BOUNDARY VALUES OF r_r, r_r^*, R_r, R_r^* SETS OF CERTAIN CLASSES OF GRAPHS

Padma M M[1], M. Jayalakshmi[2]

Dept.of Mathematical and Computational Studies, Dr.Ambedkar Institute of Technology, Bengalurue, Karnataka State, INDIA, Pin 560 056,

email: [1]padmaprakash.mm@gmail.com, [2]jayachatra@yahoo.co.in.

Abstract

Let $G(V, E)$ be an undirected, finite, connected and a simple graph. For $u \in V$, associate a vector $\Gamma(u/S) = (d(u/s_1), d(u/s_2)..., d(u/s_k))$ with respect to $S = \{s_1, s_2, ..., s_k\}$ of $V(G)$, where $d(u/v) = \frac{\sum_{u_i \in N[u]} d(u_i, v)}{deg(u)+1}$. Then the subset S is said to be rational resolving set if $\Gamma(u/S) \neq \Gamma(v/S)$ for all $u, v \in V - S$ and is denoted by r_r set. A rational resolving set S with minimum cardinality is called rational metric basis or an rmb set and its cardinality is called rational metric dimension, denoted by $rmd(G)$ or $l_{r_r}(G)$. The maximum cardinality of a minimal r_r set of graph G is called upper r_r number of G and is denoted by $u_{r_r}(G)$. A subset S of $V(G)$ is said to be an r_r^* set if S is r_r set and $\bar{S} = \{V - S\}$ is also an r_r set. The minimum and maximum cardinality of minimal r_r^* set of graph G are respectively called lower and upper r_r^* number of G, denoted by $l_{r_r^*}(G)$ and $u_{r_r^*}(G)$. A subset S of $V(G)$ is said to be an R_r set if S is an r_r set and $\bar{S} = \{V - S\}$ is not an r_r set. The minimum and maximum cardinality of minimal R_r set of G are called respectively lower and upper R_r number of G and are denoted by $l_{R_r}(G)$ and $u_{R_r}(G)$. A subset S of $V(G)$ is said to be an R_r^* set if both S and $\bar{S} = \{V - S\}$ are not r_r sets. The minimum and maximum cardinality of minimal R_r^* set of G are called respectively lower and upper R_r^* number of G, denoted by $l_{R_r^*}(G)$ and $u_{R_r^*}(G)$. In this paper we are obtaining the lower and upper r_r, r_r^*, R_r, R_r^* numbers of certain classes of graphs.

Key words: Closed Neighborhood, Rational resoloving set, Neighborhood resolving sets, Rational Metric dimension.

AMS Subject Classification number: 05C20

1 Introduction

Many networks are represented by a graph, in which vertex play an important role and it depends on its neighbors. To determine the position of a vertex in the network, we need to select the landmarks in such a way that the distance of the vertex from the landmark and the distances of its neighborhood vertices from the landmark are considered. Here $N(u) = \{x : ux \in E(G)\}$, called open neighborhood of the vertex u, $N[u] = N(u) \cup u$ is called closed neighborhood of the vertex u and $d(u, v)$ is the length of the shortest path between u and v. A subset S of the vertex set V of a connected graph G is said to be a resolving set of G if for every pair of vertices u, $v \in V - S$ there exists a vertex $w \in S$ such that $d(u, w) \neq d(v, w)$. The minimum cardinality of a resolving set S of G is called metric dimension of a graph G and is denoted by $\beta(G)$. Metric dimension was defined by

F. Harary et al. [2] and P.J. Slater [8]. For the entire survey, we refer the latest survey article by Joseph A. Gallian [5]. All the graphs considered here are undirected, finite, connected and simple. Throughout this paper P_n denote a path on n vertices with a vertex set $V = \{v_i : 1 \leq i \leq n\}$ and edge set $E = \{v_i v_{i+1} : 1 \leq i < n\}$. Similarly C_n denote a cycle on n vertices with a vertex set $V = \{v_i : 1 \leq i \leq n\}$ and edge set $E = \{v_i v_{i+1}\} \cup \{v_1 v_n\}$. We use the standard terminology, the terms not defined here may be found in [1, 3, 4].

