Topological pressure of a factor map for free semigroup actions

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Abstract. We consider finitely generated free semigroup actions on a compact metric space and we prove a relation for two topological pressures with a factor map.

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1 Introductions

Let (X, T) be a topological dynamical system (TDS for short), where X is a compact metric space and $T : X \to X$ is a surjective and continuous map. It is well-known that entropies are foundational to our current understanding of dynamical systems. In 1971, Bowen [7] considered a factor map $\pi : (X, T) \to (Y, S)$, and showed that

$$h(T) \leq h(S) + \sup_{y \in Y} h(T, \pi^{-1}y),$$

where $h(T, K)$ denote the entropy of a compact subset K of X. Topological pressure, firstly defined by Ruelle in [8], and later given some further study by Walters in [6], is a generalization of topological entropy.

In the present note, we follow [7] and introduce the topological presssure for free semigroup actions, prove a relation for two topological pressures with a factor map similar Bowen [7].

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2 Preliminaries

Let F_m^+ be the set of all finite words of symbols $0, 1, \cdots, m-1$. For every $w \in F_m^+$, $|w|$ denotes the length of w, i.e., the number of symbols in w. If $w, w' \in F_m^+$, define ww' to be the word obtained by writing w' to the right of w. With respect to this law of composition, F_m^+ is a free semigroup with m generators. We write $w \prec w'$ if there exists a word w'' such that $w' = ww''$.

Let X be a compact metric space with metric d, and $f_0, f_1, \cdots, f_{m-1}$ be continuous maps of X into itself. Then there is a free semigroup with m generators $f_0, f_1, \cdots, f_{m-1}$ acting on X. And we denote $\mathcal{F} = \{f_0, f_1, \cdots, f_{m-1}\}$. For $\omega = f_{\omega_0} \circ \cdots f_{\omega_{k-1}}$. We define the Bowen metric on X by

$$d_\omega(x, y) = \max_{\omega' \prec \omega} d(f_{\omega'}(x), f_{\omega'}(y)).$$

For any $\epsilon > 0$, a subset $E \subset X$ is called $(\omega, \epsilon, \mathcal{F})$-spanning set, if for any $x \in X$, there exists $y \in E$ such that $d_\omega(x, y) < \epsilon$. For any $\epsilon > 0$, a subset $K \subset X$ is called $(\omega, \epsilon, \mathcal{F})$-separated set, if for any $x, y \in K$, we have $d_\omega(x, y) \geq \epsilon$. For any $\omega \in F_m^+$, and $n \in \mathbb{N}$, $x \in X$, we set

$$S_n\varphi(x) := \sum_{i=0}^{n-1} \varphi(f_\omega^i x).$$

Definition 2.1. *For any $\varphi \in C(X)$,*

$$P_n(\varphi, \mathcal{F}, \epsilon) = \frac{1}{m^n} \sum_{|\omega|=n} \inf_{E_{\omega,n}} \Big\{ \sum_{x \in E_{\omega,n}} e^{S_n\varphi(x)} : \quad E_{\omega,n} \textit{ is a } (\omega, n, \epsilon) \textit{ spanning set of } X \Big\}.$$

Define the topological pressure

$$P(\mathcal{F}, \varphi) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log P_n(\varphi, \mathcal{F}, \epsilon).$$

On the other hands, we can define the topological pressure by separated set. From a standard proof, we can see that they are equivalent. Now we state the definition as follows without the proof of the equivalence.

Definition 2.2. *For any $\varphi \in C(X)$,*

$$Q_n(\varphi, \mathcal{F}, \epsilon) = \frac{1}{m^n} \sum_{|\omega|=n} \sup_{F_{\omega,n}} \Big\{ \sum_{x \in F_{\omega,n}} e^{S_n\varphi(x)} : \quad F_{\omega,n} \textit{ is a } (\omega, n, \epsilon) \textit{ separated set of } X \Big\}.$$

Define the topological pressure

$$P(\mathcal{F}, \varphi) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log Q_n(\varphi, \mathcal{F}, \epsilon).$$

We remark that if $\varphi = 0$, we denote the topological entropy $h(\mathcal{F}) = P(\mathcal{F}, \varphi)$. Clearly, the above definitions of topological pressure holds for any compact subset $K \subset X$.

