Multiple Positive solutions of Sturm-Liouville problems for second order singular and impulsive differential equations

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Abstract. In this paper,we study the existence of multiple positive solutions of nonlinear singular two-point boundary value problems for second-order impulsive differential equations.The proof is based on the theory of fixed point index in cones.

Key words. Multiple positive solutions; Singular two-point boundary value problem; Second-order impulsive differential equations; Fixed point index in cones.

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1. Introduction

Impulsive differential equations play a very important role in modern applied mathematics due to their deep physical background and broad application. In this paper,we consider the existence of multiple positive solutions of two-point boundary value problems for nonlinear second-order singular and impulsive differential equations:

$$\begin{cases} -Lu = h(x)g(x,u), \qquad x \in I', \\ -\Delta(pu')|_{x=x_k} = I_k(u(x_k)), \quad k = 1, 2, \cdots, m, \\ \Delta(pu)|_{x=x_k} = \overline{I}_k(u(x_k)), \quad k = 1, 2, \cdots, m, \\ R_1(u) = \alpha u(0) - \beta u'(0) = 0, \\ R_2(u) = \gamma u(1) + \delta u'(1) = 0, \end{cases} \tag{1.1}$$

here $Lu = (p(x)u')' + q(x)u$ is sturm-liouville operator, $I = [0, 1]$, $I' = I \setminus \{x_1, x_2, \cdots, x_m\}$ and $0 < x_1 < x_2 < \cdots < x_m < 1$ are given, $R^+ = [0, +\infty)$, $g \in C(I \times R^+, R^+)$, $I_k, \overline{I}_k \in C(R^+, R^+)$, $\Delta(pu')|_{x=x_k} = p(x_k)u'(x_k^+) - p(x_k)u'(x_k^-)$, $\Delta(pu)|_{x=x_k} = p(x_k)u(x_k^+) - p(x_k)u(x_k^-)$ $u'(x_k^+), u(x_k^+)$ ($u'(x_k^-), u(x_k^-)$) denote the right limit (left limit) of $u'(x)$ *and* $u(x)$ at $x = x_k$ respectively,$h(x) \in C(I, R^+)$ and may be singular at $x = 0$ *or* $x = 1$.

Throughout this paper, we always suppose that

(S_1) $p(x) \in C^1([0, 1], R)$, $p(x) > 0$, $q(x) \in C([0, 1], R)$, $q(x) \le 0$, $\alpha, \beta, \gamma, \delta \ge 0$, $\rho = \beta\gamma + \alpha\gamma + \alpha\delta > 0$.

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It is well known that there are abundant results about the existence of positive solutions of boundary value problems for second order impulsive differential equations. Some works can be found in $[1, 3, 6-9]$ and references therein.They,mainly investigated the case $p(x) = 1$ and $q(x) = 0$. In this paper ,we will consider the case $p(x) \neq 1$,and $q(x) \neq 0$. Here we also mention that second order dynamic inclusions on time scales with impulses has been studied in [2] .We obtain the existence results of positive solutions,by means of the fixed point index theorem in cones under some conditions on $g(x, u)$ concerning the first eigenvalue corresponding to the relevant linear operator.

To conclude the introduction,we introduce the following notation:

$$g_0 = \liminf_{u\to 0^+} \min_{x\in[a,b]} \frac{g(x,u)}{u}, \ I_0(k) = \liminf_{u\to 0^+} \frac{I_k(u)}{u}, \ \overline{I}_0(k) = \liminf_{u\to 0^+} \frac{\overline{I}_k(u)}{u}$$

$$g_\infty = \liminf_{u\to +\infty} \min_{x\in[a,b]} \frac{g(x,u)}{u}, \ I_\infty(k) = \liminf_{u\to +\infty} \frac{I_k(u)}{u}, \ \overline{I}_\infty(k) = \liminf_{u\to +\infty} \frac{\overline{I}_k(u)}{u};$$

Moreover,for the simplicity in the following discussion,we introduce the following hypotheses.

