Signed Domination Number of Directed Paths $P_m \times P_n$

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Abstract: Let *D* be a finite simple directed graph with vertex set *V(D)* and arc set *A(D)*. A function $f: V(D) \rightarrow \{-1, 1\}$ is called a signed dominating function (SDF) if $f(N_D^-[v]) \geq 1$ for each vertex $v \in V$. The weight *W*(*f*) of *f* is defined by $\sum_{v \in V} f(v)$. The signed domination number of a digraph D is $\gamma_s(D) = min\ \{W(f) \mid f$ is an *SDF* of D$\}$. Let $P_m \times P_n$ denote the cartesian product of directed paths of length m and n. In this paper, we determine the exact values of $\gamma_s(P_m \times P_n)$ for m = 8, 9, 10 and arbitrary n. Also, we give a lower bound of $\gamma_s(P_m \times P_n)$.

Keywords*: Directed graph, Directed path, Cartesian product, Signed dominating function, Signed domination number.*

1. Introduction

Throughout this paper, a digraph *D(V, A)* always means a finite directed graph without loops and multiple arcs, where *V* = *V(D)* is the vertex set and *A* = *A(D)* is the arc set. If *uv* is an arc of *D*, then say that *v* is an out-neighbor of *u* and u is an in-neighbor of *v*. For a vertex $v \in V(D)$, let $N_D^+(v)$ and $N_D^-(v)$ denote the set of out-neighbors and in-neighbors of *v*, respectively. We write $d_D^+(v) = |N_D^+(v)|$ and $d_D^-(v) = |N_D^-(v)|$ for the out-degree and in-degree of *v* in *D*, respectively (shortly $d^+(v)$, $d^-(v)$). A digraph D is r-regular if $d_D^+(v) = d_D^-(v) = r$ for any vertex v∈D. Let $N_D^+[v] = N_D^+(v) \cup \{v\}$ and $N_D^-[v] = N_D^-(v) \cup \{v\}$, be the set of v and all vertices of out-degrees and in-degrees, respectively. The maximum out-degree and in-degree of *D* are denoted by $\Delta^+(D)$ and $\Delta^-(D)$, respectively (shortly Δ^+, Δ^-). The minimum out-degree and in-degree of *D* are denoted by δ^+(D) and δ^-(D), respectively (shortly δ^+, δ^-). A *signed dominating function* of *D* is defined in [6] as function $f: V \rightarrow \{-1,1\}$ such that $f(N_D^-[v]) \geq 1$ for every vertex $v \in V$. The *signed domination number* of a directed graph D is $\gamma_s(D) = min\ \{W(f) \mid f$ is an *SDF* of $D\}$. Also, a signed *k-dominating* function (*SKDF*) of *D* is a function $f: V \rightarrow \{-1, 1\}$ such that $f(N_D^-[v]) \geq k$ for every vertex $v \in V$. The *k-signed domination number* of a digraph *D* is $\gamma_{ks}(D) = min\ \{W(f) \mid f$ is *SKDF* of $D\}$. Consult [1] for the notation and terminology which are not defined here.

The cartesian product $D_1 \times D_2$ of two digraphs D_1 and D_2 is the digraph with vertex set $V(D_1 \times D_2) = V(D_1) \times V(D_2)$ and $((u_1,u_2),(v_1,v_2)) \in A(D_1 \times D_2)$ if and only if either $u_1 = v_1$ and $(u_2, v_2) \in A(D_2)$ or $u_2 = v_2$ and $(u_1, v_1) \in A(D_1)$.