Let (X, d) and (Y, ρ) be compact spaces, and there exists a free semigroup generated by $\mathcal{F} = \{f_0, f_1, \cdots, f_{m-1}\}$ actions on X, and a free semigroup generated by $\mathcal{G} = \{g_0, g_1, \cdots, g_{m-1}\}$ actions on Y. A continuous sujective map $\pi : X \to Y$ satisfies $\pi \circ f_i = g_i \circ \pi$, for any $0 \le i \le m-1$, whcih means that $(Y, \mathcal{G})$ is a factor of $(X, \mathcal{F})$.

Theorem 2.1. *For any dynamical system $(X, \mathcal{F})$ with metric d and dynamical system $(Y, \mathcal{G})$ is a factor of $(X, \mathcal{F})$ with metric ρ and $|\mathcal{F}| = m$. Let $\varphi : Y \to \mathbb{R}$ be a continuous map, and the factor map be $\pi : X \to Y$. Then we have*

$$P(\mathcal{G}, \varphi) \le P(\mathcal{F}, \varphi \circ \pi) \le P(\mathcal{G}, \varphi \circ \pi) + \sup_{y \in Y} h(\mathcal{F}, \pi^{-1}\{y\}) + \log m.$$

where $h(\mathcal{F}, K)$ denote the entropy of a compact subset K of X.

3 Proof of main result

In the following, for any $c > 0$, we set

$$\mathrm{var}(\varphi, c) = \sup_{x,y \in X} |\{\varphi(x) - \varphi(y)| : \; d(x, y) < c\}.$$

We can assume that $a = \sup_{y \in Y} h(\mathcal{F}, \pi^{-1}(y)) < \infty$, since if $a = \infty$, the proof is finished. For $\epsilon > 0, \omega \in F_m^+$, let F_y^ω denote the maximal $(\omega, \epsilon, \mathcal{F})$ separated set of $\pi^{-1}(y)$, and $M_{F_y^\omega}(\omega, \epsilon, \mathcal{F}, \pi^{-1}(y))$ denote the cardinality, we set

$$M(n, \epsilon, \mathcal{F}, \pi^{-1}(y)) := \frac{1}{m^n} \sum_{|\omega|=n} M_{F_y^\omega}(\omega, \epsilon, \mathcal{F}, \pi^{-1}(y)).$$

By the definition of $h(\mathcal{F})$, for the above $\epsilon > 0$, let $\alpha > 0$, for any $y \in Y$, choose $m(y)$ such that

$$a + \alpha \ge h(\mathcal{F}, \pi^{-1}(y), \epsilon) \ge \frac{1}{m(y)} \log M(m(y), \epsilon, \mathcal{F}, \pi^{-1}(y)). \tag{3.1}$$

Next we define

$$D_n(\omega, z, 2\epsilon, \mathcal{F}) := \{c \in X : \; d_\omega(c, z) < 2\epsilon\},$$

where $d_\omega(c, z) = \max_{\omega' \prec \omega} d(f_\omega(c), f_\omega(z))$. Since F_y^ω is a $(\omega, \epsilon, \mathcal{F})$ spanning set of $\pi^{-1}(y)$,

$$U_y = \bigcup_{z \in F_y} D_{m(y)}(\omega, z, 2\epsilon, \mathcal{F})$$

is a open cover of $\pi^{-1}(y)$. For any $y \in Y$,

$$(X \setminus U_y) \cap \bigcap_{r>0} \pi^{-1}(\overline{B_r(y)}) = \emptyset,$$

where $B_r(y) = \{z \in Y : \ \rho(z, y) < r\}$. From the finite intersection property, there exists $W_y = B_r(y)$ such that $U_y \supset \pi^{-1}(W_y)$. Let $\{W_{y_1}, W_{y_2}, \cdots, W_{y_p}\}$ is a finite cover of Y, with the Lebesgue number δ. By the definition of $P(\mathcal{G}, \varphi \circ \pi)$, we can choose a $(\omega, \delta, \mathcal{G})$-spanning set of Y, denotes by $E_{\omega,n}$, which satisfies

$$P_n(E, \mathcal{G}, \varphi \circ \pi) = \frac{1}{m^n} \sum_{|\omega|=n} \sum_{x \in E_{\omega,n}} e^{S_n \varphi \circ \pi(x)} < P(\mathcal{G}, \varphi \circ \pi). \tag{3.2}$$