(H_1) :

$$g_0 + \frac{\sigma \sum_{k=1}^{m} (I_0(k)\phi_1(x_k) + \overline{I}_0(k)\phi_1'(x_k))}{\int_a^b \phi_1(x)h(x)dx} > \lambda_1, \quad g_\infty + \frac{\sigma \sum_{k=1}^{m} (I_\infty(k)\phi_1(x_k) + \overline{I}_\infty(k)\phi_1'(x_k))}{\int_a^b \phi_1(x)h(x)dx} > \lambda_1.$$

here $\sigma = \min\{\frac{m(a)}{m(1)}, \frac{n(b)}{n(0)}\}$ (see section 2), and $\phi_1(x)$ is the eigenfunction related to the smallest eigenvalue λ_1 of the eigenvalue problem $-L\phi = \lambda\phi h, \ R_1(\phi) = R_2(\phi) = 0$.

(H_2) : There is a $p > 0$ such that $0 \leq u \leq p$ and $0 \leq x \leq 1$ implies

$$g(x, u) \leq \eta p, \ I_k(u) \leq \eta_k p, \ \bar{I}_k(u) \leq \bar{\eta}_k p$$

here $\eta, \ \eta_k, \ \bar{\eta}_k \geq 0$, $\eta + \sum_{k=1}^{m} (\eta_k + \bar{\eta}_k) > 0$, $\eta \int_0^1 G(y, y)h(y)dy + \sum_{k=1}^{m} G(x_k, x_k)(\eta_k + \bar{\eta}_k) < 1$ and $G(x, y)$ is the Green's function of boundary value problem $-Lu = 0, \ R_1(u) = R_2(u) = 0$ (see section 2).

(H_3) : $0 < \int_1^0 G(y, y)h(y)dy < +\infty$

2. Preliminary

In this paper, we shall consider the following space

$$PC(I, R) = \{u \in C(I, R); u|_{(x_k, x_{k+1})} \in C(x_k, x_{k+1}), \ u(x_k^-) = u(x_k), \ \exists \ u(x_k^+), \ k = 1, 2, \cdots, m\}$$

$$PC'(I, R) = \{u \in C(I, R); u'|_{(x_k, x_{k+1})} \in C(x_k, x_{k+1}), \ u'(x_k^-) = u'(x_k), \ \exists \ u'(x_k^+), \ k = 1, 2, \cdots, m\}$$

with the norm $\|u\|_{PC} = \sup_{x\in[0,1]} |u(x)|$, $\|u\|_{PC'} = \max\{\|u\|_{PC}, \ \|u'\|_{PC}\}$, Then $PC(I, R)$,$PC'(I, R)$ are Banach spaces.

Definition 2.1: A function $u \in PC'(I,R) \cap C^2(I',R)$ is a solution of (1.1), if it satisfies the differential equation

$$Lu + h(x)g(x,u) = 0, \quad x \in I'$$

and the function u satisfies conditions $\Delta(pu')|_{x=x_k} = -I_k(u(x_k)), \Delta(pu)|_{x=x_k} = \overline{I}_k(u(x_k))$ and $R_1(u) = R_2(u) = 0$.

Let $Q = I \times I$ and $Q_1 = \{(x,y) \in Q | 0 \le x \le y \le 1\}, \ Q_2 = \{(x,y) \in Q | 0 \le y \le x \le 1\}$. Let $G(x,y)$ is the Green's function of the boundary value problem

$$-Lu = 0, \ R_1(u) = R_2(u) = 0.$$

Following from [4], $G(x,y)$ can be written by

$$G(x,y) := \begin{cases} \frac{m(x)n(y)}{\omega}, \ (x,y) \in Q_1, \\ \frac{m(y)n(x)}{\omega}, \ (x,y) \in Q_2. \end{cases} \tag{2.1}$$

Lemma 2.1[4]: Suppose that (S_1) holds, then the Green's function $G(x,y)$, defined by (2.1), possesses the following properties:
(i): $m(x) \in C^2(I,R)$ is increasing and $m(x) > 0, \ x \in (0,1]$.
(ii): $n(x) \in C^2(I,R)$ is decreasing and $n(x) > 0, \ x \in [0,1)$.
(iii): $(Lm)(x) \equiv 0, \ m(0) = \beta, \ m'(0) = \alpha$.
(iv): $(Ln)(x) \equiv 0, \ n(1) = \delta, \ n'(1) = -\gamma$.
(v): ω is a positive constant. Moreover, $p(x)(m'(x)n(x) - m(x)n'(x)) \equiv \omega$.
(vi): $G(x,y)$ is continuous and symmetrical over Q.
(vii): $G(x,y)$ has continuously partial derivative over $Q_1, \ Q_2$.
(viii): For each fixed $y \in I$, $G(x,y)$ satisfies $LG(x,y) = 0$ for $x \ne y, \ x \in I$. Moreover, $R_1(G) = R_2(G) = 0$ for $y \in (0,1)$.
(viiii): G'_x has discontinuous point of the first kind at $x = y$ and

$$G'_x(y+0,y) - G'_x(y-0,y) = -\frac{1}{p(y)}, \ y \in (0,1).$$

Following from Lemma2.1, it is easy to see that:

$G(x,y) \le G(y,y) = \frac{m(y)n(y)}{\omega}, \ x,y \in [0,1]$

$$G(x,y) \ge \sigma G(y,y), x \in [a,b], y \in [0,1], where\, a \in (0,t_1], b \in [t_m,1), \ 0 < \sigma = min\{\frac{m(a)}{m(1)}, \frac{n(b)}{n(0)}\} < 1 \tag{2.2}$$

Consider the linear Sturm-Liouvile problem

$$-(Lu)(x) = \lambda u(x)h(x), \quad R_1(u) = R_2(u) = 0.$$

By the Sturm-Liouvile theory of ordinary differential equations, we know that there exists an eigenfunction $\phi_1(x)$ with respect to the first eigenvalue $\lambda_1 > 0$ such that $\phi_1(x) > 0$ for $x \in (0,1)$.

Let E be a Banach space and $K \subset E$ be a closed convex cone in E. For $r > 0$, let $K_r = \{u \in K : ||u|| < r\}$ and $\partial K_r = \{u \in K : ||u|| = r\}$. The following two Lemmas are needed in our argument.

Lemma 2.2: Let $\Phi : K \to K$ be a continuous and completely continuous mapping and $\Phi u \neq u$ for $u \in \partial K_r$. Thus the following conclusions hold:
(i) If $||u|| \leq \|\Phi u\|$ for $u \in \partial K_r$, then $i(\Phi, K_r, K) = 0$;
(ii) If $\|u\| \geq \|\Phi u\|$ for $u \in \partial K_r$, then $i(\Phi, K_r, K) = 1$.

Lemma 2.3: Let $\Phi : K \to K$ be a continuous and completely continuous mapping. Suppose that the following two conditions are satisfied:

(i) $\inf\limits_{u \in \partial K_r} ||\Phi u|| > 0$; (ii) $\mu \Phi u \neq u$ for every $u \in \partial K_r$ and $\mu \geq 1$.

Then, $i(\Phi, K_r, K) = 0$.

In applications below, we take $E = C(I, R)$ and define

$$K = \{u \in C(I, R) : u(x) \geq \sigma \|u\|, x \in [a, b]\}.$$

One may readily verify that K is a cone in E.

Define an operator $\Phi : K \to K$ by

$$(\Phi u)(x) = \int_0^1 G(x, y)h(y)g(y, u(y))dy + \sum_{0<x_k<x} G(x, x_k)(I_k(u(x_k)) + \bar{I}_k(u(x_k))), \ x \in I.$$

It follows form (H_3) that ϕ is well defined.

Lemma 2.4: If (H_3) is satisfied,then $\Phi : K \to K$ is continuous and completely continuous, Moreover, $\Phi(K) \subset K$.

Proof By the property of continuous of $g(x, u)$, $I_k(x)$, $\bar{I}_k(x)$, it is easy to see that $\Phi : K \to K$ is continuous and completely continuous. Thus we only need to show $\Phi(K) \subset K$. In fact, for $u \in K$, by using inequalities (2.2) and (H_3), we have that

$$\|\Phi u\| \leq \int_0^1 G(y, y)h(y)g(y, u(y))dy + \sum_{0<x_k<x} G(x_k, x_k)(I_k(u(x_k)) + \bar{I}_k(u(x_k))) < +\infty$$

On the other hand ,for any $x \in [a, b]$,by (2.2) we obtain

$$\begin{aligned}
(\Phi u)(x) &= \int_0^1 G(x, y)h(y)g(y, u(y))dy + \sum_{0<x_k<x} G(x, x_k)(I_k(u(x_k)) + \bar{I}_k(u(x_k))) \\
&\geq \int_b^a G(x, y)h(y)g(y, u(y))dy + \sum_{0<x_k<x} G(x, x_k)(I_k(u(x_k)) + \bar{I}_k(u(x_k))) \\
&\geq \sigma \left(\int_1^0 G(y, y)h(y)g(y, u(y))dy + \sum_{0<x_k<x} G(x_k, x_k)(I_k(u(x_k)) + \bar{I}_k(u(x_k))) \right) \\
&\geq \sigma \|\Phi u\|
\end{aligned}$$

Thus, $\Phi(K) \subset K$.

