Highly Accurate Inference on the Sharpe Ratio for Autocorrelated Return Data

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Abstract

Performance measurement is an integral part of investment analysis and risk management. The Sharpe ratio is one of the most prominently used measures of performance of an investment with respect to return and risk. While most of the literature has addressed the large sample properties of the Sharpe ratio, it is important to compare the performance of these methods when only a small sample of observations is available. We propose a third-order asymptotic likelihood-based method to obtain highly accurate inference for the Sharpe ratio when returns are assumed to follow a Gaussian autoregressive process. Through real life examples, we show that results can vary vastly according to the methods used to obtain them. Results from simulation studies illustrate that our proposed method is superior to the existing methods used in the literature even with a very small number of observations.

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1 Introduction

Sharpe (1966) defined the ratio of a fund's excess return per unit of risk measured by its standard deviation, as a measure of investment performance with respect to return and risk. This ratio is known as the Sharpe ratio and it is so popular in investment analysis that investments are often ranked and evaluated based on this ratio. This paper focuses on using asymptotic likelihood theory to obtain accurate inference for the Sharpe ratio when returns are assumed to follow a Gaussian autocorrelated process.

Approximations aimed at improving the accuracy of likelihood methods have been proposed over the past three decades. Among them, Barndorff-Nielsen (1986, 1990) introduced the modified signed log likelihood ratio statistic to approximate tail probability with order of convergence $O(n^{-3/2})$. However, the Barndoff-Nielsen (1986) method requires the existence of an ancillary statistic, which, in a general setting may not exist or, even when it does exist, it may not be unique. Fraser (1988, 1990), Fraser and Reid (1995), and Fraser, Reid and Wu (1999), extended the modified signed log likelihood ratio statistic method to a general model setting.

The objective of this paper is to derive the modified signed log likelihood ratio statistic to obtain highly accurate inference for the Sharpe ratio when returns are assumed to be Gaussian autocorrelated. While most of the literature has addressed the large sample properties of the Sharpe ratio (see for instance, Lo (2002), Mertens (2002), Christie (2005), Bailey and Lopez de Prado (2012)), it is important to compare the performance of these methods when only a small sample of observations is available. Real life examples are used in this paper to illustrate that results obtained by the methods discussed in this paper can vary vastly. Simulation studies are then conducted to compare the accuracy of the methods.

Mathematically, the Sharpe ratio for a fund with an expected return μ and return standard deviation σ is

$$SR = \frac{\mu - r_f}{\sigma}, \tag{1}$$

where r_f is the risk-free rate of return of a benchmark fund. This ratio can be shown to be the slope between a risky and a risk-free fund in (μ, σ) space. According to the mean-variance theory developed by Markowitz (1952) and the Capital Asset Pricing Model (CAPM) developed by Sharpe (1964) and Lintner (1965), the Sharpe ratio of the market portfolio represents the slope of the Capital Market Line.

The natural estimator of the Sharpe ratio is given by:

$$\widehat{SR} = \frac{\hat{\mu} - r_f}{\hat{\sigma}}, \tag{2}$$

where $\hat{\mu}$ is the sample mean return of the fund and $\hat{\sigma}$ is the corresponding sample standard deviation. The return defined here is the log return of a fund

$$r_t = \log\left(\frac{p_t}{p_{t-1}}\right),$$

where p_t is the price of the fund at time t. In this paper, we consider inference for the one sample Sharpe ratio, and inference for the difference between two independent samples under a Gaussian AR(1) structure. Finally, we extend the discussion to the Gaussian AR(2) structure.

2 Likelihood-Based Inference

In this section, we review two standard first-order likelihood-based methods and then introduce a third-order likelihood-based method. Let $\mathbf{y} = (y_1, \ldots, y_n)'$ be a sample from a population with density function given by $f(\cdot, \boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is a p-dimensional parameter and $p < n$. Then the likelihood function of $\boldsymbol{\theta}$ is given as follows:

$$\mathcal{L}(\boldsymbol{\theta}; \mathbf{y}) = c \cdot \prod_{i=1}^{n} f(y_i; \boldsymbol{\theta}),$$

where $c = c(\mathbf{y}) \in (0, +\infty)$ is an arbitrary constant. The log likelihood function is defined as:

$$\ell(\boldsymbol{\theta}) \quad = \quad \ell(\boldsymbol{\theta}; \mathbf{y}) = a + \sum_{i=1}^{n} \log f(y_i; \boldsymbol{\theta}), \tag{3}$$

where $a \in \Re$ is an arbitrary constant independent of $\boldsymbol{\theta}$. Moreover, we denote $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\boldsymbol{\theta})$ to be the vector of nuisance parameters, and $\psi = \psi(\boldsymbol{\theta})$ to be our scalar parameter of interest.

2.1 First-order likelihood-based methods

One of the most commonly used techniques to obtain inference for $\psi = \psi(\boldsymbol{\theta})$ is based on the asymptotic distribution of the maximum likelihood estimator (mle) of ψ. Let $\hat{\boldsymbol{\theta}}$ be the mle of $\boldsymbol{\theta}$, which satisfies the first order conditions:

$$\ell_{\boldsymbol{\theta}}(\hat{\boldsymbol{\theta}}) = \left.\frac{\partial \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right|_{\hat{\boldsymbol{\theta}}} = \mathbf{0}.$$

With the regularity conditions stated in Cox and Hinkley (1979), $\hat{\boldsymbol{\theta}}$ is asymptotically distributed as a normal distribution with mean $\boldsymbol{\theta}$ and asymptotic variance given by the inverse of the Fisher expected information

$$var(\hat{\boldsymbol{\theta}}) \approx \left[E\left(-\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'}\right)\right]^{-1}.$$

Calculating the Fisher expected information can be complicated, but it can be estimated by the observed information evaluated at the mle. Hence, we have

$$\hat{var}(\hat{\boldsymbol{\theta}}) \approx \left[j(\hat{\boldsymbol{\theta}})\right]^{-1} = \left[-\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right]^{-1}_{\hat{\boldsymbol{\theta}}}.$$

By applying the delta method, the standardized mle of ψ can simply be stated as

$$q = q(\psi) = \frac{\hat{\psi} - \psi}{\sqrt{\hat{var}(\hat{\psi})}}.$$

This statistic is asymptotically distributed as a standard normal distribution with

$$\hat{var}(\hat{\psi}) \approx \psi'_{\boldsymbol{\theta}}(\hat{\boldsymbol{\theta}}) j^{-1}(\hat{\boldsymbol{\theta}}) \psi_{\boldsymbol{\theta}}(\hat{\boldsymbol{\theta}}),$$

where $\psi_{\boldsymbol{\theta}}(\hat{\boldsymbol{\theta}})$ is the derivative of $\psi(\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$ and is evaluated at the mle. If we define the significance function of ψ obtained from $q(\psi)$ to be

$$p(\psi) = \Phi(q(\psi)),$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution, then inference for ψ can be obtained directly from this significance function. For example, the p-value for testing $H_0 : \psi = \psi_0$ versus $H_a : \psi \neq \psi_0$ is $2 \min\{p(\psi_0), 1 - p(\psi_0)\}$. Moreover, a $(1 - \gamma)100\%$ confidence interval for ψ is (ψ_L, ψ_U) where:

$$\begin{aligned} \psi_L &= \min\left\{p^{-1}(\gamma/2),\ p^{-1}(1 - \gamma/2)\right\} \\ \psi_U &= \max\left\{p^{-1}(\gamma/2),\ p^{-1}(1 - \gamma/2)\right\}. \end{aligned}$$

Another commonly used asymptotic technique to obtain inference for ψ is based on the log likelihood ratio statistic, $2\left[\ell(\hat{\boldsymbol{\theta}}) - \ell(\hat{\boldsymbol{\theta}}_\psi)\right]$, where $\hat{\boldsymbol{\theta}}_\psi$ is the constrained mle obtained by maximizing $\ell(\boldsymbol{\theta})$ subject to the constraint $\psi(\boldsymbol{\theta}) = \psi$. One way to obtain the constrained mle is to apply the Lagrange multiplier technique which is to maximize the function

$$H(\alpha, \boldsymbol{\theta}) = \ell(\boldsymbol{\theta}) + \alpha\left[\psi(\boldsymbol{\theta}) - \psi\right],$$

with respect to $(\alpha, \boldsymbol{\theta})$, where α is the Lagrange multiplier. Hence $(\hat{\alpha}, \hat{\boldsymbol{\theta}}_\psi)$ satisfies the first order conditions:

$$\left.\frac{\partial H(\alpha, \boldsymbol{\theta})}{\partial(\alpha, \boldsymbol{\theta})}\right|_{(\hat{\alpha}, \hat{\boldsymbol{\theta}}_\psi)} = \mathbf{0}.$$

From this resultant maximization, we can define the tilted log likelihood function as

$$\tilde{\ell}(\boldsymbol{\theta}) = \ell(\boldsymbol{\theta}) + \hat{\alpha}\left[\psi(\boldsymbol{\theta}) - \psi\right].$$

This function has the property that $\tilde{\ell}(\hat{\boldsymbol{\theta}}_\psi) = \ell(\hat{\boldsymbol{\theta}}_\psi)$. The tilted log likelihood function is an important function for the proposed method, which will be discussed in the next section.