At the same time, we assume this δ is small enough such that

$$\mathrm{var}(S_{m(y_i)}\varphi, \delta) < \epsilon \tag{3.3}$$

for each $1 \leq i \leq p$. For any $y \in E_{\omega,n}$, $0 \leq j < n$, choose $c_j(y) \in \{y_1, \cdots, y_p\}$ such that

$$\overline{B}_\delta(g^{|\omega'|=j}_{\omega',\omega' \prec \omega}(y)) = \{z \in Y : \ \rho(g^{|\omega'|=j}_{\omega',\omega' \prec \omega}(y), z) < \delta\} \subset W_{c_j(y)},$$

where $g^{|\omega'|=j}_{\omega',\omega' \prec \omega}(y) = g_{\omega_0} \circ \cdots \circ g_{\omega_{j-1}}, \omega = \omega_0 \cdots \omega_{n-j} \cdots \omega_{n-1}$.

Recuresly, we define $t_0(y) = 0$ and $t_{s+1}(y) = t_s(y) + m(c_{t_s(y)}(y))$ until $t_{l+1}(y) \geq n$, and set $l(y) = l$. For $y \in E_{\omega,n}$, $x_0 \in F^\omega_{c_{t_0}(y)}, \cdots, x_l \in F^\omega_{c_{t_l}(y)}$ we consider the following set

$$\begin{aligned} &V(y; x_0, \cdots, x_l) \\ &:= \Big\{x \in X : \ \ d(f^{|\omega'|=t+t_s(y)}_{\omega',\omega' \prec \omega}(x), f^{|\omega'|=t}_{\omega',\omega' \prec \omega}(x_s)) < 2\epsilon \text{ for all } \ 0 \leq t < m(c_{t_s}(y)) \text{ and } 0 \leq s \leq l(y)\Big\} \end{aligned}$$

Then we claim that

(1a) $\mathcal{V} = \{V(y; x_0, \cdots, x_s) : \ y \in E_{\omega,n}, x_s \in F_{c_{t_s}(y)}, 0 \leq s \leq l(y)\}$ is a cover of X.

(1b) For any $\omega \in F^+_m$, with $|\omega| = n$, $(\omega, 4\epsilon, \mathcal{F})$-separated set, and any element of $\mathcal{V}$ can have at most one separated point.

Now we prove the claim as follows.

Let $x \in X$, since $E_{\omega,n}$ is a $(\omega, \delta, \mathcal{G})$-spanning set of Y, then there exists a $y \in E$ such that $\rho_\omega(y, \pi(x)) = \max_{\omega' \prec \omega} \rho(g^{|\omega'|=j}_{\omega',\omega' \prec \omega}(y), g^{|\omega'|=j}_{\omega',\omega' \prec \omega}(\pi x)) < \delta$ for each $0 \leq j < n$. For any $0 \leq s \leq l(y)$,

$$\pi \circ f^{|\omega'|=t_s(y)}_{\omega',\omega' \prec \omega}(x) = g^{|\omega'|=t_s(y)}_{\omega',\omega' \prec \omega}(\pi x) \in W_{c_s(y)}. \tag{3.4}$$

This implies that there exists $x_s \in F^\omega_{c_{t_s}(y)}$ such that $d(f^{|\omega'|=t+t_s(y)}_{\omega',\omega' \prec \omega}(x), f^{|\omega'|=t}_{\omega',\omega' \prec \omega}(x_s)) < 2\epsilon$ for all $0 \leq t < m(c_{t_s}(y))$ and $0 \leq s \leq l(y)$. So we have $x \in V(y; x_0, \cdots, x_l)$. So we get (1a).

For (1b). For any $z, c \in V(y; x_0, \cdots, x_l)$, then

$$\begin{aligned} d(f_{\omega',\omega'\prec\omega}^{|\omega'|=t+t_s(y)}(z), f_{\omega',\omega'\prec\omega}^{|\omega'|=t}(c)) \leq & d(f_{\omega',\omega'\prec\omega}^{|\omega'|=t+t_s(y)}(z), f_{\omega',\omega'\prec\omega}^{|\omega'|=t}(x_s)) \\ & + d(f_{\omega',\omega'\prec\omega}^{|\omega'|=t+t_s(y)}(x_s), f_{\omega',\omega'\prec\omega}^{|\omega'|=t}(c)) < 4\epsilon, \end{aligned}$$

for each $0 \leq t < m(c_{t_s}(y))$ and $0 \leq s \leq l(y)$.