Lemma 2.5:If u is a fixed point of the operator Φ, then u is a solution of problem (1.1).

3. Main Results

Lemma 3.1:If (H_2) is satisfied, then $i(\Phi, K_p\ , K) = 1$.

Proof Let $u \in K$ with $\|u\| = p$. It follows from (H_2) that

$$\begin{aligned} \|\Phi u\| &\leq \int_0^1 G(y,y)h(y)g(y,u(y))dy + \sum_{k=1}^m G(x_k,x_k)(I_k(u(x_k)) + \overline{I}_k(u(x_k))) \\ &\leq p[\eta \int_0^1 G(y,y)h(y)dy + \sum_{k=1}^m G(x_k,x_k)(\eta_k + \overline{\eta}_k)] < p = \|u\|. \end{aligned}$$

Thus

$$\|\Phi u\| < \|u\|, \quad \forall\ u \in \partial K_p.$$

It is obvious that $\Phi u \neq u$ for $u \in \partial K_p$. Therefore, $i(\Phi, K_p, K) = 1$, here we use Lemma2.2.

Theorem 3.1:Assume that $(H_1) - (H_3)$ are satisfied. Then problem (1.1) has at least two positive solutions u_1 and u_2 with

$$0 < ||u_1|| < p < ||u_2||.$$

Proof According to Lemma 3.1, we have that

$$i(\Phi, K_p, K) = 1. \tag{3.1}$$

Since (H_1) holds, then there exists $0 < \varepsilon < 1$ such that

$$(1-\varepsilon)[g_0 + \frac{\sigma \sum\limits_{k=1}^m (I_0(k)\phi_1(x_k) + \overline{I}_0(k)\phi_1'(x_k))}{\int_a^b \phi_1(x)h(x)dx}] > \lambda_1,$$

$$(1-\varepsilon)[g_\infty + \frac{\sigma \sum\limits_{k=1}^m (I_\infty(k)\phi_1(x_k) + \overline{I}_\infty(k)\phi_1'(x_k))}{\int_a^b \phi_1(x)h(x)dx}] > \lambda_1. \tag{3.2}$$

By the definitions of $g_0,\ I_0$, one can find $0 < r_0 < p$ such that

$$g(x,u) \geq g_0(1-\varepsilon)u,\ I_k(u) \geq I_0(k)(1-\varepsilon)u,\ \ \overline{I}_k(u) \geq \overline{I}_0(k)(1-\varepsilon)u\ \forall\ x \in [a,b],\ 0 < u < r_0.$$

Let $r \in (0, r_0)$, then for $u \in \partial K_r$, we have

$$u(x) \geq \sigma\|u\| = \sigma r > 0. \quad x \in [a,b]$$

Thus

$$\begin{aligned} (\Phi u)(\frac{1}{2}) &= \int_0^1 G(\frac{1}{2},y)h(y)g(y,u(y))dy + \sum_{0<x_k<\frac{1}{2}} G(\frac{1}{2},x_k)(I_k(u(x_k)) + \overline{I}_k(u(x_k))) \\ &\geq \int_a^b G(\frac{1}{2},y)h(y)g(y,u(y))dy + \sum_{0<x_k<\frac{1}{2}} G(\frac{1}{2},x_k)(I_k(u(x_k)) + \overline{I}_k(u(x_k))) \end{aligned}$$

$$
\begin{aligned}
&\geq \quad g_0(1-\varepsilon)\int_a^b G(\frac{1}{2},y)h(y)u(y)dy + (1-\varepsilon)\sum_{0<x_k<\frac{1}{2}} G(\frac{1}{2},x_k)I_0(k)u(x_k) \\
&+ \quad (1-\varepsilon)\sum_{0<x_k<\frac{1}{2}} G(\frac{1}{2},x_k)\overline{I}_0(k)u(x_k) \\
&\geq \quad (1-\varepsilon)\sigma r\left(g_0\int_a^b G(\frac{1}{2},y)h(x)dy + \sum_{0<x_k<\frac{1}{2}} G(\frac{1}{2},x_k)(I_0(k)+\overline{I}_0(k))\right) > 0
\end{aligned}
$$

from which we see that $\inf_{u\in\partial K_r} ||\Phi u|| > 0$, namely, hypothesis (i) of Lemma 2.3 holds. Next we show that $\mu\Phi u \neq u$ for any $u \in \partial K_r$ and $\mu \geq 1$.