2 Boundary values of r_r, r_r^*, R_r, R_r^* sets of certain classes of Graphs

Rational metric dimension of graphs were originally proposed by A. Raghavendra, B. Sooryanarayana, C. Hegde [11] . Consider a graph $G(V, E)$. For $u \in V$, associate a vector $\Gamma(u/S) = (d(u/s_1), d(u/s_2)..., d(u/s_k))$ with respect to $S = \{s_1, s_2, ..., s_k\}$ of V, where $d(u/v) = \frac{\sum_{u_i \in N[u]} d(u_i, v)}{deg(u)+1}$. Then subset S is said to be a rational resolving set if $\Gamma(x/S) \neq \Gamma(y/S)$ for all $x, y \in V - S$ and is denoted by r_r set. The minimum cardinality of a rational resolving set S is called rational metric dimension and is denoted by $rmd(G)$ or $l_{r_r}(G)$. A rational resolving set S with minimum cardinality is called rational metric basis or an rmb set. An r_r set of G is said to be minimal if no subset of it is a r_r set. Clearly minimum cardinality of a minimal r_r set is $l_{r_r}(G)$, called lower r_r number of G. Now we define the following. The maximum cardinality of a minimal r_r set of graph G is called upper r_r number of G and is denoted by $u_{r_r}(G)$. A subset S of $V(G)$ is said to be an r_r^* set if S is r_r set and $\bar{S} = \{V - S\}$ is also an r_r set. The minimum cardinality of an r_r^* set of graph G is called lower r_r^* number of G and is denoted by $l_{r_r^*}(G)$ and the maximum cardinality of a minimal r_r^* set of graph G is called upper r_r^* number of G and is denoted by $u_{r_r^*}(G)$. A subset S of $V(G)$ is said to be an R_r set if S an r_r set and $\bar{S} = \{V - S\}$ is not an r_r set. The minimum and maximum cardinality of minimal R_r sets of G are called respectively lower and upper R_r number of G and are denoted by $l_{R_r}(G)$ and $u_{R_r}(G)$. A subset S of $V(G)$ is said to be an R_r^* set if both S and $\bar{S} = \{V - S\}$ are not r_r sets. The minimum and maximum cardinality of minimal R_r^* sets of G are called respectively lower and upper R_r^* number of G and are denoted by $l_{R_r^*}(G)$ and $u_{R_r^*}(G)$. (Suppose p and q represent some graph theoretical properties like domination, resolving, rmd etc, then a subset S of $V(G)$ is said to be pq set if S is both p set and q set. If S is an arbitrary set, need not be minimal having the property p then minimum and maximum cardinality of S is denoted by $\hat{l}_p(G)$ and $\hat{u}_p(G)$.)

Remark 2.1. *For a path P_n,*

$$d(v_j/v_i) = \begin{cases} \frac{1}{2} & if \quad i = j = 1 \ or \ i = j = n \\ \frac{2}{3} & if \quad i = j \neq 1 \ or \ i = j \neq n \\ |\, j - i \,| & if \quad i \neq j \ and \ (j \neq 1 \ or \ j \neq n) \\ \frac{2|j-i|-1}{2} & if \quad i \neq j \ and \ (j = 1 \ or \ j = n). \end{cases}$$

Lemma 2.2. *For a Path P_n, a singleton set $\{v_i\}$ is an rmb set if and only if either v_i is an end vertex or a support vertex in P_n.*

Proof. From Ragavendra et al [11], $rmd(P_n) = 1$, $\{v_1\}$, $\{v_n\}$ are rmb sets. Now $\{v_2\}$ is an rmb set, because from the Remark 2.1,

$$d(v_j/v_2) = \begin{cases} \frac{1}{2} & \text{if} \quad j = 1 \\ j - 2 & \text{if} \quad 3 \leq i \leq n - 1 \\ \frac{2n-3}{2} & \text{if} \quad j = n. \end{cases}$$

Thus $\Gamma(v_i/\{v_2\}) \neq \Gamma(v_j/\{v_2\})$ for every $i \neq j$. Similarly by symmetry $\{v_{n-1}\}$ is also an *rmb* set. But for a vertex v_i which is not an end vertex or a support vertex in P_n, singleton set $\{v_i\}$ is not an *rmb* set, because $d(v_{i-1}/v_i) = 1 = d(v_{i+1}/v_i)$ which imply $\Gamma(v_{i-1}/\{v_i\}) = \Gamma(v_{i+1}/\{v_i\})$ □