With the regularity conditions given in Cox and Hinkley (1979), the log likelihood ratio statistic is asymptotically distributed as a chi-square distribution with degrees of freedom being equal to $\dim(\psi)$. Since the dimension of ψ is equal to one, we have the signed log likelihood ratio statistic:

$$R = R(\psi) = \text{sgn}(\hat{\psi} - \psi)\sqrt{2\left[\ell(\hat{\boldsymbol{\theta}}) - \ell(\hat{\boldsymbol{\theta}}_{\psi})\right]}. \tag{4}$$

This statistic is asymptotically distributed as a standard normal distribution. Thus, the significance function of ψ obtained from $R(\psi)$ is

$$p(\psi) = \Phi(R(\psi))$$

and, as in the standardized mle case, inference for ψ can be obtained from this significance function.

Theoretically, both $q(\psi)$ and $R(\psi)$ have order of accuracy $O(n^{-1/2})$. Doganaksoy and Schmee (1993) showed that $R(\psi)$ tends to give better coverage properties than $q(\psi)$. Moreover, $R(\psi)$ is a parameterization invariant method, whereas $q(\psi)$ is not. However, $q(\psi)$ is more popular in practice because of its simplicity in calculations.

2.2 Third-order likelihood-based method

In recent years, there exists various improvements to obtain more accurate inference methodologies. In particular, Barndorff-Nielsen (1986, 1990) proposed the modified signed log likelihood ratio statistic

$$R^* = R^*(\psi) = R(\psi) - \frac{1}{R(\psi)} \log \frac{R(\psi)}{Q(\psi)}. \tag{5}$$

This statistic is asymptotically distributed as a standard normal distribution with order of accuracy $O(n^{-3/2})$. Note that $R(\psi)$ is the signed log likelihood ratio statistic and $Q(\psi)$ is a special statistic that depends on an ancillary statistic. This ancillary statistic needs to be constructed on a case-by-case basis. For the special case that ψ is a component parameter of the canonical parameter of a canonical exponential family model, $Q(\psi)$ is simply the standardized mle of ψ.

Fraser (1988, 1990), Fraser and Reid (1995), and Fraser, Reid and Wu (1999) derived a systematic method to obtain $Q(\psi)$ for a general model setting. The idea is to first obtain the locally defined canonical parameter $\boldsymbol{\varphi}(\boldsymbol{\theta})$, which takes the form

$$\boldsymbol{\varphi}(\boldsymbol{\theta}) = \mathbf{V}' \frac{\partial}{\partial \mathbf{y}} \ell(\boldsymbol{\theta}) \tag{6}$$

where

$$\mathbf{V} = \left.\frac{\partial \mathbf{y}}{\partial \boldsymbol{\theta}}\right|_{\hat{\boldsymbol{\theta}}} = -\left(\frac{\partial \mathbf{z}(\boldsymbol{\theta}, \mathbf{y})}{\partial \mathbf{y}}\right)^{-1} \left.\frac{\partial \mathbf{z}(\boldsymbol{\theta}, \mathbf{y})}{\partial \boldsymbol{\theta}}\right|_{\hat{\boldsymbol{\theta}}} \tag{7}$$

with $\mathbf{z}(\boldsymbol{\theta}, \mathbf{y})$ representing a pivotal quantity. Then $Q(\psi)$ expressed in the $\boldsymbol{\varphi}(\boldsymbol{\theta})$ scale is given by

$$Q = Q(\psi) = \text{sgn}(\hat{\psi} - \psi) \frac{\left|\chi(\hat{\boldsymbol{\theta}}) - \chi(\hat{\boldsymbol{\theta}}_\psi)\right|}{\sqrt{v\hat{a}r(\chi(\hat{\boldsymbol{\theta}}) - \chi(\hat{\boldsymbol{\theta}}_\psi))}}, \tag{8}$$

where

$$\chi(\boldsymbol{\theta}) = \psi'_{\boldsymbol{\theta}}(\hat{\boldsymbol{\theta}}_\psi)\boldsymbol{\varphi}_{\boldsymbol{\theta}}^{-1}(\hat{\boldsymbol{\theta}}_\psi)\boldsymbol{\varphi}(\boldsymbol{\theta}), \tag{9}$$

$\psi_{\boldsymbol{\theta}}(\boldsymbol{\theta})$ is the derivative of $\psi(\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$, $\boldsymbol{\varphi}_{\boldsymbol{\theta}}(\boldsymbol{\theta})$ is the derivative of $\boldsymbol{\varphi}(\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$, and $|\chi(\hat{\boldsymbol{\theta}}) - \chi(\hat{\boldsymbol{\theta}}_\psi)|$ measures the departure $\left|\hat{\psi} - \psi\right|$ in $\boldsymbol{\varphi}(\boldsymbol{\theta})$ scale. The estimated variance of $(\chi(\hat{\boldsymbol{\theta}}) - \chi(\hat{\boldsymbol{\theta}}_\psi))$ is given as

$$v\hat{a}r(\chi(\hat{\boldsymbol{\theta}}) - \chi(\hat{\boldsymbol{\theta}}_\psi)) = \frac{\psi'_{\boldsymbol{\theta}}(\hat{\boldsymbol{\theta}}_\psi)\tilde{\mathbf{j}}^{-1}(\hat{\boldsymbol{\theta}}_\psi)\psi_{\boldsymbol{\theta}}(\hat{\boldsymbol{\theta}}_\psi)\left|\tilde{\mathbf{j}}(\hat{\boldsymbol{\theta}}_\psi)\right|\left|\boldsymbol{\varphi}_{\boldsymbol{\theta}}(\hat{\boldsymbol{\theta}}_\psi)\right|^{-2}}{\left|\mathbf{j}(\hat{\boldsymbol{\theta}})\right|\left|\boldsymbol{\varphi}_{\boldsymbol{\theta}}(\hat{\boldsymbol{\theta}})\right|^{-2}}, \tag{10}$$

with $\tilde{\mathbf{j}}(\hat{\boldsymbol{\theta}}_\psi)$ being the observed information calculated from the tilted likelihood function evaluated at the constrained mle $\hat{\boldsymbol{\theta}}_\psi$.

In the next section, we apply the third-order likelihood methodology for inference on the Sharpe ratio. An example is presented to illustrate the application of the proposed method along with some existing methods in the statistical literature. Simulation results are then conducted and presented to compare the accuracy of the methods discussed in this paper.

3 The One Sample Sharpe Ratio under AR(1) Returns

In this section, we consider the use of the method for inference on the one sample Sharpe ratio under AR(1) returns. Consider a fund with log returns at time t denoted by r_t, where $t = 1, 2, \ldots, T$. Under a Gaussian AR(1) assumption on this return series, we have the following basic setting:

$$\begin{cases} r_t = \mu + \epsilon_t^* & t \geq 1; \\ \epsilon_t^* = \rho\epsilon_{t-1}^* + \sigma v_t & t \geq 2; \\ v_t \sim N(0,1) & t \geq 2. \end{cases}$$

Additionally, to make this AR(1) process stationary, we require $|\rho| < 1$.

A stationary process is a stochastic process whose joint probability distribution function does not change when shifted in time. Consequently, parameters such as the mean and variance do not change over time and do not follow any trends. Thus, for a stationary AR(1) process, we have:

$$r_t \sim N\left(\mu, \frac{\sigma^2}{1-\rho^2}\right), \text{ for } t \geq 1 \text{ and} \tag{11}$$

$$cov\left(r_i, r_j\right) = \frac{\sigma^2 \rho^{|i-j|}}{1-\rho^2}, \quad \text{for any } i,j = 1,2,\ldots, \text{T.} \tag{12}$$

Both expressions in (**??**) and (**??**) are independent of t, which obviously conform to stationarity. From (**??**), the parameter of interest and its corresponding derivative are:

$$\psi\left(\boldsymbol{\theta}\right) = \frac{\mu - r_f}{\sqrt{var\left(r_t\right)}} = \frac{\mu - r_f}{\sqrt{\frac{\sigma^2}{1-\rho^2}}} \ ,$$

$$\psi_{\boldsymbol{\theta}}\left(\boldsymbol{\theta}\right) = \left(\psi_\rho\left(\boldsymbol{\theta}\right), \psi_\mu\left(\boldsymbol{\theta}\right), \psi_{\sigma^2}\left(\boldsymbol{\theta}\right)\right)',$$

where $\boldsymbol{\theta} = (\rho, \mu, \sigma^2)'$.