In the past few years, several types of domination problems in graphs have been studied [2-5, 9-10], most of those belonging to the vertex domination. In 1995, Dunbar et al. [2], have introduced the concept of signed domination number of an undirected graph. Hass and Wexler in [10], established a sharp lower bound on the signed domination number of a general graph with a given minimum and maximum degree and also of some simple grid graph. Zelinka [6] initiated the study of the signed domination numbers of digraphs. He studied a signed domination number of the digraphs which the indegrees of vertices do not exceed 1, also the acyclic tournament and the circulant tournament. Karami et al. [7] were established lower and upper bounds of the signed domination number of digraphs. Atapour et al. [8], presented some sharp lower bounds on the signed k-domination number of digraphs. Also, H. Aram et al. [11], were established upper bound of the signed k-domination number of digraphs. Shaheen and Salim [12], were calculated the signed domination numbers of cartesian product $C_m \times C_n$ for m = 3, 4, 5, 6, 7. In [13], Shaheen calculated $\gamma_s(C_m \times C_n)$ for m = 8, 9, 10 and for some general values of m, and n. Also, Shaheen [14], calculated the signed domination numbers of cartesian product $P_m \times P_n$ for m = 2, 3, 4, 5, 6, 7 and arbitrary n. In this paper, we study the signed domination number of cartesian product $P_m \times P_n$ for m, n ≥8. We mainly determine the exact values of $\gamma_s(P_8 \times P_n)$, $\gamma_s(P_9 \times P_n)$ and $\gamma_s(P_{10} \times P_n)$.

Theorem 1.1 (Zelinka [6]). Let D be a directed cycle or path with n vertices. Then $\gamma_s(D) = n$.

Lemma 1.2 (Zelinka [6]). Let D be a directed graph with n vertices. Then $\gamma_s(D) \equiv n \pmod 2$.

In [14], the following results are proved.

Theorem 1.3 (Shaheen [14]). $\gamma_s(P_2 \times P_n) = n$: $n \equiv 0 \pmod 2$ and $\gamma_s(P_2 \times P_n) = n + 1$: $n \equiv 1 \pmod 2$. $\gamma_s(P_3 \times P_n) = n + 2\lceil n/3 \rceil$. $\gamma_s(P_4 \times P_n) = 2n$: $n \equiv 0 \pmod 2$ and $\gamma_s(P_4 \times P_n) = 2n + 2$: $n \equiv 1 \pmod 2$. $\gamma_s(P_5 \times P_n) = (7n + 6)/3$: $n \equiv 0 \pmod 3$, $\gamma_s(P_5 \times P_n) = (7n + 8)/3$: $n \equiv 1 \pmod 3$ and $\gamma_s(P_5 \times P_n) = (7n + 4)/3$: $n \equiv 2 \pmod 3$. $\gamma_s(P_6 \times P_n) = (8n + 6)/3$: $n \equiv 0 \pmod 3$, $\gamma_s(P_6 \times P_n) = (8n + 10)/3$: $n \equiv 1 \pmod 3$ and $\gamma_s(P_6 \times P_n) = (8n + 8)/3$: $n \equiv 2 \pmod 3$. $\gamma_s(P_7 \times P_n) = 3n + 4$.

2. Main results

In this section we calculate the signed domination number of the cartesian product of two directed paths P_m and P_n for m = 8, 9, 10 and arbitrary n.

The vertices of a directed path P_n are always denoted by the integers {1, 2, . . ., n}, considered modulo n. The *i*th row of $V(P_m \times P_n)$ is $R_i = \{(i, j) : j = 1, 2, \ldots, n\}$ and the *j*th column $K_j = \{(i, j) : i = 1, 2, \ldots, m\}$. For any vertex $(i, j) \in V(P_m \times P_n)$, always we have the indices i and j are reduced modulo m and n, respectively. If *f* is a signed dominating function for $P_m \times P_n$, then we denote $f(K_j) = \sum_{i=1}^{m} f(i, j)$ of the weight of a column K_j and put $s_j = |f(K_j)|$. The sequence $(s_1, s_2, \ldots, s_n)$ is called a signed dominating sequence corresponding to *f*.

Remark 2.1. Since each vertex of K_1 and R_1 is 0-in-degree or 1-in-degree, then we have $f((i, 1)) = 1$ for i =1, 2, …,m and $f((1, j)) = 1$ for j =1, 2,…,n. So, always we have $s_1 = m$. Furthermore, always we consider the signed dominating sequence $(m, s_2, \ldots, s_n)$ for $\gamma_s(P_m \times P_n)$.

