Topologically Beta-Type Transitive Maps

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Abstract

In this paper, we define and introduce a new type of topological transitive map called topological β –Type transitive map and investigate some of its properties. Further, we introduce the notions of β- minimal mapping. We have proved that every topological β-type transitive map is a topological transitive map as every open set is β-open set but the converse not necessarily true, and that every β-type minimal map is a minimal map as every open set is β-open set, but the converse not necessarily true.

Keywords: Topological β-type transitive, β- minimal maps, β- continuous, β- dense β-open sets.

I. Introduction

Semi-open sets, preopen sets, α-sets, and β-open sets play an important role in the researches of generalizations of continuity in topological spaces. By using these sets we introduced and studied various modifications of transitivity, such as alpha-transitivity and beta-transitivity. The study of semi open sets and semi continuity in topological spaces was initiated by Levine [5]. Bhattacharya and Lahiri [7] introduced the concept of semi generalized closed sets in topological spaces analogous to generalized closed sets which was introduced by Levine [4]. Throughout this paper, the word "space " will mean topological space on which no separation axioms are assumed, unless otherwise mentioned. The collections of semi-open, semi-closed sets and α–sets in (X,τ) will be denoted by $SO(X,\tau)$, $SC(X,\tau)$ and τ^{α} respectively. Ogata N. [6] has shown that τ^{α} is a topology on X with the following properties: $\tau \subseteq \tau^{\alpha}$, $(\tau^{\alpha})^{\alpha} = \tau^{\alpha}$ and $S \in \tau^{\alpha}$ if and only if $S = U \setminus N$ where $U \in \tau$ and N is nowhere dense $(i.e.\ Int(Cl(N)) = \varphi)$ in (X,τ) . Hence $\tau = \tau^{\alpha}$ if and only if every nowhere dense (nwd) set in (X,τ) is closed. Hence every transitive map implies α-transitive. Also if $LC(X,\tau) = LC(X,\tau^{\alpha})$ then every transitive map implies α-transitive; and this structure also occurs if $SO(X,\tau) \subseteq LC(X,\tau)$ .Clearly every α–set is semi-open and every nwd set in (X,τ) is semi-closed. Andrijevic [1] has observed that $SO(X,\tau^{\alpha}) = SO(X,\tau)$, and that $N \subseteq X$ is nwd in (X,τ^{α}) if and only if N is nwd in (X,τ). A subset B of a space X is said to be regular open [14] (resp. regular closed [14]) if B = Int (Cl(B)) (resp. B = Cl(Int(B))). . A subset S of a space X is said to be regular open [14] (resp. regular closed [14]) if S = Int(Cl(S)) (resp. S = Cl(Int(S))). A point x of X is called a θ-cluster [15] point of A if Cl(U) ∩ A ≠ ∅ for every open set U of X containing x. The set of all θ-cluster points of A is called the θ-closure [15] of A and is denoted by $Cl_{\theta}(A)$. A set A is said to be θ-closed if A = $Cl_{\theta}(A)$. The complement of a θ-closed set is said to be θ-open [11] .It is known that a subset U of a space X is θ-open if and only if for any x ∈ U, there exists an open set V in X such that x ∈ V ⊂ Cl(V) ⊂ U. A subset S of a space X is said to be semi-open [10] (resp. preopen [12], α-open [13], semi-preopen [9] or β-open [1A]) if

$S \subset Cl(Int(S))$ (resp. $S \subset Int(Cl(S))$, $S \subset Int(Int(S)))$, $S \subset Cl(Int(Cl(S)))$). The family of all semi-open (resp. pre-open, α-open, β-open) subsets of X is denoted by SO(X)(resp. PO(X), τ^{α} (or αO(X)), βO(X)) The complement of a semi-open (resp. pre-open, α-open, β-open) set is said to be semi-closed (resp. pre-closed, α-closed, β-closed). If S is a subset of a space X, then the β-closure of S, denoted by βCl(S), is the smallest β-closed set containing S. The semi-closure (resp. pre-closure, α-closure, β-closure) of S is similarly defined and is denoted by $Cl_s(S)$ (resp. $Cl_p(S)$, Cl_α (S), Cl_β (S)). The β-interior of S, denoted by Int_β (S), is the largest β-open set contained in S. A subset S is said to be β-regular if it is β-open and β-closed. The family of all β-closed (resp. β-regular) subsets of X is denoted by βC(X) (resp. βR(X)) and the family of all β-open (resp. β-regular) subsets of X containing a point x ∈ X is denoted by βO(X; x) (resp. βR(X; x)).