Theorem 2.3. *For a Path P_n, a subset $S = \{v_i, v_j\}$, $\forall\ i, j$ with $3 \leq i < j \leq n-2$ of $V(P_n)$ is a minimal r_r set.*

Proof. Let $S = \{v_i, v_j\}$, $3 \leq i < j \leq n-2$ be a subset of $V(P_n)$. Let x, y be any two vertices of P_n. Since $3 < i < n-2$, from Lemma 2.2, $\{v_i\}$ is not an r_r set, which imply $d(x/v_i) = d(y/v_i)$, for some x, y of $V(P_n)$. Let $d(x/v_i) = d(y/v_i)$ for $x = v_l$ and $y = v_m$ for some l, m with $1 \leq l, m \leq n$. Without loss of generality, consider $l < m$. Consider the following cases.

Case 1: $l = 1$.
From Remark 2.1, $d(v_l/v_j) = \frac{2|j-l|-1}{2}$ and $d(v_m/v_j) =| j - m |$ as $l < m$. Therefor $d(v_l/v_j) \neq d(v_m/v_j)$ as $d(v_m/v_j)$ is an integer whereas $d(v_l/v_j)$ is not an integer. Hence $\Gamma(x/S) \neq \Gamma(y/S)$

Case 2: $l \neq 1$
From Remark 2.1, $d(v_l/v_j) =| j - l |$ and $d(v_m/v_j) =| j - m |$ as $l \neq 1 \Rightarrow m \neq 1$.
Suppose $d(v_l/v_j) = d(v_m/v_j)$, then $| j - m |=| j - l |$ which imply $j - m = -(j - l)$, because $j-m \neq j-l$ as $l \neq m$. But $j-m = -(j-l) \Rightarrow 2j = m+l$. Similarly we have $d(v_l/v_i) = d(v_m/v_i)$ $\Rightarrow 2i = m+l$. Combining we have $2j = m+l$ and $2i = m+l$ imply $i = j$ which is not possible. Therefore $d(v_l/v_i) \neq d(v_m/v_i)$ and hence $\Gamma(x/S) \neq \Gamma(y/S)$. Other cases follow by symmetry.

Therefore $\forall\ i, j$ with $3 \leq i < j \leq n-2$, $\{V_i\}$, $\{V_j\}$ are not r_r sets, but $S = \{v_i, v_j\}$ of $V(P_n)$ is an r_r set which imply $S = \{v_i, v_j\}$ is a minimal r_r set. □

Corollary 2.4. *For a Path P_n, $n \geq 2$, any k-element subset S of $V(P_n)$ for $k \geq 2$ is an r_r set,but not minimal, because either S contain end vertices or support vertices or a subset $\{v_i, v_j\}$ with $3 \leq i < j \leq n-2$.*

Corollary 2.5. *For a Path P_n, $n \geq 6$, a subset $\{v_i, v_j\}$ of $V(P_n)$ with $3 \leq i < j \leq n-2$ is a minimal r_r set with maximum cardinality from Corollary 2.4.*

Theorem 2.6. *For a Path P_n, $l_{r_r}(P_n) = 1$ for $n \geq 1$ and*

$$u_{r_r}(P_n) = \begin{cases} 1 & \text{if} \quad n \leq 5 \\ 2 & \text{if} \quad n \geq 6 \end{cases}$$

Proof. $\{v_1\}$ is one of the r_r set with minimum cardinality. Therefore $l_{r_r}(P_n) = 1$.
To find $u_{r_r}(P_n)$, consider the following cases.

Case 1: $n \leq 4$.
From Lemma 2.2, every singleton subset of $V(P_n)$ is a minimal r_r set which imply $u_{r_r}(P_n) = 1$.

Case 2: $n = 5$.
From Lemma 2.2, every singleton subset of $V(P_5)$ except $\{v_3\}$ is a minimal r_r set and hence no 2-element subset of $V(P_5)$ is an r_r set which imply $u_{r_r}(P_5) = 1$

Case 3: $n \geq 6$.
From Corollary 2.5, a subset $\{v_i, v_j\}$ with $3 \leq i < j \leq n-2$ is a minimal r_r set with maximum cardinality Therefore $u_{r_r}(P_n) = 2$.