If this is not true, then there exist $u_0 \in \partial K_r$ and $\mu_0 \geq 1$ such that $\mu_0\Phi u_0 = u_0$. Note that $u_0(x)$ satisfies

$$
\begin{cases}
Lu_0 + \mu_0 h(x)g(x,u_0(x)) = 0, & x \in I', \\
-\Delta(pu_0')|_{x=x_k} = \mu_0 I_k(u_0(x_k)), & k = 1,2,\cdots,m, \\
\Delta(pu_0)|_{x=x_k} = \mu_0 \overline{I}_k(u_0(x_k)), & k = 1,2,\cdots,m, \\
\alpha u_0(0) - \beta u_0'(0) = 0, \\
\gamma u_0(1) + \delta u_0'(1) = 0.
\end{cases}
\tag{3.3}
$$

Multiply equation (3.3) by $\phi_1(x)$ and integrate from a to b, note that

$$
\begin{aligned}
&\int_a^b \phi_1(x)[(p(x)u_0'(x))' + q(x)u_0(x)]dx = \int_a^{x_1} \phi_1(x)[(p(x)u_0'(x))' + q(x)u_0(x)]dx \\
+ &\sum_{k=1}^{m-1}\int_{x_k}^{x_{k+1}} \phi_1(x)[(p(x)u_0'(x))' + q(x)u_0(x)]dx + \int_{x_m}^b \phi_1(x)[(p(x)u_0'(x))' + q(x)u_0(x)]dx \\
= \ &\phi_1(x_1)p(x_1)u_0'(x_1-0) - \int_a^{x_1} p(x)u_0'(x)\phi_1'(x)dx + \int_a^{x_1} q(x)u_0(x)\phi_1(x)dx \\
+ &\sum_{k=1}^{m-1}[\phi_1(x_{k+1})p(x_{k+1})u_0'(x_{k+1}-0) - \phi_1(x_k)p(x_k)u_0'(x_k+0) - \int_{x_k}^{x_{k+1}} p(x)u_0'(x)\phi_1'(x)dx \\
+ &\int_{x_k}^{x_{k+1}} q(x)u_0(x)\phi_1(x)dx] - \phi_1(x_m)p(x_m)u_0'(x_m+0) - \int_{x_m}^b p(x)u_0'(x)\phi_1'(x)dx \\
+ &\int_{x_m}^b q(x)u_0(x)\phi_1(x)dx \\
= \ &-\sum_{k=1}^{m}\Delta(p(x_k)u_0'(x_k))\phi_1(x_k) - \int_a^b p(x)\phi_1'(x)u_0'(x)dx + \int_a^b q(x)\phi_1(x)u_0(x)dx
\end{aligned}
$$

.

Also note that

$$
\begin{aligned}
&\int_a^b p(x)\phi_1'(x)u_0'(x)dx = \int_a^{x_1} p(x)\phi_1'(x)du_0(x) + \sum_{k=1}^{m-1}\int_{x_k}^{x_{k+1}} p(x)\phi_1'(x)du_0(x) \\
+ &\int_{x_m}^b p(x)\phi_1'(x)du_0(x)
\end{aligned}
$$

$$
\begin{aligned}
&= -\sum_{k=1}^{m} \Delta(p(x_k)u_0(x_k))\phi_1'(x_k) - \int_a^b u_0(x)(p(x)\phi_1'(x))'dx \\
&= -\sum_{k=1}^{m} \Delta(p((x_k)u_0(x_k))\phi_1'(x_k) + \int_a^b u_0(x)q(x)\phi_1(x)dx + \lambda_1 \int_a^b h(x)\phi_1(x)u_0(x)dx
\end{aligned}
$$

$$
\begin{aligned}
\int_a^b \phi_1(x)[(p(x)u_0'(x))' + q(x)u_0(x)]dx &= -\sum_{k=1}^{m} \Delta(p(x_k)u_0'(x_k))\phi_1(x_k) \\
&+ \sum_{k=1}^{m} \Delta(p(x_k)u_0(x_k))\phi_1'(x_k) - \lambda_1 \int_a^b h(x)\phi_1(x)u_0(x)dx \\
&= \sum_{k=1}^{m} \mu_0(I_k(u_0(x_k))\phi_1(x_k) + \overline{I}_k(u_0(x_k))\phi_1'(x_k)) - \lambda_1 \int_a^b u_0(x)h(x)\phi_1(x)dx
\end{aligned}
$$