In this paper, we will define a new class of topological transitive maps called topological β - transitive and a new class of minimal maps called β - minimal maps. We will also study some of their properties.

II. Preliminaries and Definitions

In this section, we recall some of the basic definitions. Let X be a topological space and $A \subset X$. The interior (resp. closure) of A is denoted by Int(A) (resp. Cl(A).

Definition 2.1 A subset A of a topological space (X, τ) is called β-open [2] or semi-preopen [3] if $A \subseteq Cl(Int(Cl(A)))$**.** The compliment of a β-open set is called β-closed [2].

Definition 2.2 [2] Let A be a subset of a space X then β-closure of A defined as the intersection of all β – closed sets containing A is denoted by Cl_β (A).

Lemma 2.3 [2] The following statement hold for a subset A of a topological space (X,τ):
A is β-closed if and only if $A = Cl_\beta(A)$.
Note that the family τ^{β} of β –open sets in X always forms a topology on X, this topology is finer than τ.

Definition 2.4 The β- closure of a set A is the intersection of all β-closed sets containing A and is denoted by $Cl_\beta(A)$.

Remark 2.5: For any subset A of the space X, $A \subset Cl_\beta(A) \subset Cl(A)$

Definition 2.6 Let (X,τ) be a topological any space, A subset of X, The $\text{int}_\beta(A) = \cup\{U : U$ is β–open and $U \subset A\}$.

Remark 2.7 A subset A is β –open if and only if $\text{int}_\beta(A) = A$.

Proof: The proof is obvious from the definition.

Definition 2.8 Let (X, τ) and (Y, σ) be two topological spaces, recall that a map $f : X \to Y$ is said to be β –continuous if for each open set H of Y, $f^{-1}(H)$ is β–open in X.

Definition 2.9 The intersection of all β-closed sets containing A is called the β-closure of A and is denoted by $Cl_\beta(A)$.. The β-interior ($Int_\beta(A)$) of a subset A ⊂ X is the union of all β-open sets contained in A.

Lemma 2.10.([2], [3])**.** The following hold for a subset A of a topological space X:

(1) $Int_\beta(A) = A \cap Cl(Int(Cl(A)))$;

(2) $Cl_\beta(A) = A \cup Int(Cl(Int(A)))$;

(3) $x \in Cl_\beta(A) \leftrightarrow O \cap A \neq \phi \;\; for \;\; every \;\; O \in \beta O(X)$;

(4) $Cl_\beta(X \setminus A) = X \setminus Int_\beta(A)$;

(5) A is β-closed if and only if $A = Cl_\beta(A)$.

III. New Types of Transitivity and Minimal Systems

In this section, we define a new class of topologically transitive maps that are called β-type transitive maps on a space (X, τ), and we study some of their properties and prove some results associated with these new definitions. We will also define and introduce a new class of β-type minimal maps.

Topological transitivity is a global characteristic of dynamical systems. By a dynamical system (X, f) [8] we mean a topological space X together with a continuous map $f : X \to X$. The space X is sometimes called the phase space of the system. A set $A \subseteq X$ is called $f - inveriant$ if $f(A) \subseteq A$.

A dynamical system (X, f) is called *minimal* if X does not contain any non-empty, proper, closed $f - inveriant$ subset. In such a case we also say that the map f itself is minimal. Thus, one cannot simplify the study of the dynamics of a minimal system by finding its nontrivial closed subsystems and studying first the dynamics restricted to them.