□

Theorem 2.7. *For a Path P_n, $l_{r_r^*}(P_n) = 1$ for $n > 1$ and*

$$u_{r_r^*}(P_n) = \begin{cases} 1 & if \quad n \leq 5 \\ 2 & if \quad n \geq 6 \end{cases}$$

Proof. $S = \{v_1\}$ is an r_r set and $\bar{S} = V - S = \{v_2, v_3, ..., v_n, \}$ is also an r_r set as it contain the end vertex v_n. Hence S is r_r^* set with minimum cardinality. Therefor $l_{r_r^*}(P_n) = 1$.
To find $u_{r_r^*}(P_n)$, consider the following cases.

Case 1: $n \leq 5$.
From Lemma 2.2, a singleton subset $S = \{v_1\}$ or $\{v_2\}$ or $\{v_{n-1}\}$ or $\{v_n\}$ is an r_r set and for any S, $\bar{S} = V - S$ is also an r_r set and no k-element subset for $k \geq 2$ of $V(P_n)$ is an r_r set which imply S is a minimal r_r^* set with maximum cardinality. Therefore $u_{r_r^*}(P_n) = 1$.

Case 2: $n \geq 6$.
From Corollary 2.5, a subset $S = \{v_i, v_j\}$ with $3 \leq i < j \leq n - 2$ of $V(P_n)$ and $\bar{S} = V - S$ are r_r sets and no k-element subset for $k \geq 3$ of $V(P_n)$ is an r_r set which imply S is an minimal r_r^* set with maximum cardinality. Therefore $u_{r_r^*}(P_n) = 2$.

□

Theorem 2.8. *For a Path P_n,*

$$l_{R_r}(P_n) = u_{R_r}(P_n) = \begin{cases} 0 & if \quad 1 < n \leq 4 \\ n-1 & if \quad n \geq 5 \end{cases}$$

Proof. To find $l_{R_r}(P_n)$, consider the following cases.

Case 1: $1 < n \leq 4$.
From Lemma 2.2, every singleton subset of $V(P_n)$ is an r_r set which imply for any k with $1 \leq k \leq 3$, a k-element subset S of $V(P_n)$ is an r_r set and for any S, $\bar{S} = V - S$ is also an r_r set which imply S is not an R_r set and therefore $l_{R_r}(P_n) = u_{R_r}(P_n) = 0$.

Case 2: $n \geq 5$.
From Lemma 2.2, every k-element subset of $V(P_n)$ for $k \geq 2$ is an r_r set and every singleton subset $\{v_i\}$, $3 \leq i \leq n - 2$ of $V(P_n)$ is not an r_r set, which imply a subset S of $V(P_n)$ is an R_r set, only if $\bar{S} = V - S$ is a singleton subset $\{v_i\}$, $3 \leq i \leq n - 2$ of $V(P_n)$. Therefore $l_{R_r}(P_n) = u_{R_r}(P_n) = n - 1$.

□

Theorem 2.9. *For a Path P_n, $n > 1$, $l_{R_r^*}(P_n) = u_{R_r^*}(P_n) = 0$*

Proof. For any k-element subset S of $V(P_n)$ with $1 \leq k < n - 1$, either S or $V - S$ contain atleast one end vertex which imply either S or $V - S$ is always an r_r set. Therefore there exists no R_r^* set for P_n and hence $l_{R_r^*}(P_n) = u_{R_r^*}(P_n) = 0$. □

Theorem 2.10. *For a complete graph K_n, $n > 2$, (when $n = 2$, $K_n = P_n$)*

(i) $l_{r_r}(K_n) = u_{r_r}(K_n) = n - 1$

(ii) $l_{r_r^*}(K_n) = u_{r_r^*}(K_n) = 0$

(iii) $l_{R_r}(K_n) = u_{R_r}(K_n) = n - 1$

(iv) $l_{R_r^*}(K_n) = u_{R_r^*}(K_n) = 2$

Proof. From Ragavendra et al [11], $rmd(K_n) = n - 1$ and any $(n-1)$-element subset S of $V(K_n)$ is a minimal r_r set.

(i) $rmd(K_n) = n - 1 \Rightarrow l_{r_r}(K_n) = n - 1$ and there exists no minimal r_r set with cardinality greater than $n - 1$ which imply $u_{r_r}(K_n) = n - 1$.

