\tilde{A}$ with positive entries in positions (1, 2), (2, 3), …, (n– 1,n).

Denote by A_{n-1} the leading principal submatrix that is obtained by deleting the last row and the last column in A.

Lemma 1

A_{n-1} is a con-s-normal matrix.

Proof

Equating the last diagonal entries of the two matrices in relation **(1)**, we see that

$$|\gamma_n| = \beta_n. \qquad \textbf{. . . (8)}$$

Equating the leading principal submatrices of order $n-1$ in **(1)**, we have

$$A_{n-1}A_{n-1}^{\theta} + \beta_n^2 e_{n-1}e_{n-1}^{s} = \overline{A_{n-1}^{\theta}A_{n-1}} + |\gamma_n|^2 e_{n-1}e_{n-1}^{s} \qquad \textbf{. . . (9)}$$

Here, e_{n-1} is the last coordinate column vector in the space C^{n-1}. Equalities **(8)** and **(9)** prove the lemma.

Corollary 1

All the leading principal submatrices of a con-s-normal matrix A of form **(2)** are also con-s-normal.

Remark

A similar assertion is valid for trailing submatrices, that is, for submatrices counted off the right lower corner of A. Moreover, any principal submatrix lying at the intersection of successive rows and columns of matrix **(2)** is con-s-normal.

Now, we equate in **(1)** the entries in the positions $(n-2, n)$ and $(n-1, n)$, which gives

$$\beta_{n-1}\overline{\gamma}_n = \beta_n\gamma_{n-1}, \qquad \textbf{. . . (10)}$$

$$\alpha_{n-1}\overline{\gamma}_n + \overline{\alpha}_n\beta_n = \alpha_{n-1}\beta_n + \overline{\alpha}_n\gamma_n$$

Or $$\alpha_{n-1}(\overline{\gamma}_n - \beta_n) = \overline{\alpha}_n(\gamma_n - \beta_n). \qquad \textbf{. . . (11)}$$

Using **Lemma 1** and its corollary recursively, we obtain the relations

$$\beta_{j-1}\overline{\gamma}_j = \beta_j\gamma_{j-1}, \qquad j = 3,4,\ldots,n, \qquad \textbf{. . . (12)}$$

$$\alpha_{j-1}(\overline{\gamma}_j - \beta_j) = \overline{\alpha}_j(\gamma_j - \beta_j), \qquad j = 2,3,\ldots,n. \qquad \textbf{. . . (13)}$$

According to **(12)**, a choice of $\beta_2,\ldots,\beta_n$ and γ_n uniquely determines $\gamma_2,\ldots,\gamma_{n-1}$. Note that γ_n must obey condition **(8)**.

If $\gamma_n = \beta_n$, then **(12)** implies the equalities

$$\gamma_j = \beta_j, \qquad j = 2,3,\ldots,n-1.$$

In this case, A is a s-symmetric matrix and relations **(13)** impose no limitations on its diagonal entries $\alpha_1,\ldots,\alpha_n$.

If $\gamma_n = -\beta_n$, then the equalities

$$\gamma_j = -\beta_j, \qquad j = 2,3,\ldots,n-1, \qquad \ldots \textbf{(14)}$$

are derived from **(12)** and the equalities

$$\alpha_{j-1} = \bar{\alpha}_j, \qquad j = 2,3,\ldots,n, \qquad \ldots \textbf{(15)}$$

are derived from **(13)**.

Thus, the choice of α_n determines the entire diagonal of A.

Finally, assume that

$$\gamma_n = \beta_n e^{i\phi}, \qquad \phi \in (-\pi,\pi), \quad \phi \neq 0. \qquad \ldots \textbf{(16)}$$

Relations **(12)** yield

$$\gamma_{n-1} = \beta_{n-1}e^{-i\phi}, \qquad \gamma_{n-2} = \beta_{n-2}e^{i\phi}, \qquad \gamma_{n-3} = \beta_{n-3}e^{-i\phi}\ldots \qquad \ldots \textbf{(17)}$$

Similarly to the preceding case, all the diagonal entries have the same modulus. Define ψ by the formula

$$\psi = arg\left(e^{i\phi} - 1\right). \qquad \ldots \textbf{(18)}$$

Choosing α_n, we find from **(13)** that

$$\alpha_{n-1} = \bar{\alpha}_n e^{i2\psi}, \quad \alpha_{n-2} = \bar{\alpha}_{n-1}e^{-i2\psi} = \alpha_n e^{-i4\psi}, \quad \alpha_{n-3} = \bar{\alpha}_n e^{i6\psi},\ldots \qquad \ldots \textbf{(19)}$$

For instance, if $n = 3$ and $\varphi = \pi/2$, we have

$$\psi = arg\left(i-1\right) = \frac{3}{4}\pi$$

And $\qquad \gamma_2 = -\beta_2 i, \qquad \alpha_2 = -i\bar{\alpha}_3, \qquad \alpha_1 = -\alpha_3.$

In the case described by relations **(16)–(19)**, conjugate-s-normal matrices of form **(2)** cannot be reduced to s-symmetric or s-skew symmetric matrices.