Given a point x in a system (X, f), $O_f(x) = \{x, f(x), f^2(x), ...\}$ denotes its orbit (by an orbit we mean a forward orbit even if f is a homeomorphism) and $\omega_f(x)$ denotes its ω -limit set, i.e. the set of limit points of the sequence $x, f(x), f^2(x), ...$. The following conditions are equivalent:

- (X, f) is β- minimal,
- every orbit is β-dense in X ,
- $\omega_f(x) = X$ for every x ∈ X .

A minimal map f is necessarily surjective if X is assumed to be Hausdorff and compact.

Now, we will study the Existence of minimal sets. Given a dynamical system (X, f) , a set $A \subseteq X$ is called a *minimal set* if it is non-empty, closed and invariant and if no proper subset of A has these three properties. So, $A \subseteq X$ is a minimal set if and only if $(A, f|A)$ is a minimal system. A system (X, f) is minimal if and only if X is a minimal set in (X, f).

The basic fact discovered by G. D. Birkhoff (1920) is that in any compact system (X, f) there are minimal sets. This follows immediately from the Zorn's lemma. Since any orbit closure is invariant, we get that *any compact orbit closure contains a minimal set*. This is how compact minimal sets may appear in non-compact spaces. Two minimal sets in (X, f) either are disjoint or coincide. A minimal set A is strongly $f - inveriant$, i.e. $f(A) = A$. Provided it is compact Hausdorff

we define a function $f : X \to X$ is called βr-homeomorphism if f is β-irresolute bijective and $f^{-1} : X \to X$ is β-irresolute.

Definition 3.1 Two topological dynamical systems $f : X \to X$, $x_{n+1} = f(x_n)$ and $g : Y \to Y$, $y_{n+1} = g(y_n)$ are said to be topologically βr-conjugate if there is βr-homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$ *(i.e. h(f(x)) = g(h(x)).* We will call h a topological βr-conjugacy.

Remark 3.2 Let (X, f) and (Y, g) be two topological dynamical systems.

If $\{x_{0,}\ x_1, x_2, \ldots\}$ denotes an orbit of $x_{n+1} = f(x_n)$ then $\{\ y_0 = h(x_0),\ \ y_1 = h(x_1),$ $y_2 = h(x_2), \ldots\ \}$ yields an orbit of g since $y_{n+1} = h(x_{n+1}) = h(f(x_n)) = g(h(x_n) = g(y_n)$.
In particular, h maps periodic orbits of f onto periodic orbits of g.
We introduced and defined the new type of topologically transitive in such a way that it is preserved under topologically βr- conjugation

Definition 3.3 Let (X,τ) be a topological space, and $f : X \to X$ β-irresolute map, then f is said to be a topologically beta-transitive map if for every pair of β-open sets U and V in X there is a positive integer n such that $f^n(U) \cap V \neq \phi$

Definition 3.4 Let (X, τ) be a topological space. Recall that a subset A of X is called β-dense in X if $Cl_\beta(A) = X$.

Remark 3.5 Any β-dense subset in X intersects any β-open set in X.

Proof: Let A be an β-dense subset in X, then by definition, $Cl_\beta(A) = X$, and let U be a non-empty β-open set in X. Suppose that A∩U=ϕ. Therefore $B = U^c = X - U$ is β-closed and $A \subset U^c$ =B. So $Cl_\beta(A) \subset Cl_\beta(B)$, i.e. $Cl_\beta(A) \subset$ B, but $Cl_\beta(A) = X$, so X⊂B, this contradicts that U ≠ φ

Definition 3.6. Let (X, τ) and (Y, σ) be two topological spaces, a map $f : X \to Y$ is called β-irresolute [16] if $f^{-1}(V)$ is β-open for every β-open set V in Y. equivalently, if $f^{-1}(V)$ is β-closed for every β-closed set V in Y

Definition 3.7. A function $f : X \to X$ is called a βr-homeomorphism if f is a bijective β- irresolute and f^{-1} is β-irresolute.