For any $(\omega, 4\epsilon, \mathcal{F})$-separated set $H_{\omega,n}$ of X, we estimate the upper bound of

$$\sum_{x \in F_{\omega,n}} e^{S_n\varphi\circ\pi(x)}$$

By the definition of $V(y; x_0, \cdots, x_l)$, we can denote

$$\mathcal{V}_y := \{V(y; x_0, \cdots, x_l) : x_i \in F^{\omega}_{c_{t_i(y)}} : 0 \leq i \leq l\}.$$

Set $\mathcal{V} = \cup_{y \in E_{\omega,n}} \mathcal{V}_y$, where

$$\#\mathcal{V}_y = \prod_{i=0}^{l=l(y)} M(\omega_{c_{t_i(y)}}, m(c_{t_i}(y)), \epsilon, \mathcal{F}, \pi^{-1}(y)) \leq e^{n(a+\alpha)} m^n, \tag{3.5}$$

where $A = \max\{m(y_1), \cdots, m(y_p)\}$. For each $x \in H_{\omega,n}$, there exist $y \in E_{\omega,n}$ and $x_0 \in F^{\omega}_{c_{t_1}(y)}, \cdots, x_l \in F^{\omega}_{c_{t_l}(y)}$ such that $x \in V(y; x_0, \cdots, x_l)$. Hence,

$$\begin{aligned} S_n\varphi\circ\pi(x) \leq & \sum_{i=0}^{l} S_{m(c_{t_i(y)}(y))}\varphi\circ\pi(x_i) + n\text{var}(\varphi\circ\pi, 2\epsilon) \\ & \leq S_{m(c_{t_i(y)}(y)))}\varphi(y_{c_{t_i(y)}(y)}) + m(c_{t_i(y)}(y))\text{var}(\varphi\circ\pi, 2\epsilon) \end{aligned}$$

Furhtermore, by (3.3)

$$S_{m(c_{t_i(y)}(y))}\varphi\circ(\pi(x_i)) \leq S_{m(c_{t_i(y)}(y)))}\varphi(f_{\omega}^{t_i(y)}(y)) + \epsilon + m(c_{t_i(y)}(y))\text{var}(\varphi\circ\pi, 2\epsilon) \tag{3.6}$$

for $0 \leq i \leq l(y)$. Hence,

$$\begin{aligned} S_n\varphi\circ\pi(x) \leq & \sum_{i=0}^{l} \Big(S_{m(c_{t_i(y)}(y)))}\varphi(f_{\omega}^{t_i(y)}(y)) + \epsilon + m(c_{t_i(y)}(y))\text{var}(\varphi\circ\pi, 2\epsilon)\Big) + n\text{var}(\varphi\circ\pi, 2\epsilon) \\ \leq & S_n\varphi(y) + l(y)\epsilon + 2n\text{var}(\varphi\circ\pi, 2\epsilon) \end{aligned} \tag{3.7}$$

By (3.7)

$$\sum_{x \in H_{\omega,n}} e^{S_n\varphi\circ\pi(x)} \leq \sum_{y \in E_{\omega,n}} \#\mathcal{V}_y \exp\Big(S_n\varphi(y) + l(y)\epsilon + 2n\text{var}(\varphi\circ\pi, 2\epsilon)\Big) \tag{3.8}$$

From (3.8),

$$\frac{1}{m^n} \sum_{|\omega|=n} \sum_{x \in H_{\omega,n}} e^{S_n\varphi\circ\pi(x)} \leq \frac{1}{m^n} \sum_{|\omega|=n} \sum_{y \in E_{\omega,n}} \#\mathcal{V}_y \exp\Big(S_n(y) + l(y)\epsilon + 2n\text{var}(\varphi\circ\pi, 2\epsilon)\Big).$$

From (3.5),

$$
\begin{aligned}
&\frac{1}{m^n}\sum_{|\omega|=n}\sum_{x\in H_{\omega,n}} e^{S_n\varphi\circ\pi(x)}\\
\leq&\frac{1}{m^n}\sum_{|\omega|=n}\sum_{y\in E_{\omega,n}} e^{n(a+\alpha)}m^n\exp\Big(S_n(y)+l(y)\epsilon+2n\mathrm{var}(\varphi\circ\pi,2\epsilon)\Big).
\end{aligned}
$$

Let $n\to\infty$ and $\epsilon\to 0$, and connect with (3.2) we finish the proof.

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