So we obtain

$$
\begin{aligned}
\lambda_1 \int_a^b u_0(x)h(x)\phi_1(x)dx &= \sum_{k=1}^{m} \mu_0(I_k(u_0(x_k))\phi_1(x_k) + \overline{I}_k(u_0(x_k))\phi_1'(x_k)) \\
&+ \mu_0 \int_a^b \phi_1(x)h(x)g(x,u_0(x))dx \\
&\geq (1-\varepsilon)\sum_{k=1}^{m} u_0(x_k)(I_0(k)\phi_1(x_k) + \overline{I}_0(k)\phi_1'(x_k)) \\
&+ (1-\varepsilon)g_0 \int_a^b \phi_1(x)u_0(x)h(x)dx
\end{aligned}
$$

Since $u_0(x) \geq \sigma||u_0|| = \sigma r$, we have $\int_a^b \phi_1(x)u_0(x)h(x)dx > 0$ and $\sum_{k=1}^{m} u_0(x_k)(I_0(k)\phi_1(x_k) + \overline{I}_0(k)\phi_1'(x_k)) > 0$. So from the above inequality we see that $\lambda_1 > (1-\varepsilon)g_0$.
Thus

$$
\begin{aligned}
[\lambda_1 - (1-\varepsilon)g_0] \int_a^b u_0(x)h(x)\phi_1(x)dx &\geq (1-\varepsilon)\sum_{k=1}^{m}(I_0(k)\phi_1(x_k) + \overline{I}_0(k)\phi_1'(x_k))u_0(x_k) \\
&\geq (1-\varepsilon)\sigma r \sum_{k=1}^{m}(I_0(k)\phi_1(x_k) + \overline{I}_0(k)\phi_1'(x_k)).
\end{aligned}
$$

Since $\int_a^b u_0(x)h(x)\phi_1(x)dx \leq r\int_a^b \phi_1(x)h(x)dx$, we have

$$
[\lambda_1 - (1-\varepsilon)g_0] \int_a^b h(x)\phi_1(x)dx \geq (1-\varepsilon)\sigma \sum_{k=1}^{m}(I_0(k)\phi_1(x_k) + \overline{I}_0(k)\phi_1'(x_k)),
$$

which contradicts (3.2) again. Hence Φ satisfies the hypotheses of Lemma 2.3 in K_r. Thus

$$
i(\Phi, K_r, K) = 0. \tag{3.4}
$$

On the other hand, from (H_1), there exists $H > p$ such that

$$
g(x,u) \geq g_\infty(1-\varepsilon)u,\ I_k(u) \geq I_\infty(k)(1-\varepsilon)u,\ \overline{I}_k(u) \geq \overline{I}_\infty(k)(1-\varepsilon)u,\ \forall\, x \in [a,b],\ u \geq H. \tag{3.5}
$$

Let
$C = \max\limits_{0\le u\le H}\max\limits_{a\le x\le b}|g(x,u)-g_\infty(1-\varepsilon)u| + \sum\limits_{k=1}^{m}\max\limits_{0\le u\le H}|I_k(u)-I_\infty(k)(1-\varepsilon)u| + \sum\limits_{k=1}^{m}\max\limits_{0\le u\le H}|\overline{I}_k(u)-\overline{I}_\infty(k)(1-\varepsilon)u|$. It is clear that

$$g(x,u)\ge g_\infty(1-\varepsilon)u-C,\ \ I_k(u)\ge I_\infty(k)(1-\varepsilon)u-C,\ \ \overline{I}_k(u)\ge \overline{I}_\infty(k)(1-\varepsilon)u-C,\ \forall\, x\in[a,b],\ u\ge 0. \tag{3.6}$$