Remark 2.2. Since $f((1, j)) = 1$ and $f[(i, j)] \geq 1$, then the case $f((i, j)) = f((i+1, j)) = -1$ is not possible. So, $s_j \geq 0$ for j = 1, …, n. Furthermore, s_j is odd if m is odd and even when m is even.

Remark 2.3: Let *f* is a $\gamma_s(P_m \times P_n)$-function. Then $f[(r, s)] \geq 1$ for each $1 \leq r \leq m$ and each $1 \leq s \leq n$. If $(i, j) \notin V\{K_1 \cup R_1 \cup K_n \cup R_m\}$ and $f((i, j)) = -1$, then $f((i\pm1, j)) = f((i, j\pm1)) = 1$ because $f[(i, j)] \geq 1$, $f((i+1, j-1)) = 1$ because $f[(i+1, j)] \geq 1$ and $f((i-1, j+1)) = 1$ because $f[(i, j+1)] \geq 1$. On the other hand, if $f((i\pm1, j)) = f((i, j\pm1)) = 1$, $f((i-1, j-1)) = 1$ and $f((i+1, j+1)) = 1$, then we must have $f((i, j)) = -1$ since *f* is a minimum signed dominating function.

For the remainder of this section, let *f* be a signed domination function of $P_m \times P_n$ with signed dominating sequence $(m, s_2, \ldots, s_n)$. We need the following Lemma:

Lemma 2.4. If $s_j = k$ then $s_{j-1}, s_{j+1} \geq m - 2k$. Furthermore, $s_{j-1} + s_j \geq m - k$ and $s_j + s_{j+1} \geq m - k$.

Proof. Let $s_j = k$, then there are $(m – k)/2$ of vertices in K_j which get value -1. By Remark 2.3, K_{j+1} include at least $2(m – k)/2$ of vertices which get the value 1 and at most $m – (m –k) = k$ of vertices which has value -1. Hence, $s_{j+1} \geq m – 2k$. Furthermore, $s_j + s_{j+1} \geq m – k$. By the same argument (with considering $f(1, j) = 1$ for all j), we get $s_{j-1} \geq m – 2k$ and $s_{j-1} + s_j \geq m – k$. □

To determine $\gamma_s(P_8 \times P_n)$ we need the following proposition. Here, we have $s_j = 0,2,4,6$, or 8.

Proposition 2.5. For $j > 1$, the case $(s_j, s_{j+1}, s_{j+2}) = (2, 4, 2)$ is not possible. Furthermore, there are four cases for $(s_j, s_{j+1}, s_{j+2}) = (2, 4, 4)$ as follows:

1. $f(2, j) = f(5, j) = f(8, j) = f(3, j+1) = f(6, j+1) = f(4, j+2) = f(7, j+2) = -1$ and $f(i, k) = 1$ otherwise for $k = j, j+1, j+2$.
2. $f(2, j) = f(5, j) = f(7, j) = f(3, j+1) = f(8, j+1) = f(4, j+2) = f(6, j+2) = -1$ and $f(i, k) = 1$ otherwise for $k = j, j+1, j+2$.
3. $f(2, j) = f(4, j) = f(7, j) = f(5, j+1) = f(8, j+1) = f(2, j+2) = f(6, j+2) = -1$ and $f(i, k) = 1$ otherwise for $k = j, j+1, j+2$.
4. $f(2, j) = f(4, j) = f(7, j) = f(5, j+1) = f(8, j+1) = f(3, j+2) = f(6, j+2) = -1$ and $f(i, k) = 1$ otherwise for $k = j, j+1, j+2$.