3. ON THE MULTIPLICITY OF CON-s-EIGEN VALUES

If irreducible matrix **(2)** is s-normal, then all of its s-eigen values are simple, which follows from the relation $rank\left(A - zI_n\right) \geq n-1 \quad \forall z \in \mathbb{C}$.

This consideration is inapplicable to con-s-normal matrices. For instance, the Jordan block $J_n(0)$ with zero on the main diagonal has the rank n–1 and, at the same time, an s-eigen value of multiplicity n.

In general, con-s-normal matrices are not s-normal. Moreover, the con-s-eigen values rather than s-eigen values are invariants of s-unitary congruences. We recall their definition as given in **[2]**.

With a matrix $A \in M_n(C)$, we associate the matrices

$$A_L = \overline{A}A \quad \textbf{. . . (20)}$$

And $$A_R = A\overline{A}. \quad \textbf{. . . (21)}$$

Although, in general, the products AB and BA need not be similar, $\overline{A}A$ is always similar to $A\overline{A}$ (see **[3, Section 4.6]**). Therefore, in the subsequent discussion of the secondary spectral properties of these matrices, it will be suffices to consider only one of them, say, A_L.

The secondary spectrum of A_L has two remarkable properties:

(a) It is s-symmetric about the real axis. Moreover, the s-eigen values λ and $\overline{\lambda}$ have the same multiplicity.

(b) The negative s-eigen values of A_L (if any) are necessarily of even algebraic multiplicity.

Let, $$\lambda_S(A_L) = \{\lambda_1, ..., \lambda_n\} \quad \textbf{. . . (22)}$$

be the secondary spectrum of A_L.

Definition 7

The con-s-eigen values of A are the n scalars $\mu_1,\ldots,\mu_n$ introduced as follows:

(a) If $\lambda_i \in \lambda_s(A_L)$ does not lie on the negative real semiaxis, then the corresponding con-s-eigen value μ_i is defined as the square root of λ_i with a nonnegative real part:

$$\mu_i = \lambda_i^{1/2}, \qquad Re\,\mu_i \geq 0. \quad \textbf{. . . (23)}$$

The multiplicity of μ_i is set equal to that of λ_i.

(b) With a real negative s-eigenvalue $\lambda_i \in \lambda_s(A_L)$, we associate two conjugate purely imaginary con-s-eigen values

$$\mu_i = \pm \lambda_i^{1/2}. \quad \textbf{. . . (24)}$$

The multiplicity of each con-s-eigen value is set equal to half the multiplicity of λ_i.

The set

$$C\lambda_S(A) = \{\mu_1, ..., \mu_n\} \quad \textbf{. . . (25)}$$

is called the conjugate secondary spectrum of A.

For a s-symmetric A, we have $\overline{A} = A^{\theta}, A_L = A^{\theta} A;$ thus, the con-s-eigen values of A are identical to its s-singular values.

If A is s-skew symmetric, then

$$\overline{A} = -A^{\theta}, \qquad A_L = -A^{\theta} A.$$

As noted above, every negative s-eigen value λ of A_L has an even multiplicity. It gives rise to two purely imaginary con-s-eigen values $\mu = \pm i\sqrt{|\lambda|}$ of half the multiplicity.

The most important property of s-normal matrices is that every matrix of this class can be transformed into a secondary diagonal matrix by a proper s-unitary similarity transformation. The secondary diagonal entries of the transformed matrix are the s-eigen values of A. This spectral theorem for normal matrices has the following counterpart in the theory of unitary congruences **[6, 7]**.

Theorem 1

Every con-s-normal matrix $A \in M_n(C)$ can be brought by a proper s-unitary congruence transformation to a block diagonal form with diagonal blocks of orders 1 and 2. The 1-by-1 blocks are nonnegative con-s-eigen values of A. Each 2-by-2 block corresponds to a pair of complex conjugate con-s-eigen values $\mu_j = \rho_j e^{i\theta_j}$, $\overline{\mu}_j$ and has the form

$$\begin{bmatrix} 0 & \rho_j \\ \rho_j e^{-i2\theta_j} & 0 \end{bmatrix} \qquad \textbf{. . . (26)}$$

Or

$$\begin{bmatrix} 0 & \mu_j \\ \overline{\mu}_j & 0 \end{bmatrix}. \qquad \textbf{. . . (27)}$$

Every matrix $A \in M_n(C)$ can be represented in the form

$$A = S + K, \qquad \textbf{. . . (28)}$$

where

$$S = \frac{1}{2}\left(A + A^S\right), \qquad K = \frac{1}{2}\left(A - A^S\right). \qquad \textbf{. . . (29)}$$

Matrices **(29)** are called the real and imaginary parts of A, respectively.