Definition 3.8. [16]If f and g are β-irresolute, then the composition g ∘ f is also β-irresolute. The identity function I : (X, τ) → (X, τ) is β-irresolute

Definition 3.9 A subset A of a topological space (X,τ) is said to be nowhere β-dense, if its β-closure has an empty β-interior, that is, $\text{int}_\beta(Cl_\beta(A)) = \phi$.

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IV. β-Minimal functions:

In this section, we introduce a new definition on minimal functions called β-minimal and we study some new theorems associated with this new definition.

A dynamical system (X, f) is called β-*minimal* if X does not contain any non-empty, proper, β-closed f-invariant subset. In such a case we also say that the map f itself is β-minimal. Another definition of β-minimal function is that if the orbit of every point x in X is β-dense in X then the map f is said to be β-minimal..
Let us introduce and study an equivalent new definition.

Definition 4.1 (β-minimal)Let X be a topological space and f be β-irresolute function on X. Then (X, f) is called β-minimal system (or f is called β-minimal function on X) if one of the three equivalent conditions hold:

(1) The orbit of each point of X is β-dense in X.

(2) $Cl_\beta(O_f(x)) = X$ for each x ϵ X .

(3) Given x ϵ X and a nonempty β-open U in X, there exists n ϵ N such that $f^n(x) \in U$.

Theorem 4.2. For (X, f) the following statements are equivalent:

(1) f is an β-minimal function.

(2) If E is a β-closed subset of X with $f(E) \subset E,$ we say E is invariant. Then E=ϕ or E=X.

(3) If U is a nonempty β-open subset of X, then $\bigcup_{n=0}^{\infty} f^{-n}(U) = X$.

Proof:

(1) $\Rightarrow$(2): If A ≠ ϕ, let x ϵ A. Since A is invariant and β-closed, i.e. $Cl_\beta(A) = A$ so $Cl_\beta(O_f(x)) \subset A$.. On other hand $Cl_\beta(O_f(x)) = X$.Therefore A = X.

(2)$\Rightarrow$(3) Let A=X\ $\bigcup_{n=0}^{\infty} f^{-n}(U)$. Since U is nonempty, A ≠ X and Since U is β-open and f is β-irresolute, A is β-closed. Also $f(A) \subset A$, so A must be ϕ.

(3) $\Rightarrow$(1): Let x ϵ X and U be a nonempty β-open subset of X. Since x ϵ X $= \bigcup_{n=0}^{\infty} f^{-n}(U)$ .Therefore x ϵ $f^{-n}(U)$ for some n>0. So $f^n(x) \in U$

Proposition .4.3 if $f: X \to X$ *and* $g: Y \to Y$ are topologically βr-conjugate. Then

(1) f is β-type transitive if and only if g is β-type transitive;

(2) f is β-minimal if and only if g is β-minimal;

(3) f is topologically β-mixing if and only if g is topologically β-mixing.

Proof (1)

Assume that $f: X \to X$ *and* $g: Y \to Y$ are topological dynamical systems which are topologically βr-conjugated by $h: X \to Y$. Suppose f is β-type transitive. Let A, B be β-open subsets of the topological space Y (*to show* $g^n(A) \cap B \neq \varphi$ *for some* $n > 0$). $U = h^{-1}(A)$ *and* $V = h^{-1}(B)$ are β-open subsets of X since h *is an* $\beta-$*irresolute.*

Then there exists some n>0 such that $f^n(U) \cap V \neq \varphi$ *since* f is β–type transitive. Thus (as $f \circ h^{-1} = h^{-1} \circ g$ *implies* $f^n \circ h^{-1} = h^{-1} \circ g^n$),

$\phi \neq f^n(h^{-1}(A)) \cap h^{-1}(B) = h^{-1}(g^n(A)) \cap h^{-1}(B)$

Therefore, $h^{-1}(g^n(A) \cap B) \neq \phi$ *implies* $g^n(A) \cap B \neq \phi$ since h^{-1} *is invertible*.