The proof comes immediately by drawing those cases. □

Theorem 2.6. For $n \geq 5$, is

$$\gamma_s(P_8 \times P_n) = \begin{cases} \dfrac{10n + 12}{3} & \text{if } n \equiv 0 \pmod 3, \\ \dfrac{10n + 14}{3} & \text{if } n \equiv 1 \pmod 3, \\ \dfrac{10n + 10}{3} & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof. We define a function
$f(2, 3j-1) = f(5, 3j-1) = f(8, 3j-1) = -1$ for $1 \leq j \leq \lceil (n-1)/3 \rceil$, $f(3, 3j) = f(6, 3j) = -1$ for $1 \leq j \leq \lceil (n-2)/3 \rceil$, $f(4, 3j+1) = f(7, 3j+1) = -1$ for $1 \leq j \leq \lceil (n-3)/3 \rceil$ and $f(i, j) = 1$ otherwise. We have f is a SDF for $P_8 \times P_n$ (For an illustration $\gamma_s(P_8 \times P_7)$, see Figure 1). Therefore,

$\gamma_s(P_8 \times P_n) \leq 8n – 2(3n/3 + 2n/3 + 2(n-3)/3) = (10n + 12)/3$ when $n \equiv 0 \pmod 3$. (1)
$\gamma_s(P_8 \times P_n) \leq 8n – 2(3(n-1)/3 + 2(n-1)/3 + 2(n-1)/3) = (10n + 14)/3$ when $n \equiv 1 \pmod 3$. (2)
$\gamma_s(P_8 \times P_n) \leq 8n – 2(3(n+1)/3 + 2(n-2)/3 + 2(n-2)/3) = (10n + 10)/3$ when $n \equiv 2 \pmod 3$. (3)

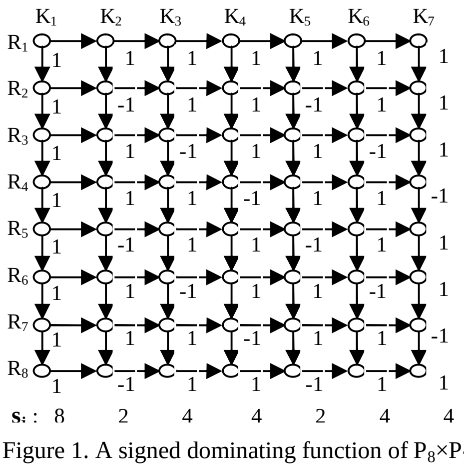

Figure 1. A signed dominating function of $P_8 \times P_7$.

By Remarks 2.1 and 2.2, $s_1 = 8$ and $s_j = 0, 2, 4, 6$ or 8. By Lemma 2.4, if $s_j = 0$ then $s_{j-1}, s_{j+1} = 8$ where $1 < j < n$, and if $s_j = 2$ then $s_{j+1} \geq 4$. By Proposition 2.5, if $s_j = 2$ then $s_{j+1} \geq 4$, furthermore

$s_j + s_{j+1} + s_{j+2} \geq 10$ where $j \leq n-2$. When $s_j = 0$, we can modifying the sequence $(s_1, \ldots, s_n)$ to $(s'_1, \ldots, s'_n)$ (is not necessarily a signed dominating sequence of $P_8 \times P_n$) as follows:

For $3 \leq j \leq n-2$, if $s_j = 0$ then we put:
$s'_j = s_j + 4$, $s'_{j-1} = s_{j-1} - 2$, $s'_{j+1} = s_{j+1} - 2$ and $s'_j = s_j$, otherwise.
(If $s_j = 0$ for $j = 2$ or $n-1$, then $s'_j = s_j + 4$ and $s'_{j+1} = s_{j+1} - 4$, also if $s_n = 0$ then $s'_n = s_n + 4$ and $s'_{n-1} = s'_{n-1} - 4$). The obtained sequence $(s'_1, \ldots, s'_n)$ has required properties $s_j \geq 2$ for all j and if $s'_j = 2$ then $s_{j+1}, s_{j+2} \geq 4$. Hence, $s_j + s_{j+1} + s_{j+2} \geq 10$ for $2 \leq j \leq n-2$. By minimality of the signed domination number of $P_8 \times P_n$, we can assume the following order $(s_j, s_{j+1}, s_{j+2}) = (2, 4, 4)$. Thus