For a clearer understanding of our testing procedure, it is useful to rewrite our model in reduced matrix formulations. We do so as follows:

$$\mathbf{r} = \mu\mathbf{1} + \sigma\boldsymbol{\epsilon} = \mu\mathbf{1} + \boldsymbol{\epsilon}^*,$$

where $\mathbf{1}$ is the one vector, and

$$\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_T)' \sim N\left(\mathbf{0}, \boldsymbol{\Omega} = \left(\frac{\rho^{|i-j|}}{1-\rho^2}\right)_{ij}\right).$$

Additionally, we will require the inverse matrix of $\boldsymbol{\Omega}$, its Cholesky decomposition, and its derivative matrix. Specifically,

$$\mathbf{A} = (a_{ij}) = \boldsymbol{\Omega}^{-1} = \begin{pmatrix} 1 & -\rho & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & -\rho & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\rho & 1+\rho^2 & -\rho \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\rho & 1 \end{pmatrix} = \mathbf{L}'\mathbf{L} \ ,$$

and

$$\mathbf{L} = (l_{ij}) = \begin{pmatrix} \sqrt{1-\rho^2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\rho & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\rho & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\rho & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\rho & 1 \end{pmatrix}. \tag{13}$$

Notice that the dependence of matrices $\mathbf{A}$ and $\mathbf{L}$ are only on the parameter ρ. We define $A_\rho = \left(\frac{da_{ij}}{d\rho}\right)$, $A_{\rho\rho} = \left(\frac{d^2 a_{ij}}{d\rho^2}\right)$, and $L_\rho = \left(\frac{dl_{ij}}{d\rho}\right)$.

The probability density function of $\mathbf{r}$ is given by

$$\begin{aligned} f(\mathbf{r};\boldsymbol{\theta}) &= (2\pi)^{-\frac{n}{2}} \left|\sigma^2\boldsymbol{\Omega}\right|^{-\frac{1}{2}} e^{-\frac{1}{2}(\mathbf{r}-\mu\cdot\mathbf{1})'(\sigma^2\boldsymbol{\Omega})^{-1}(\mathbf{r}-\mu\cdot\mathbf{1})} \\ &= \prod_{i=2}^{n} \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(r_i-\mu-\rho(r_{i-1}-\mu))^2}\right) \frac{1}{\sqrt{2\pi\frac{\sigma^2}{1-\rho^2}}} e^{-\frac{1}{2\frac{\sigma^2}{1-\rho^2}}(r_1-\mu)^2}. \end{aligned}$$

The log likelihood function can be written as

$$\ell(\boldsymbol{\theta}) = a - \frac{n}{2}\log\sigma^2 + \frac{1}{2}\log\left(1-\rho^2\right) - \frac{1}{2\sigma^2}(\mathbf{r}-\mu\mathbf{1})'\mathbf{A}(\mathbf{r}-\mu\mathbf{1}).$$

The first and second derivatives of $\ell(\boldsymbol{\theta})$ with respect to the parameter $\boldsymbol{\theta}$ can be obtained in the normal fashion. And hence the quantities: $\hat{\boldsymbol{\theta}}, j(\hat{\boldsymbol{\theta}}), \hat{\boldsymbol{\theta}}_\psi, \tilde{j}(\hat{\boldsymbol{\theta}})$, and $R(\psi)$ can be obtained.

3.1 Existing methods

The following is a summary of some of the popular methods in the literature used to obtain inference for the one sample Sharpe ratio:

1. Lo (2002) showed that, for large sample size, the distribution of the estimator of the Sharpe ratio can be derived using the mle and delta method:

$$\widehat{SR} = \hat{\psi} = \psi(\hat{\boldsymbol{\theta}}) \xrightarrow{d} N\left(\psi(\boldsymbol{\theta}), \psi'_{\boldsymbol{\theta}}(\hat{\boldsymbol{\theta}}) var(\hat{\boldsymbol{\theta}}) \psi_{\boldsymbol{\theta}}(\hat{\boldsymbol{\theta}})\right). \tag{14}$$

   where $var(\hat{\boldsymbol{\theta}})$ is the inverse of the Fisher expected information.

2. Alternatively, we can replace the Fisher expected information from the result above with the observed information evaluated at $\hat{\boldsymbol{\theta}}$:

$$\widehat{SR} = \hat{\psi} = \psi\left(\hat{\boldsymbol{\theta}}\right) \xrightarrow{d} N\left(\psi(\boldsymbol{\theta}), \psi'_{\boldsymbol{\theta}}(\hat{\boldsymbol{\theta}}) \mathbf{j}^{-1}(\hat{\boldsymbol{\theta}}) \psi_{\boldsymbol{\theta}}(\hat{\boldsymbol{\theta}})\right). \tag{15}$$

3. The signed log likelihood ratio statistic:

$$R(\psi) = \text{sgn}(\hat{\psi} - \psi)\sqrt{2[l(\hat{\boldsymbol{\theta}}) - l(\hat{\boldsymbol{\theta}}_\psi)]} \xrightarrow{d} N(0,1).$$

4. Van Belle (2002) noted that under a null of $\mu = 0$, the t-statistic has a standard error of approximately

$$\sqrt{(1+\rho)/(1-\rho)}.$$

With the restriction that the method is applicable only when $\mu = 0$, it cannot be applied to obtain confidence intervals for $SR = \psi$. Hence, we will not include this method in our numerical studies.

3.2 Application of the third-order likelihood-based method

In order to apply the third-order likelihood-based method discussed in Section 2, we need the pivotal quantity $\mathbf{z}$. For this problem,

$$\mathbf{z} = \left(\boldsymbol{\sigma}^2\boldsymbol{\Omega}\right)^{-\frac{1}{2}}(\mathbf{r} - \mu\mathbf{1}) = \frac{\mathbf{L}\,(\mathbf{r} - \mu\mathbf{1})}{\sigma} = \begin{pmatrix} \sqrt{1-\rho^2}\epsilon_1 \\ \epsilon_2 - \rho\epsilon_1 \\ \vdots \\ \epsilon_n - \rho\epsilon_{n-1} \end{pmatrix}$$

is a pivotal quantity. We note this pivotal quantity coincides with the standard quantity used to estimate the parameters of an AR(1) model in the literature (see for example Hamilton (1994)). Consequently, $\mathbf{V}$ can be constructed from (**??**) and takes the form

$$\mathbf{V} = -\left(\frac{\partial \mathbf{z}}{\partial \mathbf{r}}\right)^{-1} \frac{\partial \mathbf{z}}{\partial \boldsymbol{\theta}}\bigg|_{\hat{\boldsymbol{\theta}}} = \begin{pmatrix} -\hat{\mathbf{L}}^{-1}\hat{\mathbf{L}}_\rho\,(\mathbf{r} - \hat{\mu}\mathbf{1}) & \mathbf{1} & \frac{\mathbf{r}-\hat{\mu}\cdot\mathbf{1}}{2\hat{\sigma}^2} \end{pmatrix}.$$

Note that $\mathbf{V}$ is a matrix of sample returns $\mathbf{r}$ and is not related to the parameter $\boldsymbol{\theta}$. Finally, the locally defined canonical parameter at the data can be obtained from (**??**) and it takes the form

$$\boldsymbol{\varphi}(\boldsymbol{\theta}) = (\varphi_1(\boldsymbol{\theta}), \varphi_2(\boldsymbol{\theta}), \varphi_3(\boldsymbol{\theta}))' = \mathbf{V}' \cdot \frac{\partial}{\partial \mathbf{r}} \ell(\boldsymbol{\theta}) = \mathbf{V}'\left(-\frac{1}{\sigma^2}\mathbf{A}\,(\mathbf{r} - \mu\mathbf{1})\right).$$