For a conjugate-s-normal matrix A, decomposition **(28)**, **(29)** has a number of special properties. We need the property stated in the following proposition.

Theorem 2

Let A be a conjugate-s-normal matrix with decomposition **(28), (29)**. Then, the con-s-eigen values of S (respectively, K) are the real (respectively, imaginary) parts of the con-s-eigen values of A.

Corollary 2

If a conjugate-s-normal matrix A has a pair of complex con-s-eigen values $\mu = \sigma + iK,\ \bar{\mu},$ then σ is a multiple con-s-eigenvalue of $S = (A + A^S)/2$. The number of real con-s-eigen values of A is equal to the multiplicity of zero as a con-s-eigen value of $K = (A - A^S)/2$.

We return to conjugate-s-normal matrices of form **(2)** that satisfy relations **(16)–(19)**. In this case, it is easy to see that matrices **(29)** are tridiagonal along with A.

Lemma 2

The multiplicity of each con-s-eigen value of S (respectively, K) is at most two.

Proof

For definiteness, we consider S. The con-s-eigen values of this matrix are nonnegative scalars whose squares are the conventional s-eigen values of the five-diagonal matrix

$$S_L = \overline{S}S = S^{\theta}S.$$

The irreducibility of S ensures that all the entries of S_L lying on the diagonal $i - j = 2$ are nonzero. It follows that

$$rank\left(S_L - xI_n\right) \geq n-2 \qquad \forall x \in \mathbb{R}.$$

Therefore, the multiplicity of each s-eigen value of the Hermitian matrix S_L is at most two.

Corollary 3

A con-s-normal matrix A described by relations **(16)–(19)** has at most two real con-s-eigen values. All the pairs of conjugate con-s-eigen values of A have distinct real parts. The corresponding con-s-eigen values of $S = (A + A^S)/2$ are double. By contrast, all the nonzero con-s-eigen values of $K = (A - A^S)/2$ are simple.

These assertions are direct implications of **Lemma 2**, **Theorem 2**, and **Corollary 2**.

Corollary 3 makes obvious the following ultimate conclusion: the con-s-spectrum of a con-s-normal matrix described by relations **(16)–(19)** cannot be located on a line in the complex plane.

We conclude this section by a small illustration of the facts given above. It is easy to verify that the tridiagonal matrix

$$A = \begin{pmatrix} 1 & 1+i & 0 \\ 1-i & 1 & -1+i \\ 0 & -1-i & 1 \end{pmatrix}$$

is conjugate-s-normal.

Its s-symmetric part

$$S = \begin{pmatrix} 1 & i & 0 \\ -i & 1 & i \\ 0 & -i & 1 \end{pmatrix}$$

has the simple con-s-eigen value 1 and the double con-s-eigen value is also 1. The nonzero con-s-eigen values of the s-skew symmetric part

$$K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

are equal to $\pm i\sqrt{2}$. Thus,

$$C\lambda_S(A) = \left\{1, 1+i\sqrt{2}, 1-i\sqrt{2}\right\}.$$

4. ON THE REPRESENTATIONS OF THE TRANSPOSED MATRIX

Returning to representation **(5)**, we recall how **Proposition 1** can be proved. Assume that the degree k of the polynomial f in **(5)** is greater than one. Then, it is easy to see that the entries of A lying on the diagonals $i - j = k$ and $j - i = k$ must be nonzero. This, however, contradicts the fact that $f(A)$ must be the tridiagonal matrix A^{θ}.

The following assertion proved in **[4]** can be considered an analogue of representation **(5)** for con-s-normal matrices.

Theorem 3

A matrix $A \in M_n(C)$ is con-s-normal if and only if

$$A^S = f(A_R)A = Af(A_L) \qquad \textbf{. . . (30)}$$

for a polynomial f with real coefficients. This polynomial can be chosen so that its degree is less than n.

Suppose that $A \neq 0$ and the polynomial f in **(30)** has a zero degree; that is,

$$A^S = \alpha A.$$

A comparison of the norms of the left- and right-hand sides reveals that |α| = 1. Furthermore, it is easy to verify that the equality

$$A^S = e^{i\phi} A$$

is possible only for $\varphi = \pi k$, $k \in \mathbb{Z}$. Thus, in this case, A is either s-symmetric or s-skew symmetric.

Now, we show that, for a con-s-normal matrix described by relations **(16)–(19)**, the degree k of f in representation **(30)** must be at least $[n/2]$. Indeed, assuming the contrary, that is,

$$0 < k \leq \left[\frac{n}{2}\right] - 1 \leq \frac{n-2}{2},$$

we observe that $A\,(A_L)^k$ is the only monomial in the matrix $A\,f(A_L)$ that has nonzero entries on the diagonal $i - j = 1 + 2k$ (which does not exceed $n - 1$) and on the diagonal $j - i = 1 + 2k$. The same diagonals must be nonzero in $A\,f(A_L)$. This contradicts the fact that $A\,f(A_L)$ must be the tridiagonal matrix A^S.

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