For $n \equiv 0 \pmod 3$:

$$\gamma_s(P_8 \times P_n) = s_1 + \sum_{j=2}^{n-2} s_j + s_{n-1} + s_n \geq 8 + 10(n-3)/3 + 6 = (10n+12)/3.$$

For $n \equiv 1 \pmod 3$:

$$\gamma_s(P_8 \times P_n) = s_1 + \sum_{j=2}^{n} s_j \geq 8 + 10(n-1)/3 = (10n+14)/3.$$

For $n \equiv 2 \pmod 3$:

$$\gamma_s(P_8 \times P_n) = s_1 + \sum_{j=2}^{n-1} s_j + s_n \geq 8 + 10(n-2)/3 + 2 = (10n+10)/3.$$

These together with (1), (2) and (3) the proof of Theorem 2.6 is complete. □

Now, we consider the signed domination number of $P_9 \times P_n$. Here, we have s_j is odd and $1 \leq s_j \leq 9$.

Proposition 2.7. There is one possible for (s_j, s_{j+1}) (3, 3), it is
$f(2, j) = f(5, j) = f(8, j) = f(3, j+1) = f(6, j+1) = f(9, j+1) = -1$, otherwise $f(i, d) = 1$ for $d = j, j+1$.

The proof comes immediately by drawing those cases. □

Proposition 2.8. If $s_j = 3$ then $s_j + s_{j+1} + s_{j+2} \geq 11$.

Proof. By Remark 2.2, s_j = 1, 3, 5, 7 or 9. We have $s_j = 3$, by Lemma 2.4, $s_{j+1} \geq 3$. If $s_{j+1} \geq 7$, we obtained the required. If $s_{j+1} = 5$ then by Lemma 2.4, we have $s_{j+2} \geq 3$ (otherwise $s_{j+1} \geq 7$), again gets the required. Let $s_{j+1} = 3$, by Proposition 2.7 is $f(4, j+2) = f(7, j+2) = -1$, therefore $s_{j+2} \geq 5$. Finally, for all cases we conclude that $s_j + s_{j+1} + s_{j+2} \geq 11$. □

Theorem 2.9. For $n \geq 5$, is

$$\gamma_s(P_9 \times P_n) = \begin{cases} \dfrac{11n+12}{3} & \text{if } n \equiv 0 \pmod 3, \\ \dfrac{11n+16}{3} & \text{if } n \equiv 1 \pmod 3, \\ \dfrac{11n+14}{3} & \text{if } n \equiv 2 \pmod 3). \end{cases}$$

Proof. We define a function f as follows:
$f(2, 3j-1) = f(5, 3j-1) = f(8, 3j-1) = -1$ for $1 \leq j \leq \lceil (n-1)/3 \rceil$, $f(3, 3j) = f(6, 3j) = f(9, 3j) = -1$ for $1 \leq j \leq \lceil (n-2)/3 \rceil$, $f(4, 3j+1) = f(7, 3j+1) = -1$ for $1 \leq j \leq \lceil (n-3)/3 \rceil$ and $f(i, j) = 1$ otherwise. We have f is a SDF for $P_9 \times P_n$. See Figure 2, For an illustration $\gamma_s(P_9 \times P_9)$. Therefore,

$\gamma_s(P_9 \times P_n) \leq 9n - 2(3n/3 + 3n/3 + 2(n-3)/3) = (11n + 12)/3$ when $n \equiv 0 \pmod 3$). (4)
$\gamma_s(P_9 \times P_n) \leq 9n - 2(3(n-1)/3 + 3(n-1)/3 + 2(n-1)/3) = (11n + 16)/3$ when $n \equiv 1 \pmod 3$). (5)

$\gamma_s(P_9 \times P_n) \le 9n - 2(3(n+1)/3 + 3(n-2)/3 + 2(n-2)/3) = (11n+14)/3$ when $n \equiv 2 \pmod 3$. (6)