(ii) Since from (i), any r_r set contain minimum $n - 1$ elements, for any subset S of $V(K_n)$, both S and $\bar{S} = V - S$ cannot contain minimum $n - 1$ elements. Hence there exist no r_r^* set for K_n and therefore $l_{r_r^*}(K_n) = u_{r_r^*}(K_n) = 0$.

(iii) Since from (i), any minimal r_r set S contain minimum $n - 1$ elements, imply $\bar{S} = V - S$ contain exactly one element and hence $\bar{S}$ is not an r_r set. Therefore S is a minimal R_r set with minimum and maximum cardinality which imply $l_{R_r}(K_n) = u_{R_r}(K_n) = n - 1$.

(iv) Since from (i), any subset of $V(K_n)$ containing $n - 1$ elements is an r_r set, if S is a singleton subset of $V(K_n)$, then $\bar{S} = V - S$ contain $n - 1$ elements which imply S is a non r_r set and $\bar{S} = V - S$ is an r_r set so that S is not an R_r^* set. But if S is 2-element subset of $V(K_n)$, then $\bar{S} = V - S$ contain $n - 2$ elements which imply both S and $\bar{S} = V - S$ are non r_r sets so that S is an R_r^* set and is minimal. Therefore $l_{R_r^*}(K_n) = u_{R_r^*}(K_n) = 2$.

□

Theorem 2.11. *For a star graph $K_{1,n}$, $n > 2$, (when $n = 2$, $K_{1,n} = P_{n+1}$)*

(i) $l_{r_r}(K_{1,n}) = u_{r_r}(K_{1,n}) = n - 1$

(ii) $l_{r_r^*}(K_{1,n}) = u_{r_r^*}(K_{1,n}) = 0$

(iii) $l_{R_r}(K_{1,n}) = u_{R_r}(K_{1,n}) = n - 1$

(iv) $l_{R_r^*}(K_{1,n}) = u_{R_r^*}(K_{1,n}) = 2$

Proof. From Ragavendra et al [11], $rmd(K_{1,n}) = n-1$ and any $(n-1)$-element subset S of $V(K_{1,n})$ containing only pendent vertices is a minimal r_r set.

(i) $rmd(K_{1,n}) = n - 1 \Rightarrow l_{r_r}(K_{1,n}) = n - 1$ and there exists no minimal r_r set with cardinality greater than $n - 1$ which imply $u_{r_r}(K_{1,n}) = n - 1$.

(ii) Since any r_r set must contain minimum $n-1$ elements, both S and $\bar{S} = V - S$ cannot contain minimum $n - 1$ elements. Hence there exists no r_r^* set for $K_{1,n}$ and therefore $l_{r_r^*}(K_{1,n}) = u_{r_r^*}(K_{1,n}) = 0$.

(iii) Any r_r set S contain minimum $n-1$ elements, imply $\bar{S} = V - S$ contain maximum 2 elements and hence $\bar{S}$ is not an r_r set. Also any r_r set of $V(K_{1,n})$ with greater cardinality cannot be minimal. Therefore any r_r set with $n - 1$ elements is a minimal R_r set with minimum and maximum cardinality which imply $l_{R_r}(K_{1,n}) = u_{R_r}(K_{1,n}) = n - 1$.

(iv) Since for R_r^* set, both S and $\bar{S}$ should not contain $n-1$ pendent vertices, any 2-element subset of $V(K_{1,n})$ containing only pendent vertices is a minimal R_r^* set with minimum and maximum cardinality. Therefore $l_{R_r^*}(K_{1,n}) = u_{R_r^*}(K_{1,n}) = 2$.

□

Theorem 2.12. *For a cycle C_n, $n > 3$, (when $n = 3$, $C_n = K_n$)*

(i) $l_{r_r}(C_n) = u_{r_r}(C_n) = 2$

(ii) $l_{r_r^*}(C_n) = u_{r_r^*}(C_n) = 2$

(iii)

$$l_{R_r}(C_n) = u_{R_r}(C_n) = \begin{cases} n-1 & \textit{if} \quad n \textit{ is odd or } n = 4 \\ n-2 & \textit{if} \quad n \textit{ is even and } n \neq 4 \end{cases}$$

(iv)

$$l_{R_r^*}(C_n) = u_{R_r^*}(C_n) = \begin{cases} 2 & \textit{if} \quad n = 4 \\ 0 & \textit{if} \quad n > 4 \end{cases}$$

Proof. From Ragavendra et al [11], $rmd(C_n) = 2$. Any 2-element subset S of $V(C_n)$ (non diagonal elements when n is even) is a minimal r_r set.