Choose $R > R_0 := \max\{\frac{H}{\sigma}, p\}$ and let $u\in\partial K_R$. Since $u(x)\ge\sigma||u|| = \sigma R > H$ for $x\in[a,b]$, from (3.5) we see that

$$g(x,u(x))\ge g_\infty(1-\varepsilon)u(x)\ge\sigma g_\infty(1-\varepsilon)R,\ \forall\ x\in[a,b].$$
$$I_k(u(x_k)\ge\sigma I_\infty(k)(1-\varepsilon)R,\ \ \overline{I}_k(u(x_k)\ge\sigma\overline{I}_\infty(k)(1-\varepsilon)R.$$

Essentially the same reasoning as above yields $\inf\limits_{u\in\partial K_R}||\Phi u||>0$. Next we show that if R is large enough, then $\mu\Phi u\ne u$ for any $u\in\partial K_R$ and $\mu\ge 1$. In fact, if there exist $u_0\in\partial K_R$ and $\mu_0\ge 1$ such that $\mu_0\Phi u_0 = u_0$, then $u_0(x)$ satisfies equation (3.3) .

Multiply equation (3.3) by $\phi_1(x)$ and integrate from a to b, using integration by parts in the left side to obtain

$$\begin{aligned}\lambda_1\int_a^b u_0(x)h(x)\phi_1(x)dx &= \sum_{k=1}^{m}\mu_0(I_k(u_0(x_k))\phi_1(x_k)+\overline{I}_k(u_0(x_k))\phi_1'(x_k))\\ +\ \ \mu_0&\int_a^b\phi_1(x)h(x)g(x,u_0(x))dx\\ \ge\ \ (1-\varepsilon)&\sum_{k=1}^{m}(I_\infty(k)\phi_1(x_k)+\overline{I}_\infty(k)\phi_1'(x_k))u_0(x_k)+(1-\varepsilon)g_\infty\int_a^b u_0(x)\phi_1(x)h(x)dx\\ -\ \ C&\left(\sum_{k=1}^{m}(\phi_1(x_k)+\phi_1'(x_k))+\int_a^b\phi_1(x)h(x)dx\right).\end{aligned}$$

If $g_\infty\le\lambda_1$, then we have

$$\begin{aligned}&[\lambda_1-(1-\varepsilon)g_\infty]\int_a^b u_0(x)h(x)\phi_1(x)dx + C\left(\sum_{k=1}^{m}(\phi_1(x_k)+\phi_1'(x_k))+\int_a^b\phi_1(x)h(x)dx\right)\\ \ge\ \ &(1-\varepsilon)\sum_{k=1}^{m}(I_\infty(k)\phi_1(x_k)+\overline{I}_\infty(k)\phi_1'(x_k))u_0(x_k).\end{aligned}$$

thus

$$\begin{aligned}&\|u_0\|[\lambda_1-(1-\varepsilon)g_\infty]\int_a^b h(x)\phi_1(x)dx + C\left(\sum_{k=1}^{m}(\phi_1(x_k)+\phi_1'(x_k))+\int_a^b\phi_1(x)h(x)dx\right)\\ \ge\ \ &(1-\varepsilon)\sigma\|u_0\|\sum_{k=1}^{m}(I_\infty(k)\phi_1(x_k)+\overline{I}_\infty(k)\phi_1'(x_k)).\end{aligned}$$

and

$$\|u_0\|\le\frac{C\left(\sum\limits_{k=1}^{m}(\phi_1(x_k)+\phi_1'(x_k))+\int_a^b\phi_1(x)h(x)dx\right)}{(1-\varepsilon)\sigma\sum\limits_{k=1}^{m}(I_\infty(k)\phi_1(x_k)+\overline{I}_\infty(k)\phi_1'(x_k))-[\lambda_1-(1-\varepsilon)g_\infty]\int_a^b\phi_1(x)h(x)dx} =: \bar{R}. \tag{3.7$_a$}$$

If $g_\infty > \lambda_1$, we can choose $\varepsilon > 0$ such that $(1-\varepsilon)g_\infty > \lambda_1$, then we have

$$\begin{aligned} C\left(\sum_{k=1}^{m}(\phi_1(x_k)+\phi_1'(x_k))+\int_a^b \phi_1(x)h(x)dx\right) &\geq [(1-\varepsilon)g_\infty-\lambda_1]\int_a^b \phi_1(x)u_0(x)h(x)dx \\ &\geq [(1-\varepsilon)g_\infty-\lambda_1]\sigma\|u_0\|\int_a^b \phi_1(x)h(x)dx. \end{aligned}$$