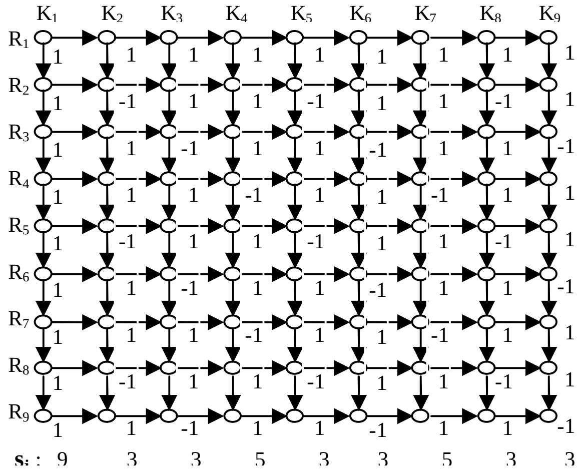

Figure 2. A signed dominating function of $P_9 \times P_9$.

By Remarks 2.1 and 2.2, $s_1 = 9$ and $s_j = 1, 3, 5, 7$ or 9. If $s_j \ge 3$ for all j, then by Proposition 2.8, we obtained the required. Let $s_j = 1$ for some j. By Lemma 2.4, we have $s_j, s_{j+1} \ge 7$. Then by modifying the sequence $(s_1, \ldots, s_n)$ to $(s'_1, \ldots, s'_n)$ (is not necessarily a signed dominating sequence of $P_9 \times P_n$ as follows:

For $3 \le j \le n-2$, if $s_j = 1$ then we put:
$s'_j = s_j + 4$, $s'_{j-1} = s_{j-1} - 2$, $s'_{j+1} = s_{j+1} - 2$ and $s'_j = s_j$, otherwise.
(If $s_j = 1$ for $j = 2$ or $n-1$, then $s'_j = s_j + 4$ and $s'_{j+1} = s_{j+1} - 4$, also if $s_n = 1$ then $s'_n = s_n + 4$ and $s'_{n-1} = s'_{n-1} - 4$). The obtained sequence $(s'_1, \ldots, s'_n)$ has required properties $s_j \ge 3$ for all j. Furthermore $s_j + s_{j+1} + s_{j+2} \ge 11$ for $2 \le j \le n-2$. By minimality of the signed domination number of $P_9 \times P_n$, we can assume the following order $(s_j, s_{j+1}, s_{j+2}) = (3, 3, 5)$. Thus
For $n \equiv 0 \pmod 3$:

$$\gamma_s(P_9 \times P_n) = s_1 + \sum_{j=2}^{n-2} s_j + s_{n-1} + s_n \ge 9 + 11(n-3)/3 + 6 = (11n+12)/3.$$

For $n \equiv 1 \pmod 3$:

$$\gamma_s(P_9 \times P_n) = s_1 + \sum_{j=2}^{n} s_j \ge 9 + 11(n-1)/3 = (11n+16)/3.$$

For $n \equiv 2 \pmod 3$:

$$\gamma_s(P_9 \times P_n) = s_1 + \sum_{j=2}^{n-1} s_j + s_n \ge 9 + 11(n-2)/3 + 3 = (11n+14)/3.$$

These together with (4), (5) and (6) the proof of Theorem 2.9 is complete. □

Next, we consider the signed domination number of $P_{10} \times P_n$. Here, we have $s_j = 0, 2, 4, 6, 8$ or 10.

Proposition 2.10. There are only six possibilities for $(s_j, s_{j+1}) = (2, 6)$. Furthermore, the case $(s_j, s_{j+1}, s_{j+2}) = (2, 6, 2)$ is not possible.