In addition, we also need the first order derivatives of the canonical parameter:

$$\boldsymbol{\varphi}_{\boldsymbol{\theta}}(\boldsymbol{\theta}) = \mathbf{V}'\left(\frac{\partial^2}{\partial \mathbf{r}\partial \boldsymbol{\theta}}\ell(\boldsymbol{\theta})\right) = \mathbf{V}'\begin{pmatrix} \frac{\partial^2\ell(\boldsymbol{\theta})}{\partial \mathbf{r}\partial\rho} & \frac{\partial^2\ell(\boldsymbol{\theta})}{\partial \mathbf{r}\partial\mu} & \frac{\partial^2\ell(\boldsymbol{\theta})}{\partial \mathbf{r}\partial\sigma^2} \end{pmatrix},$$

where

$$\frac{\partial^2\ell(\boldsymbol{\theta})}{\partial \mathbf{r}\partial\rho} = -\frac{1}{\sigma^2}\mathbf{A}_\rho\,(\mathbf{r} - \mu\mathbf{1}),\ \frac{\partial^2\ell(\boldsymbol{\theta})}{\partial \mathbf{r}\partial\mu} = \frac{1}{\sigma^2}\mathbf{A}\mathbf{1},\ \text{and}\ \frac{\partial^2\ell(\boldsymbol{\theta})}{\partial \mathbf{r}\partial\sigma^2} = \frac{1}{\sigma^4}\mathbf{A}\,(\mathbf{r} - \mu\mathbf{1}).$$

Given the above information, the signed log likelihood ratio statistic $R(\psi)$ can be constructed from (**??**), and $Q(\psi)$ can be obtained from (**??**), and finally, the proposed third-order likelihood approximation based on the Barndorff-Nielsen method $R^*(\psi)$ can be obtained from (**??**). Hence, the significance function of ψ can be obtained.

3.3 Numerical study

We will first consider a real life data problem and illustrate results obtained by the methods discussed in this paper are quite different. Hence, we use simulation studies to compare the accuracy of these methods.

3.3.1 Real life example

The data used for this example are taken from Ruppert (2004, page 113) and given in Table **??**. The data is a series of size 40 representing daily closing prices and returns for GE common stock in January and February 2000. Ruppert (2004, page 124) had tested and validated that these returns follow a Gaussian autoregressive process of order one. Moreover, the risk-free rate r_f is set as the average daily return of the 3-Month Treasury Bill during the above period. This value is 0.0001 for January and 0.0002 for February.[1]

Table **??** reports 95% confidence intervals for the Sharpe ratio separately for the January GE returns and February GE returns. Upon inspection of this table, we find the resulting confidence intervals obtained from the various methods to be quite different. Moreover, $p(\psi)$ for some specific ψ values are reported in Tables **??** and **??**. We can see that the results vary across the methods. For example, for the GE January returns data, suppose we are interested in testing:

$$H_0 : SR = \psi \leq 0 \text{ versus } H_a : SR = \psi > 0$$

The corresponding p-value from our proposed method is 0.1028. Using this p-value, we would not reject the null at a 90% significance level, while all other methods would in fact reject the null at a 90% significance level. Given the difference of confidence intervals produced by each method, it is naturally of statistical interest to study the accuracy of each method. To achieve this, simulation studies are performed.

3.3.2 Simulation study

Simulation studies to assess the performance of our third-order likelihood-based method relative to the existing methods are presented in this section. For some combinations of: $n = 26, 52$, $\mu - r_f = -1, 0, 1$, $\sigma^2 = 1$, and $\rho = -0.5, 0.5$, 10,000 Monte Carlo replications are performed from an AR(1) process with parameters $\boldsymbol{\theta} = (\rho, \mu, \sigma^2)'$. For each generated sample, the 95% confidence

[1]Source: Board of Governors of the Federal Reserve System (US), 3-Month Treasury Bill: Secondary Market Rate [DTB3], retrieved from FRED, Federal Reserve Bank of St. Louis https://research.stlouisfed.org/fred2/series/DTB3 on May 3, 2016.

interval for the Sharpe ratio is calculated. Note that, theoretically, the standard error (SE) for this simulation study can be calculated from the Bernoulli distribution and it is $\sqrt{0.05(0.95)/10000}$. The performance of each method is evaluated using the five criteria listed below:

1. The central coverage probability (CP): Proportion of the true Sharpe ratio that falls within the 95% confidence interval;

2. The lower tail error rate (LE): Proportion of the true Sharpe ratio that falls below the lower limit of the 95% confidence interval;

3. The upper tail error rate (UE): Proportion of the true Sharpe ratio that falls above the upper limit of the 95% confidence interval;

4. The average bias (AB): Defined as $AB = \frac{|LE-0.025|+|UE-0.025|}{2}$.

5. The average bias per unit of standard error (AB/SE). Notice that AB and AB/SE essentially provide the same information except the latter is easier for visualizing how each method compares to the nominal values.

The desired nominal values are 0.95, 0.025, 0.025, 0, and 0 respectively. These values reflect the desired properties of the accuracy and symmetry of the interval estimates of the Sharpe ratio.

The results are recorded in Table **??**. Both of Lo's methods (expected information (**Lo(exp)**) and observed information (**Lo(obs)**) share very similar results but they produce results that are far from the desired values. Results from the signed log likelihood ratio (**Likelihood Ratio**) are slightly better but they are still not satisfactory, especially when the sample size is small. The proposed method (**Proposed**) gives excellent results and outperforms all the other methods using all of our simulation criteria.

To examine how sample size affects the performance of each method, simulation studies under various combinations of the parameters and sample sizes are conducted. As in above, 10,000 Monte Carlo samples are generated and we obtain the 95% confidence interval for the Sharpe ratio for each combination of parameters and sample sizes. We present a visualization of our results. Figure **??** presents two plots with sample size varying from 6 to 100. Sample size is plotted versus AB/SE for the parameter settings: $\mu - r_f = 0$, $\sigma^2 = 1$, $\rho = 0.5$ and $\mu - r_f = 0$, $\sigma^2 = 1$, and $\rho = -0.5$. It is clearly evident that the proposed method outperforms all the existing methods discussed in this paper; this holds true even for an extremely small sample size of $n = 8$. The various other methods may need up to a sample size of over 100 or 200 (depending on the value of ρ) to give a reasonable AB/SE. Other combinations of parameters have also been calculated and the results are visually very similar to what has been presented in this particular case. Results are available from the authors upon request.

Similarly, to study the effect of ρ on the accuracy of each method, we performed simulation studies. Figure **??** presents a plot of ρ versus AB/SE with: $n = 26, \mu - r_f = 0$, and $\sigma^2 = 1$. From this graph, we can see how visually striking the proposed method is when compared to the other methods. The proposed method gives extremely accurate results even when $|\rho|$ is as high as 0.9.

4 Two Sample Comparison of the Sharpe Ratio under Independent AR(1) Returns

In this section, we consider two independent returns, each following a stationary Gaussian autoregressive process of order one. Our aim is to obtain inference for the difference between the two Sharpe ratios. Mathematically, let:

$$\mathbf{r}_k = \mu_k \mathbf{1}_k + \sigma_k \boldsymbol{\epsilon}_k = \mu_k \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + \sigma_k \begin{pmatrix} \epsilon_{k,1} \\ \vdots \\ \epsilon_{k,T_k} \end{pmatrix},$$

$$\epsilon_k \sim N\left(\mathbf{0}, \boldsymbol{\Omega}_k = \left(\frac{\rho_k^{|i-j|}}{1-\rho_k^2}\right)_{ij}\right),$$

where $k = 1, 2$. The inverse matrix of $\boldsymbol{\Omega}_k$, its Cholesky decomposition, and its derivatives matrix are given in the previous subsection.

Let $\boldsymbol{\theta} = (\boldsymbol{\theta}_1', \boldsymbol{\theta}_2')' = (\rho_1, \mu_1, \sigma_1^2, \rho_2, \mu_2, \sigma_2^2)'$. The parameter of interest is $\psi(\boldsymbol{\theta}) = \psi$, where

$$\psi(\boldsymbol{\theta}) = \psi_1(\boldsymbol{\theta}_1) - \psi_2(\boldsymbol{\theta}_2) = \frac{\mu_1 - r_{f1}}{\sqrt{\frac{\sigma_1^2}{1-\rho_1^2}}} - \frac{\mu_2 - r_{f2}}{\sqrt{\frac{\sigma_2^2}{1-\rho_2^2}}}.$$

The log likelihood function for return k is given as

$$\ell_k(\boldsymbol{\theta}_k) = a_k - \frac{n_k}{2}\log\sigma_k^2 + \frac{1}{2}\log(1-\rho_k^2) - \frac{1}{2\sigma_k^2}(\mathbf{r}_k - \mu_k\mathbf{1}_k)'\mathbf{A}_k(\mathbf{r}_k - \mu_k\mathbf{1}_k).$$

Thus, the joint log likelihood function is

$$\begin{aligned}\ell(\boldsymbol{\theta}) = a - \frac{n_1}{2}\log\sigma_1^2 + \frac{1}{2}\log\left(1-\rho_1^2\right) - \frac{1}{2\sigma_1^2}(\mathbf{r}_1 - \mu_1\mathbf{1}_1)'\mathbf{A}_1(\mathbf{r}_1 - \mu_1\mathbf{1}_1) \\ - \frac{n_2}{2}\log\sigma_2^2 + \frac{1}{2}\log\left(1-\rho_2^2\right) - \frac{1}{2\sigma_2^2}(\mathbf{r}_2 - \mu_2\mathbf{1}_2)'\mathbf{A}_2(\mathbf{r}_2 - \mu_2\mathbf{1}_2). \quad (16)\end{aligned}$$

All the basic required likelihood-based quantities can be derived as in Section 3. The details are available in Qi (2016).