(i) $rmd(C_n) = 2 \Rightarrow l_{r_r}(C_n) = 2$. Also any k-element subset of $V(C_n)$ for $k \geq 3$ contain 2-element subset which is an r_r set, which imply any 2-element subset of $V(C_n)$ is a minimal r_r set with maximum cardinality. Hence $u_{r_r}(C_n) = 2$

(ii) Since an r_r set of C_n must contain minimum 2 elements, any S of $V(C_n)$ with both S and $\bar{S} = V - S$ containing minimum 2 elements (non diagonal elements when n is even) is an r_r^* set, out of which exactly 2-element subset S is a minimal r_r^* set with minimum and maximum cardinality. Hence $l_{r_r^*}(C_n) = u_{r_r^*}(C_n) = 2$

(iii) Consider the following cases.

Case i When n is odd or $n = 4$
Every k-element subset of $V(C_n)$ for $k \geq 2$ is an r_r set and every singleton subset $\{v_i\}$ of $V(C_n)$ is not an r_r set, which imply a subset S of $V(C_n)$ is an R_r set, only if $\bar{S} = V - S$ is a singleton subset, that is S contain minimum $n-1$ elements. Therefore $l_{R_r}(C_n) = u_{R_r}(C_n) = n - 1$.

Case ii When n is even and $n \neq 4$
Since any two diagonally opposite vertices of $V(C_n)$ is a non r_r set, choose S of $V(C_n)$ such that $\bar{S} = V - S$ contain two diagonally opposite vertices of $V(C_n)$. Then S is minimal R_r set with minimum and maximum cardinality $n-2$. Therefore $l_{R_r}(C_n) = u_{R_r}(C_n) = n-2$.

(iv) Consider the following cases.

Case i When $n = 4$
$S = \{v_1, v_3\}$ and $\bar{S} = V - S = \{v_2, v_4\}$ are not r_r sets which imply S is an R_r^* set and hence $l_{R_r^*}(C_n) = u_{R_r^*}(C_n) = 2$.

Case ii When $n > 4$
For any subset S of $V(C_n)$, either S or $V - S$ contain atleast two elements (non diagonal elements when n is even) which imply either S or $V - S$ is always an r_r set. Therefore there exists no R_r^* set for C_n and hence $l_{R_r^*}(C_n) = u_{R_r^*}(C_n) = 0$.

□

Acknowledgment: Authors are very much thankful to the Management and the Principal of Dr. Ambedkar Institute of Technology, Bangalore for their constant support and encouragement during the preparation of this paper.

References

[1] Buckley F and Harary F, *Distance in Graphs*, Addison-Wesley,(1990).

[2] F. Harary, R.A. Meltar, On the metric dimension of a graph, Ars. Combinatoria 2 (1976), 191-195.

[3] F. Harary, Graph theory, Narosa Publishing House, New Delhi, 1969.

[4] Hartsfield Gerhard and Ringel, *Pearls in Graph Theory*, Academic Press, USA, 1994.

[5] Joseph A. Gallian, *A dynamic survey of graph labeling*, The Electronic Journal of Combinatorics, # DS6,(2009),1-219.

[6] Samir Khuller, Balaji Raghavachari, Azriel Rosenfeld, Landmarks in graphs, Discrete Appl. Math. 70(1996)217-229.

[7] A. Sebo, E. Tannier, On Metric generators of graphs, Math. Opr. Res. 29(2004), 383-393.

[8] Peter J. Slater, Dominating and references in a grap, J. Math, Phys.Sci. 22 (1988), 445-455.

[9] B. Sooryanarayana, on the metric dimension of graph, Indian journal of Pure and Appl. Math.29

[10] B. Sooryanarayana and Shanmukha, A note on the metric dimension, Far. East. J. Appl. Math. 3 (2001), 331-339.

[11] A. Raghavendra, B. Sooryanarayana, C. Hegde *Rational metric dimension of graphs*, Communication in Optimization theorey, 2014, ISSN: 2051-2953.

[12] Joe Ryan, *Exclusive sum labeling of graphs: A survey*, AKEC International Journal of Graphs.Combinatorics.,Vol 6, No. 1, (2009) 113-126.