Proof (2)

Assume that $f: X \to X$ *and* $g: Y \to Y$ are topological dynamical systems which are topologically βr-conjugated by $h: Y \to X$. Thus, h is βr-homeomorphism (that is, h is bijective and thus invertible and both h and h^{-1} are β-irresolute) and $h \circ g = f \circ h$ that is, the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{g} & Y \\ h\downarrow & & \downarrow h \\ X & \xrightarrow[f]{} & X \end{array}$$

We show that if g is β-minimal, then f is β-minimal. We want to show that for any $x \in X$, $O_f(x)$ *is* $\beta-dense$. Since h is surjective, there exists x ϵX such that $y = h^{-1}(x)$. Since g is β-minimal, $O_g(y)$ *is* $\beta-dense$. For any non-empty β- open subset U of X, $h^{-1}(U)$ is an β-open set in X since h^{-1} is β-irresolute because h is an βr-homeomorphism and it is non-empty since h is invertible. By β-density of $O_g(y)$ there exist k in N such that $g^k(y) \in h^{-1}(U) \Leftrightarrow h(g^k(y)) \in U$

Since h is βr-conjugacy; as $f \circ h = h \circ g$ implies $f^k \circ h = h \circ g^k$

so $f^k(h(y)) = h(g^k(y)) \in U$ thus $O_f(h(y))$ intersects U. This holds for any non-empty β -open set U and thus shows that $O_f(x) = O_f(h(y))$ is β-dense

Proof(3)

We only prove that if g is topologically β-mixing then f is also topologically β-mixing. Let U,V be two β-open subsets of X. We have to show that there is N>0 such that for any n>N, $f^n(U) \cap V \neq \phi.$

$h^{-1}(U)$ and $h^{-1}(V)$ are two β-open sets since h is β-irresolute. If g is topologically β-mixing then there is N > 0 such that for any n>M, $g^n(h^{-1}(U)) \cap h^{-1}(V) \neq \phi.$ Therefore there exits $x \in g^n(h^{-1}(U)) \cap h^{-1}(V)$. That is, $x \in g^n(h^{-1}(U))$ *and* $x \in h^{-1}(V)$ if and only if $x = g^n(y)$ *for* $y \in h^{-1}(U)$ *and* $h(x) \in V$. Thus, since $h \circ g^n = f^n \circ h$, so that, $h(x) = h(g^n(y) = f^n(h(y)) \in f^n(U)$ and we have $h(x) \in V$ that is $f^n(U) \cap V \neq \phi.$

So, f is β-mixing.

V. Conclusion

The main results are the following:

Definition 5.1 Let (X, τ) be a topological space, and $f : X \to X$ β-irresolute map, then f is said to be a topologically beta-transitive map if for every pair of β-open sets U and V in X there is a positive integer n such that $f^n(U) \cap V \neq \phi$.

Definition 5.2 (β-minimal)Let X be a topological space and f be β-irresolute function on X. Then (X, f) is called β-minimal system (or f is called β-minimal function on X) if one of the three equivalent conditions hold:

(1) The orbit of each point of X is β-dense in X.

(2) $Cl_\beta(O_f(x)) = X$ for each x ϵ X .

(3) Given x ϵ X and a nonempty β-open U in X, there exists n ϵ N such that $f^{n}(x) \in U$.

Theorem 5.3. For (X, f) the following statements are equivalent:

(1) f is an β-minimal function.

(2) If E is a β-closed subset of X with $f(E) \subset E,$ we say E is invariant. Then E=ϕ or E=X.

(3) If U is a nonempty β-open subset of X, then $\bigcup_{n=0}^{\infty} f^{-n}(U) = X$.

Proposition 5.4 if $f : X \to X$ *and* $g : Y \to Y$ are topologically βr-conjugate. Then

(1) f is β-type transitive if and only if g is β-type transitive;

(2) f is β-minimal if and only if g is β-minimal;

(3) f is topologically β-mixing if and only if g is topologically β-mixing.

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