4.1 Existing methods

As in Section 3, the existing methods are:

1. Extending the method in Lo (2002). Each sample will have its own estimator of the Sharpe ratio with asymptotic distribution given in (**??**), the asymptotic distribution of the difference of two Sharpe ratio estimators is therefore

$$\begin{aligned} \psi\left(\hat{\boldsymbol{\theta}}\right) &= \hat{\psi} = \widehat{SR}_1 - \widehat{SR}_2 \\ &\xrightarrow{d} N\left(\psi\left(\boldsymbol{\theta}\right), \psi'_{1\boldsymbol{\theta}_1}(\hat{\boldsymbol{\theta}}_1) var(\hat{\boldsymbol{\theta}}_1)\psi_{1\boldsymbol{\theta}_1}(\hat{\boldsymbol{\theta}}_1) + \psi'_{2\boldsymbol{\theta}_2}(\hat{\boldsymbol{\theta}}_2) var(\hat{\boldsymbol{\theta}}_2)\psi_{2\boldsymbol{\theta}_2}(\hat{\boldsymbol{\theta}}_2)\right) \end{aligned}$$

   where $var(\hat{\boldsymbol{\theta}}_k)$ is taken to be the inverse of the Fisher expected information obtained from $\ell_k(\boldsymbol{\theta}_k)$.

2. The Fisher expected information from the result above is replaced with the observed information evaluated at mle to obtain

$$\begin{aligned} \psi\left(\hat{\boldsymbol{\theta}}\right) &= \hat{\psi} = \widehat{SR}_1 - \widehat{SR}_2 \\ &\xrightarrow{d} N\left(\psi\left(\boldsymbol{\theta}\right), \psi'_{1\boldsymbol{\theta}_1}(\hat{\boldsymbol{\theta}}_1) j_1^{-1}(\hat{\boldsymbol{\theta}}_1)\psi_{1\boldsymbol{\theta}_1}(\hat{\boldsymbol{\theta}}_1) + \psi'_{2\boldsymbol{\theta}_2}(\hat{\boldsymbol{\theta}}_2) j_2^{-1}(\hat{\boldsymbol{\theta}}_2)\psi_{2\boldsymbol{\theta}_2}(\hat{\boldsymbol{\theta}}_2)\right) \end{aligned}$$

   where $j_k(\hat{\boldsymbol{\theta}}_k)$ is the expected expected information obtained from $\ell_k(\boldsymbol{\theta}_k)$ evaluated at $\hat{\boldsymbol{\theta}}_k$.

3. The signed log likelihood ratio statistic in (**??**):

$$R(\psi) = sgn(\hat{\psi} - \psi)\sqrt{2[l(\hat{\boldsymbol{\theta}}) - l(\hat{\boldsymbol{\theta}}_\psi)]}\ .$$

4.2 Application of the third-order method

The pivotal quantity $\mathbf{z}$ for this problem is chosen to coincide with the standard quantity used to estimate the parameters of an AR(1) model in the literature (see for example Hamilton (1994)):

$$\mathbf{z} = \begin{pmatrix} \sigma_1^2\boldsymbol{\Omega}_1 & \mathbf{0} \\ \mathbf{0} & \sigma_2^2\boldsymbol{\Omega}_2 \end{pmatrix}^{-\frac{1}{2}} \begin{pmatrix} \mathbf{r}_1 - \mu_1\mathbf{1}_1 \\ \mathbf{r}_2 - \mu_2\mathbf{1}_2 \end{pmatrix} = \begin{pmatrix} \frac{\mathbf{L}_1(\mathbf{r}_1 - \mu_1\mathbf{1}_1)}{\sigma} \\ \frac{\mathbf{L}_2(\mathbf{r}_2 - \mu_2\mathbf{1}_2)}{\sigma_2} \end{pmatrix}.$$

Hence, $\mathbf{V}$ can be constructed from (**??**) as:

$$\begin{aligned}
\mathbf{V} &= -\left(\frac{\partial \mathbf{z}}{\partial \mathbf{r}}\right)^{-1} \frac{\partial \mathbf{z}}{\partial \boldsymbol{\theta}}\bigg|_{\hat{\boldsymbol{\theta}}} \\
&= \begin{pmatrix} -\hat{\mathbf{L}}_1^{-1}\hat{\mathbf{L}}_{\rho_1}\left(\mathbf{r}_1 - \hat{\mu}_1\mathbf{1}_1\right) \quad \mathbf{1}_1 \quad \frac{\mathbf{r}_1-\hat{\mu}_1\mathbf{1}_1}{2\hat{\sigma}_1^2} & \mathbf{0} \\ \mathbf{0} & -\hat{\mathbf{L}}_2^{-1}\hat{\mathbf{L}}_{\rho_2}\left(\mathbf{r}_2 - \hat{\mu}_2\mathbf{1}_2\right) \quad \mathbf{1}_2 \quad \frac{\mathbf{r}_2-\hat{\mu}_2\mathbf{1}_2}{2\hat{\sigma}_2^2} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{V}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_2 \end{pmatrix}.
\end{aligned}$$

Finally, $\boldsymbol{\varphi}(\boldsymbol{\theta})$ can be obtained by (**??**) and is given as

$$\boldsymbol{\varphi}\left(\boldsymbol{\theta}\right) = \left(-\frac{1}{\sigma_1^2}\left(\mathbf{r}_1 - \mu_1\mathbf{1}_1\right)' \mathbf{A}_1\mathbf{V}_1, -\frac{1}{\sigma_2^2}\left(\mathbf{r}_2 - \mu_2\mathbf{1}_2\right)' \mathbf{A}_2\mathbf{V}_2\right)'.$$

With the above information, $R^*(\psi)$ can be obtained from (**??**).

4.3 Numerical Study

As in the previous section, we will first consider a real life data problem and illustrate how discordant the results can be based on the method used. We will then use simulation studies to compare the accuracy of the methods.

4.3.1 Real life example

The data used for this example are the same data as in the previous example and are recorded in Table **??**. However, this time, our interest is in comparing the performance of GE common stock in January 2000 with its performance in February 2000. We conduct this inference by evaluating the difference between the Sharpe ratios. The 95% confidence intervals obtained from the various methods discussed in this section, for the difference between the Sharpe ratios in January and February under a Gaussian AR(1) returns process, are presented in Table **??**. Again by looking at this table, it is clear the resulting confidence intervals are very different from each other. Table **??** records the p-values for various ψ values. For example, for testing $H_0 : \psi = SR_{Jan} - SR_{Feb} \leq 0.3$ vs $H_a : \psi > 0.3$ at 5% level of significance, both the **Lo(obs)** and **Likelihood Ratio** method will reject H_0 because the corresponding $p(0.3) < 0.05$, whereas the **Proposed** method gives a contradictory result because its $p(0.3) > 0.05$. Hence it is important to compare the accuracy of the methods. We do this through simulation studies.

4.3.2 Simulation study

To compare the performance of the proposed method relative to the existing methods, simulation studies are performed. For various combinations of: $n_1, n_2 = 20, 30$, $\mu - r_f = -1, 0, 1$, $\sigma^2 = 1$, and $\rho = -0.2, 0.7$, 10,000 Monte Carlo replications are generated. For each generated sample, the

95% confidence interval for the difference of the Sharpe ratios is calculated based on the methods discussed in this section. As in Section 3, the performance of a method is judged using the same five simulation criteria. The results are recorded in Table **??**. Both of Lo's methods with the expected information and with the observed information share very similar simulation results and they are far from the desired values. Results from the signed log likelihood ratio method are slightly better but still far from the desired values. The proposed method gives extremely accurate results regardless of the choice of the parameter values.