Proof. By the drawing, we have only these cases for $(s_j, s_{j+1}) = (2, 6)$ which are:
1. $f(2, j) = f(4, j) = f(6, j) = f(9, j) = -1$ and $f(7, j+1) = f(10, j+1) = -1$,
2. $f(2, j) = f(4, j) = f(7, j) = f(9, j) = -1$ and $f(5, j+1) = f(10, j+1) = -1$,
3. $f(2, j) = f(4, j) = f(7, j) = f(10, j) = -1$ and $f(5, j+1) = f(8, j+1) = -1$,
4. $f(2, j) = f(5, j) = f(7, j) = f(9, j) = -1$ and $f(3, j+1) = f(10, j+1) = -1$,
5. $f(2, j) = f(5, j) = f(7, j) = f(10, j) = -1$ and $f(3, j+1) = f(8, j+1) = -1$,
6. $f(2, j) = f(5, j) = f(8, j) = f(10, j) = -1$ and $f(3, j+1) = f(6, j+1) = -1$.

We note that $(s_j, s_{j+1}, s_{j+2}) = (2, 6, 2)$ is not possible, this for all the previous cases. □

Proposition 2.11. If $s_j = 2$ the $s_j + s_{j+1} + s_{j+2} \geq 12$ for $2 \leq j \leq n-2$.

Proof. By Lemma 2.4, if $s_j = 2$ then $s_{j-1}, s_{j+1} \geq 6$. If $s_{j+1} \geq 8$, we obtained the required (when $s_{j+1} = 8$ is $s_{j+2} \geq 2$). Let $s_{j+1}= 6$, by applying Proposition 2.10 we get the required.□

Theorem 2.12. For $n \geq 5$, is $\gamma_s(P_{10}\times P_n) = 4n + 6$.

Proof. We define a function f as follows:
$f(2, 3j-1) = f(5, 3j-1) = f(8, 3j-1) = -1$ for $1 \leq j \leq \lceil (n-1)/3 \rceil$, $f(3, 3j) = f(6, 3j) = f(9, 3j) = -1$ for $1\leq j \leq \lceil (n-2)/3 \rceil$, $f(4, 3j+1) = f(7, 3j+1) = f(10, 3j+1) = -1$ for $1\leq j \leq \lceil (n-3)/3 \rceil$ and $f(i, j) = 1$ otherwise. We note that f is a SDF for $P_{10}\times P_n$. Furthermore, $s_j = 4$ for $j = 2, \ldots, n$. Hence

$$\gamma_s(P_{10} \times P_n) = s_1 + \sum_{j=2}^{n} s_j \leq 10 + 4(n-1) = 4n + 6. \qquad (7)$$

We will prove that $\gamma_s(P_{10}\times P_n) \geq 4n + 6$. For this we need the following claim.

Claim A. For $k \geq 3$ is $\sum_{j}^{j+k} s_j \geq 4(k+1)$ where $j \geq 2$.

Proof. Here, we consider the Claim at least for four columns. By Remarks 2.1 and 2.2, $s_1 = 10$ and $s_j = 0, 2, 4, 8$ or 10. If $s_j \geq 4$ for all j, then we get the required.

Assume that $s_j \leq 2$ for some $2 \leq j \leq n$. By Lemma 2.4, we have $s_{j-1}, s_{j+1} \geq 10 - 2s_j$, i.e. $s_{j-1}, s_{j+1} \geq 10$ when $s_j = 0$ and $s_{j-1}, s_{j+1} \geq 6$ when $s_j = 2$. From Propositions 2.10 and 2.11, we have $(s_j, s_{j+1}, s_{j+2}) = (2, 6, 2)$ is not possible and $s_j + s_{j+1} + s_{j+2} \geq 12$. Here we aim to calculate the summation $\sum_{j}^{j+k} s_j$. If $s_j \leq 2$ where $3 \leq j \leq n-1$, we can modifying the sequence $(\ldots, s_{j-1}, s_j, s_{j+1}, \ldots)$ to $(\ldots, s_{j-1} - (4 - s_j)/2, 4, s_{j+1} - (4 - s_j)/2, \ldots)$. While, if $s_j \leq 2$ for $j = 2$ or n, then we put $s_j = 4$ and $s_{j+1} = s_{j+1} - (4 - s_j)$ and $s_{j-1} = s_{j-1} - (4 - s_j)$ for $j = 2$ or n, respectively. We repeat this process if necessary (for each $s_j \leq 2$), eventually leading to a sequence set which has the same summation of the basic sequence with $s_j \geq 4$ for all j. Note s_1 is still equal 10. Hence, $\sum_{j}^{j+k} s_j \geq 4(k+1)$. □

Now by applying Claim A, we get $\gamma_s(P_{10} \times P_n) = s_1 + \sum_{j=2}^{n} s_j \geq 10 + 4(n-1) = 4n + 6$.