Thus

$$\|u_0\| \leq \frac{C\left(\sum\limits_{k=1}^{m}(\phi_1(x_k)+\phi_1'(x_k))+\int_a^b \phi_1(x)h(x)dx\right)}{[(1-\varepsilon)g_\infty-\lambda_1]\sigma\int_a^b \phi_1(x)h(x)dx} =: \bar{R}. \tag{3.7$_b$}$$

Let $R > \max\{p, \bar{R}\}$, then for any $u \in \partial K_R$ and $\mu \geq 1$, we have $\mu\Phi u \neq u$. Hence hypothesis (ii) of Lemma 2.3 is satisfied and

$$i(\Phi, K_R\ , K) = 0. \tag{3.8}$$

In view of (3.1), (3.4) and (3.8), we obtain

$$i(\Phi, K_R \setminus \bar{K}_p, K) = -1,\ i(\Phi, K_p \setminus \bar{K}_r, K) = 1.$$

Then Φ has fixed points u_1 and u_2 in $K_p \setminus \bar{K}_r$ and $K_R \setminus \bar{K}_p$, respectively, which means $u_1(x)$ and $u_2(x)$ are positive solution of the problem (1.1) and $0 < \|u_1\| < p < \|u_2\|$.

Corollary 3.1:The conclusion of Theorem 3.1 is valid if (H_1) is replaced by

$$(H_1^*) \qquad g_0 = \infty \ \ or \ \ \sum_{k=1}^{m} I_0(k)\phi_1(x_k) = \infty \ \ or \ \ \sum_{k=1}^{m} \overline{I}_0(k)\phi_1'(x_k) = \infty;$$

and

$$g_\infty = \infty \ \ or \ \ \sum_{k=1}^{m} I_\infty(k)\phi_1(x_k) = \infty \ \ or \ \ \sum_{k=1}^{m} \overline{I}_\infty(k)\phi_1'(x_k) = \infty.$$

Example:

$$\begin{cases} Lu + \frac{1}{2}(u^{\frac{1}{3}} + u^3) = 0, \quad x \in I', \\ -\Delta(pu')|_{x=x_k} = c_k u(x_k), \quad c_k \geq 0, \\ \Delta(pu)|_{x=x_k} = d_k u(x_k), \quad d_k \geq 0, \\ R_1(u) = \alpha u(0) - \beta u'(0) = 0, \\ R_2(u) = \gamma u(1) + \delta u'(1) = 0, \end{cases} \tag{3.9}$$

here $Lu = (p(x)u')' + q(x)u$. Assume that (S_1) is satisfied. Then problem (3.9) has at least two positive solutions u_1 and u_2 with

$$0 < ||u_1|| < 1 < ||u_2||$$

provided

$$1 < \frac{1}{d}\left(1 - \sum_{k=1}^{m} G(x_k, x_k)(c_k + d_k)\right), \quad d = \int_0^1 G(y,y)dy. \tag{3.10}$$

Proof To see this we will apply Theorem 3.1 (or Corollary 3.1)

By (3.10), $\eta > 0$ is chosen such that

$$1 < \eta < \frac{1}{d}(1 - \sum_{k=1}^{m} G(x_k, x_k)(c_k + d_k)).$$

Set

$$g(x,u) = \frac{1}{2}(u^{\frac{1}{3}} + u^3).$$

Note

$$g_0 = \infty, \qquad g_\infty = \infty,$$

so (H_1) (or (H_1^*)) holds.

Let $\eta_k = c_k, \bar{\eta}_k = d_k$, then $\eta,\ \eta_k,\ \bar{\eta}_k$ satisfy

$$\eta \int_0^1 G(y,y)dy + \sum_{k=1}^{m} G(x_k, x_k)(\eta_k + \overline{\eta}_k) < 1.$$

Let $p = 1$, then for $0 \le u \le p$, we have

$$g(x,u) = \frac{1}{2}(u^{\frac{1}{3}} + u^3) \le \frac{1}{2} + \frac{1}{2} < \eta p = \eta,$$

and

$$I_k(u) = c_k u = \eta_k u \le \eta_k p, \quad \overline{I}_k(u) = d_k u = \overline{\eta}_k u \le \overline{\eta}_k p$$

thus (H_2) holds. The result now follows from Theorem 3.1(or Corollary3.1)

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