5 Likelihood Methodology for the One Sample Sharpe Ratio under AR(2) Returns

We extend the third-order methodology to the case of stationary Gaussian AR(2) returns. Consider a fund with returns at time t denoted by r_t, where $t = 1, 2, \ldots, T$. Under a Gaussian AR(2) assumption on this return series, we have the following basic setting:

$$\begin{cases} r_t = \mu + \epsilon_t^* & t \geq 1; \\ \epsilon_t^* = \rho_1 \epsilon_{t-1}^* + \rho_2 \epsilon_{t-2}^* + \sigma v_t & t \geq 3; \\ v_t \sim N(0, 1) & t \geq 2. \end{cases}$$

Additionally, to make this AR(2) process stationary, we assume:

$$\begin{cases} \rho_2 - \rho_1 < 1 \\ \rho_2 + \rho_1 < 1 \\ |\rho_2| < 1 \end{cases}$$

The calculations of each required quantity for the third-order likelihood-based method are simply a direct extension of the methodology illustrated in the previous sections. Details of these calculations are available in Ji (2016).

We provide a very basic simulation study to compare the accuracy of the methods discussed in this paper. Given the **Lo(obs)** and **Lo(exp)** methods from our AR(1) analyses produced very similar poor results, we only include the results from the observed information case here and label it **Lo(obs)**. 10,000 Monte Carlo replications are performed for a variety of parameter values for the AR(2) model. Without loss of generality, the risk-free rate of return was assumed to be 0. In Table **??** we provide only a representative sample of our simulation results. Criteria for comparison are those stated in Section 3. The results are consistent with those obtained in the previous sections: the proposed method outperformed both the Lo's method and the signed log likelihood ratio method.

6 Conclusion

The objective of this paper was to apply the third-order likelihood-based method to obtain highly accurate inference for the Sharpe ratio when returns are assumed to follow a Gaussian autoregressive process. We considered the one and two sample cases for an AR(1) process and shed light on the one sample AR(2) case. Through simulation analyses, we showed that our proposed method is superior to the existing methods in the literature. Future research could include the evaluation of more general time series models such as stationary autoregressive moving average (ARMA) models.

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Table 1: GE Daily Closing Prices and Daily Returns

Date	Closing prices	Return	Date	Closing prices	Return
1/3/2000	50.4792	-0.02143497	2/1/2000	45.1667	0.007408923
1/4/2000	48.6771	-0.036352677	2/2/2000	45.2812	0.002531846
1/5/2000	48.2604	-0.008597345	2/3/2000	45.8438	0.012348031
1/6/2000	48.2604	0	2/4/2000	47.2708	0.030652803
1/7/2000	49.3854	0.023043485	2/7/2000	46.2708	-0.021381676
1/10/2000	50.8646	0.029512366	2/8/2000	45.8229	-0.009727126
1/11/2000	50.5521	-0.006162713	2/9/2000	45.2917	-0.011660173
1/12/2000	50.6354	0.001646449	2/10/2000	45.0104	-0.006230219
1/13/2000	51.3229	0.01348611	2/11/2000	45.1458	0.003003678
1/14/2000	50.6979	-0.012252557	2/14/2000	44.9167	-0.005087589
1/18/2000	49.3958	-0.026019089	2/15/2000	45.4896	0.012674065
1/19/2000	49.5312	0.002737374	2/16/2000	45.2188	-0.005970799
1/20/2000	48.7292	-0.016324334	2/17/2000	44.2708	-0.021187612
1/21/2000	48.6979	-0.000642532	2/18/2000	42.8125	-0.033495201
1/24/2000	47.0625	-0.034159402	2/22/2000	42.5104	-0.007081364
1/25/2000	46.2292	-0.017864872	2/23/2000	43.5521	0.024209171
1/26/2000	46.8438	0.01320703	2/24/2000	43.1667	-0.008888558
1/27/2000	46.4688	-0.008037543	2/25/2000	42.7708	-0.009213738
1/28/2000	45.6875	-0.016956382	2/28/2000	43.0417	0.006313786
1/31/2000	44.8333	-0.018873571	2/29/2000	44.0208	0.022492836

Table 2: 95% Confidence Intervals for the Sharpe Ratio for January and February GE Returns

Method	CI for SR of January GE Returns	CI for SR of February GE Returns
Lo(exp)	(-1.0352, 0.1492)	(-0.5640, 0.4707)
Lo(obs)	(-1.0326, 0.1467)	(-0.5683, 0.4750)
Likelihood Ratio	(-1.0748, 0.2044)	(-0.5868, 0.5573)
Proposed	**(-1.0690, 0.2801)**	**(-0.6043, 0.5953)**

Table 3: p-values for the Sharpe Ratio under AR(1) GE January Returns

ψ	-1.2	-1.1	-1	-0.9	-0.8	-0.5	-0.3	-0.1
Lo(exp)	0.9939	0.9852	0.9674	0.9348	0.8813	0.5749	0.3181	0.1282
Lo(obs)	0.9941	0.9855	0.9680	0.9356	0.8823	0.5752	0.3173	0.1271
Likelihood Ratio	0.9890	0.9787	0.9600	0.9279	0.8763	0.5751	0.3180	0.1335
Proposed	**0.9890**	**0.9793**	**0.9621**	**0.9327**	**0.8856**	**0.6059**	**0.3554**	**0.1631**

ψ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
Lo(exp)	0.0713	0.0362	0.0167	0.0070	0.0026	0.0009	0.0003	0.0001
Lo(obs)	0.0705	0.0356	0.0163	0.0068	0.0025	0.0009	0.0003	0.0001
Likelihood Ratio	0.0796	0.0457	0.0257	0.0143	0.0080	0.0045	0.0026	0.0016
Proposed	**0.1028**	**0.0628**	**0.0377**	**0.0226**	**0.0137**	**0.0084**	**0.0054**	**0.0036**

Table 4: p-values for the Sharpe Ratio under AR(1) GE February Returns

ψ	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1
Lo(exp)	0.9978	0.9933	0.9820	0.9570	0.9096	0.8314	0.7194	0.5801
Lo(obs)	0.9977	0.9929	0.9812	0.9557	0.9078	0.8294	0.7177	0.5794
Likelihood Ratio	0.9953	0.9896	0.9774	0.9527	0.9069	0.8307	0.7194	0.5798
Proposed	**0.9941**	**0.9876**	**0.9742**	**0.9481**	**0.9009**	**0.8243**	**0.7145**	**0.5790**

ψ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
Lo(exp)	0.4299	0.2893	0.1751	0.0946	0.0453	0.0192	0.0072	0.0023
Lo(obs)	0.4304	0.2908	0.1771	0.0964	0.0467	0.0200	0.0076	0.0025
Likelihood Ratio	0.4308	0.2950	0.1875	0.1121	0.0640	0.0354	0.0193	0.0104
Proposed	**0.4351**	**0.3032**	**0.1978**	**0.1223**	**0.0728**	**0.0423**	**0.0244**	**0.0141**