By using this result together with (7), the proof of Theorem 2.12 is complete. □

Lemma 2.13. $\gamma_s(P_m \times P_n) \geq \max\{\frac{m(n+1)}{3}, \frac{n(m+1)}{3}\}$.

Proof. By Lemma 2.4, we have $s_{j-1} + 2s_j \geq m$ and $2s_j + s_{j+1} \geq m$. This implies that $s_{j-1} + 4s_j + s_{j+1} \geq 2m$. Then the following equations are true:

$s_2 + 4s_3 + s_4 \geq 2m$,
$s_3 + 4s_4 + s_5 \geq 2m$,
$s_4 + 4s_5 + s_6 \geq 2m$,

…………………,
…………………,
$s_{n-4} + 4s_{n-3} + s_{n-2} \geq 2m$,
$s_{n-3} + 4s_{n-2} + s_{n-1} \geq 2m$,
$s_{n-2} + 4s_{n-1} + s_n \geq 2m$. Thus

$6(s_1 + s_2 + \ldots + s_n) - (6s_1 + 5s_2 + s_3 + s_{n-1} + 5s_n) \geq (n - 3)\, 2m$.

But, we have $s_1 = m$, $2s_2 + s_3 \geq m$ and $s_{n-1} + 2s_n \geq m$ (By Lemma 2.4). Hence

$6\gamma_s(P_m \times P_n) \geq (n - 3)\, 2m + 6\, m + m + m + 3s_2 + 3s_n$. Then

$\gamma_s(P_m \times P_n) \geq m(n + 1)/3$.

Also, by changing the rows by columns gets:

$\gamma_s(P_m \times P_n) \geq n(m + 1)/3$. So,

$$\gamma_s(P_m \times P_n) \geq \max\{\frac{m(n+1)}{3}, \frac{n(m+1)}{3}\}. \qquad \square$$

3. Conclusions

This paper determined that exact value of the signed domination number of $P_m \times P_n$ for m = 8, 9, 10 and arbitrary n. By using same technique methods, our hope eventually lead to determination $\gamma_s(P_m \times P_n)$ for general m and n.

Based on the results in this paper and [14], we arrive to the following conjecture:

Conjecture 3. 1.

1. For $m \equiv 0 \pmod 3$ and $m, n \geq 3$, is
$$\gamma_s(P_m \times P_n) = \left\lceil \frac{(m+2)n}{3} \right\rceil + 2\left\lfloor \frac{m}{3} \right\rfloor - 2 : n \equiv 0, 2 \pmod 3,$$
$$\gamma_s(P_m \times P_n) = \left\lceil \frac{(m+2)n}{3} \right\rceil + 2\left\lfloor \frac{m}{3} \right\rfloor - 1 : n \equiv 1 \pmod 3.$$

2. For $m \equiv 1 \pmod 3$ and $m, n > 4$, is
$$\gamma_s(P_m \times P_n) = \left\lceil \frac{(m+2)n}{3} \right\rceil + 2\left\lfloor \frac{m}{3} \right\rfloor.$$

3. For $m \equiv 2 \pmod 3$ and $m, n > 2$, is
$$\gamma_s(P_m \times P_n) = \left\lceil \frac{(m+2)n}{3} \right\rceil + 2\left\lfloor \frac{m}{3} \right\rfloor : n \equiv 0,1 \pmod 3,$$
$$\gamma_s(P_m \times P_n) = \left\lceil \frac{(m+2)n}{3} \right\rceil + 2\left\lfloor \frac{m}{3} \right\rfloor - 1 : n \equiv 2 \pmod 3.$$

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