Table 5: Simulation Results for Sharpe Ratio Under AR(1) Returns

Setting	Method	CP	LE	UE	AB	AB/ER
$n = 52$	Lo(exp)	0.9172	0.0414	0.0414	0.0164	10.25
$\mu = 0$	Lo(obs)	0.9172	0.0415	0.0413	0.0164	10.25
$\sigma^2 = 1$	Likelihood Ratio	0.9389	0.0301	0.0310	0.0056	3.47
$\rho = 0.5$	**Proposed**	**0.9522**	**0.0238**	**0.0240**	**0.0011**	**0.69**
$n = 52$	Lo(exp)	0.9471	0.0253	0.0276	0.0015	0.91
$\mu = 0$	Lo(obs)	0.9472	0.0253	0.0275	0.0014	0.87
$\sigma^2 = 1$	Likelihood Ratio	0.9458	0.0259	0.0283	0.0021	1.31
$\rho = -0.5$	**Proposed**	**0.9492**	**0.0243**	**0.0265**	**0.0011**	**0.69**
$n = 52$	Lo(exp)	0.9226	0.0527	0.0247	0.0140	8.75
$\mu = 1$	Lo(obs)	0.9235	0.0527	0.0238	0.0145	9.03
$\sigma^2 = 1$	Likelihood Ratio	0.9398	0.0364	0.0238	0.0063	3.94
$\rho = 0.5$	**Proposed**	**0.9481**	**0.0255**	**0.0264**	**0.0009**	**0.59**
$n = 52$	Lo(exp)	0.9515	0.0274	0.0211	0.0032	1.97
$\mu = 1$	Lo(obs)	0.9518	0.0274	0.0208	0.0033	2.06
$\sigma^2 = 1$	Likelihood Ratio	0.9534	0.0282	0.0184	0.0049	3.06
$\rho = -0.5$	**Proposed**	**0.9467**	**0.0261**	**0.0272**	**0.0017**	**1.03**
$n = 52$	Lo(exp)	0.9216	0.0232	0.0552	0.0160	10.00
$\mu = -1$	Lo(obs)	0.9225	0.0230	0.0545	0.0158	9.84
$\sigma^2 = 1$	Likelihood Ratio	0.9382	0.0228	0.0390	0.0081	5.06
$\rho = 0.5$	**Proposed**	**0.9497**	**0.0248**	**0.0255**	**0.0004**	**0.22**
$n = 52$	Lo(exp)	0.9531	0.0202	0.0267	0.0033	2.03
$\mu = -1$	Lo(obs)	0.9528	0.0205	0.0267	0.0031	1.94
$\sigma^2 = 1$	Likelihood Ratio	0.9544	0.0178	0.0278	0.0050	3.13
$\rho = -0.5$	**Proposed**	**0.9476**	**0.0256**	**0.0268**	**0.0012**	**0.75**
$n = 26$	Lo(exp)	0.8769	0.0578	0.0653	0.0366	22.84
$\mu = 0$	Lo(obs)	0.8780	0.0578	0.0642	0.0360	22.50
$\sigma^2 = 1$	Likelihood Ratio	0.9174	0.0383	0.0443	0.0163	10.19
$\rho = 0.5$	**Proposed**	**0.9462**	**0.0249**	**0.0289**	**0.0020**	**1.25**
$n = 26$	Lo(exp)	0.9465	0.0278	0.0257	0.0018	1.09
$\mu = 0$	Lo(obs)	0.9474	0.0274	0.0252	0.0013	0.81
$\sigma^2 = 1$	Likelihood Ratio	0.9443	0.0287	0.0270	0.0029	1.78
$\rho = -0.5$	**Proposed**	**0.9489**	**0.0258**	**0.0253**	**0.0005**	**0.34**
$n = 26$	Lo(exp)	0.8964	0.0724	0.0312	0.0268	16.75
$\mu = 1$	Lo(obs)	0.8966	0.0724	0.0310	0.0267	16.69
$\sigma^2 = 1$	Likelihood Ratio	0.9264	0.0476	0.0260	0.0118	7.38
$\rho = 0.5$	**Proposed**	**0.9497**	**0.0256**	**0.0247**	**0.0005**	**0.28**
$n = 26$	Lo(exp)	0.9526	0.0260	0.0214	0.0023	1.44
$\mu = 1$	Lo(obs)	0.9533	0.0264	0.0203	0.0031	1.91
$\sigma^2 = 1$	Likelihood Ratio	0.9568	0.0260	0.0172	0.0044	2.75
$\rho = -0.5$	**Proposed**	**0.9496**	**0.0243**	**0.0261**	**0.0009**	**0.56**

Table 6: 95% Confidence Intervals for the Difference between Sharpe Ratios for January and February GE Returns

Method	CI for Difference of SR
Lo(obs)	(-1.1836, 0.3910)
Likelihood Ratio	(-1.2300, 0.4159)
Proposed	**(-1.2239, 0.4762)**

Table 7: p-values for the Difference between Sharpe Ratios for January and February GE Returns

ψ	-1.5	-1.4	-1.2	-1.1	0	0.3	0.4	0.5	0.6
Lo(obs)	0.9970	0.9938	0.9773	0.9601	0.1619	0.0415	0.0237	0.0128	0.0066
Likelihood Ratio	0.9941	0.9897	0.9710	0.9531	0.1626	0.0448	0.0271	0.0160	0.0093
Proposed	**0.9941**	**0.9898**	**0.9719**	**0.9551**	**0.1878**	**0.0572**	**0.0361**	**0.0222**	**0.0134**

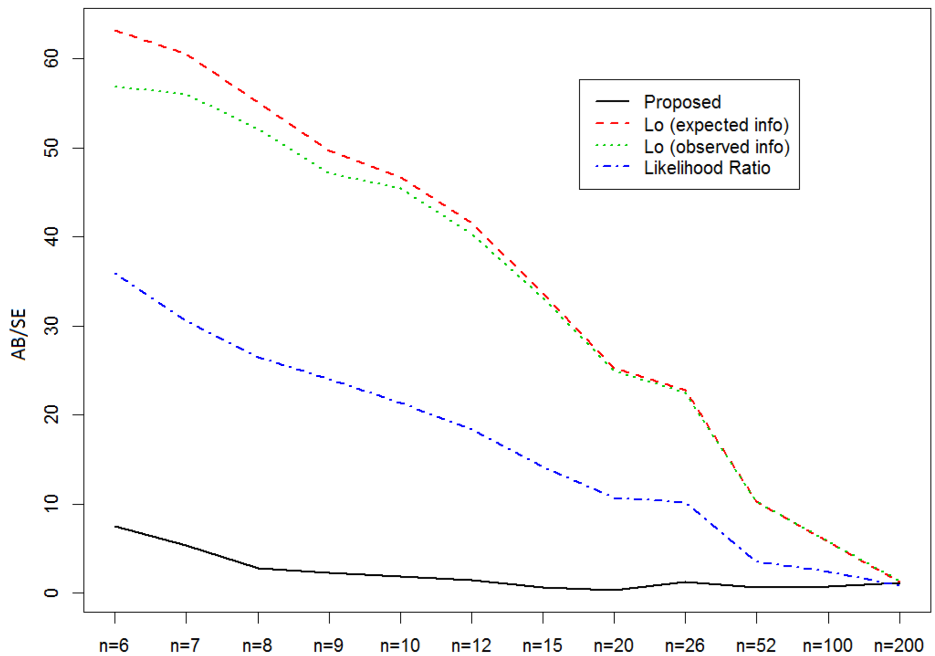
Table 8: Simulation Results for the Difference of Sharpe Ratios under AR(1) Returns.

Setting	Method	CP	LE	UE	AB	AB/SE
$n_1 = 20,\ \rho_1 = -0.2$	Lo(exp)	0.8744	0.0640	0.0616	0.0378	23.63
$n_2 = 30,\ \rho_2 = 0.7$	Lo(obs)	0.8759	0.0630	0.0611	0.0371	23.16
$\mu_1 = \mu_2 = 0$	Likelihood Ratio	0.9129	0.0450	0.0421	0.0186	11.59
$\sigma_1^2 = \sigma_2^2 = 1$	**Proposed**	**0.9479**	**0.0278**	**0.0243**	**0.0018**	**1.09**
$n_1 = 20,\ \rho_1 = 0.7$	Lo(exp)	0.8443	0.0783	0.0774	0.0529	33.03
$n_2 = 30,\ \rho_2 = 0.7$	Lo(obs)	0.8471	0.0767	0.0762	0.0515	32.16
$\mu_1 = \mu_2 = 0$	Likelihood Ratio	0.8893	0.0543	0.0564	0.0304	18.97
$\sigma_1^2 = \sigma_2^2 = 1$	**Proposed**	**0.9494**	**0.0248**	**0.0258**	**0.0005**	**0.31**
$n_1 = 20,\ \rho_1 = -0.2$	Lo(exp)	0.8935	0.0462	0.0603	0.0283	17.66
$n_2 = 30,\ \rho_2 = 0.7$	Lo(obs)	0.8955	0.0451	0.0594	0.0273	17.03
$\mu_1 = -1,\ \mu_2 = 0$	Likelihood Ratio	0.9225	0.0326	0.0449	0.0138	8.59
$\sigma_1^2 = \sigma_2^2 = 1$	**Proposed**	**0.9530**	**0.0252**	**0.0218**	**0.0017**	**1.06**
$n_1 = 20,\ \rho_1 = 0.7$	Lo(exp)	0.8483	0.0485	0.1032	0.0509	31.78
$n_2 = 30,\ \rho_2 = 0.7$	Lo(obs)	0.8516	0.0473	0.1011	0.0492	30.75
$\mu_1 = -1,\ \mu_2 = 0$	Likelihood Ratio	0.8901	0.0387	0.0712	0.0300	18.72
$\sigma_1^2 = \sigma_2^2 = 1$	**Proposed**	**0.9473**	**0.0235**	**0.0292**	**0.0029**	**1.78**
$n_1 = 20,\ \rho_1 = -0.2$	Lo(exp)	0.8707	0.0394	0.0899	0.0397	24.78
$n_2 = 30,\ \rho_2 = 0.7$	Lo(obs)	0.8720	0.0386	0.0894	0.0390	24.38
$\mu_1 = 0,\ \mu_2 = 1$	Likelihood Ratio	0.9144	0.0312	0.0544	0.0178	11.13
$\sigma_1^2 = \sigma_2^2 = 1$	**Proposed**	**0.9499**	**0.0258**	**0.0243**	**0.0008**	**0.47**
$n_1 = 20,\ \rho_1 = -0.2$	Lo(exp)	0.8377	0.0811	0.0812	0.0562	35.09
$n_2 = 20,\ \rho_2 = 0.7$	Lo(obs)	0.8407	0.0799	0.0794	0.0547	34.16
$\mu_1 = \mu_2 = 0$	Likelihood Ratio	0.8937	0.0529	0.0534	0.0282	17.59
$\sigma_1^2 = \sigma_2^2 = 1$	**Proposed**	**0.9491**	**0.0247**	**0.0262**	**0.0008**	**0.47**
$n_1 = 20,\ \rho_1 = 0.7$	Lo(exp)	0.8264	0.0858	0.0878	0.0618	38.63
$n_2 = 20,\ \rho_2 = 0.7$	Lo(obs)	0.8330	0.0817	0.0853	0.0585	36.56
$\mu_1 = \mu_2 = 0$	Likelihood Ratio	0.8790	0.0594	0.0616	0.0355	22.19
$\sigma_1^2 = \sigma_2^2 = 1$	**Proposed**	**0.9502**	**0.0239**	**0.0259**	**0.0010**	**0.62**
$n_1 = 20,\ \rho_1 = -0.2$	Lo(exp)	0.8570	0.0651	0.0779	0.0465	29.06
$n_2 = 20,\ \rho_2 = 0.7$	Lo(obs)	0.8597	0.0637	0.0766	0.0452	28.22
$\mu_1 = -1,\ \mu_2 = 0$	Likelihood Ratio	0.9004	0.0429	0.0567	0.0248	15.50
$\sigma_1^2 = \sigma_2^2 = 1$	**Proposed**	**0.9495**	**0.0272**	**0.0233**	**0.0020**	**1.22**
$n_1 = 20,\ \rho_1 = 0.7$	Lo(exp)	0.8303	0.0616	0.1081	0.0599	37.41
$n_2 = 20,\ \rho_2 = 0.7$	Lo(obs)	0.8352	0.0592	0.1056	0.0574	35.88
$\mu_1 = -1,\ \mu_2 = 0$	Likelihood Ratio	0.8782	0.0474	0.0744	0.0359	22.44
$\sigma_1^2 = \sigma_2^2 = 1$	**Proposed**	**0.9480**	**0.0248**	**0.0272**	**0.0012**	**0.75**
$n_1 = 20,\ \rho_1 = -0.2$	Lo(exp)	0.8416	0.0487	0.1097	0.0542	33.88
$n_2 = 20,\ \rho_2 = 0.7$	Lo(obs)	0.8445	0.0477	0.1078	0.0528	32.97
$\mu_1 = 0,\ \mu_2 = 1$	Likelihood Ratio	0.8994	0.0355	0.0651	0.0253	15.81
$\sigma_1^2 = \sigma_2^2 = 1$	**Proposed**	**0.9511**	**0.0241**	**0.0248**	**0.0006**	**0.34**

Table 9: Simulation Results for Sharpe Ratio Under AR(2) Returns

Setting	Method	CP	LE	UE	AB	AB/SE
$n = 50$,						
$\mu = 0$,	Lo(obs)	0.9402	0.0289	0.0309	0.0049	3.0625
$\sigma^2 = 1$,	Likelihood Ratio	0.9433	0.0284	0.02830	0.0034	2.0938
$\rho_1 = -0.5$,	**Proposed**	**0.9511**	**0.0241**	**0.0248**	**0.0006**	**0.3438**
$\rho_2 = -0.5$						
$n = 50$,						
$\mu = 0$,	Lo(obs)	0.9387	0.0309	0.0304	0.0057	3.5313
$\sigma^2 = 1$,	Likelihood Ratio	0.9410	0.0313	0.0277	0.0045	2.8125
$\rho_1 = 0.5$,	**Proposed**	**0.9495**	**0.0264**	**0.0241**	**0.0012**	**0.7188**
$\rho_2 = -0.5$						
$n = 50$,						
$\mu = 0$,	Lo(obs)	0.9461	0.0221	0.0318	0.0049	3.0313
$\sigma^2 = 1$,	Likelihood Ratio	0.9422	0.0284	0.0294	0.0039	2.4375
$\rho_1 = -0.8$,	**Proposed**	**0.9463**	**0.0254**	**0.0283**	**0.0019**	**1.1563**
$\rho_2 = -0.2$						
$n = 50$,						
$\mu = 0$,	Lo(obs)	0.9219	0.0397	0.0384	0.0141	8.7813
$\sigma^2 = 1$,	Likelihood Ratio	0.9335	0.0335	0.0330	0.0083	5.1563
$\rho_1 = 0.8$,	**Proposed**	**0.9479**	**0.0254**	**0.0267**	**0.0011**	**0.6562**
$\rho_2 = -0.2$						
$n = 50$,						
$\mu = 0$,	Lo(obs)	0.9088	0.0380	0.0532	0.0206	12.8750
$\sigma^2 = 1$,	Likelihood Ratio	0.9291	0.0298	0.0411	0.0105	6.5313
$\rho_1 = 0.4$,	**Proposed**	**0.9459**	**0.0242**	**0.0299**	**0.0029**	**1.7813**
$\rho_2 = 0.2$						
$n = 50$,						
$\mu = 0$,	Lo(obs)	0.9380	0.0291	0.0329	0.0060	3.7500
$\sigma^2 = 1$,	Likelihood Ratio	0.9443	0.0275	0.0282	0.0029	1.7813
$\rho_1 = -0.4$,	**Proposed**	**0.9506**	**0.0240**	**0.0254**	**0.0007**	**0.4375**
$\rho_2 = 0.2$						
$n = 50$,						
$\mu = 0$,	Lo(obs)	0.9095	0.0448	0.0457	0.0203	12.6563
$\sigma^2 = 1$,	Likelihood Ratio	0.9304	0.0351	0.0345	0.0098	6.1250
$\rho_1 = -0.2$,	**Proposed**	**0.9472**	**0.0268**	**0.0260**	**0.0014**	**0.8750**
$\rho_2 = 0.5$						

Figure 1: The Effect of Sample Size on AB/SE under AR(1) Returns (Upper graph: $\rho = -0.5$; Lower graph: $\rho = 0.5$)



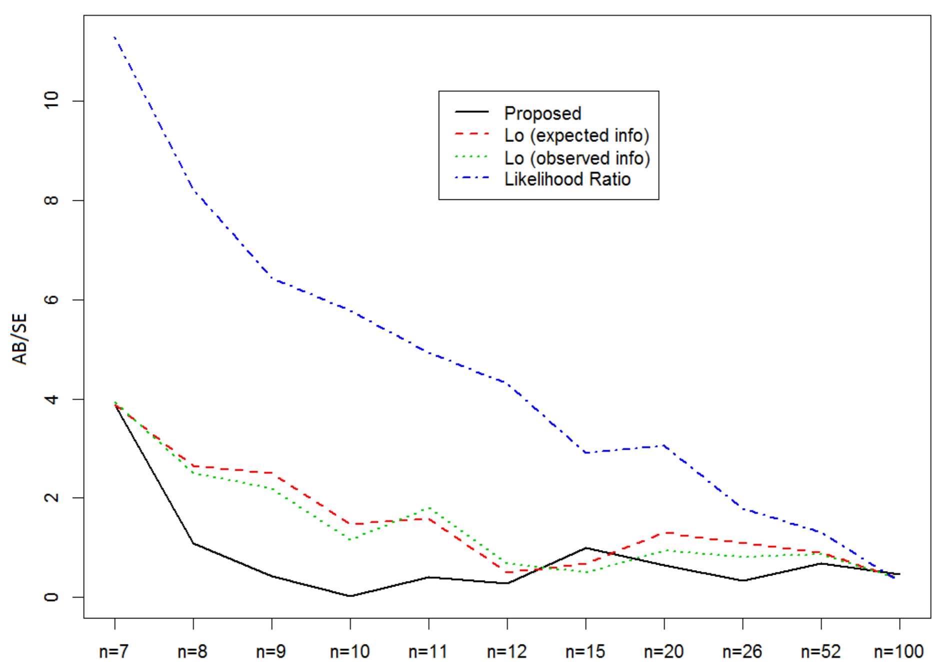


Figure 2: The Effect of ρ on AB/ER when $\psi = 0$