ON THE PARAMETRIC FACTORIZATION AND ANALYSIS OF A CUBIC ROOT MEAN FOURTH ORDER RUNGE – KUTTA FORMULA

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**Abstract**

A special type of fourth order Runge-Kutta Formula (RKF) for the solution of Initial Value Problems (IVPs) in Ordinary Differential Equation, whose step functions are made to obey the usual characteristics of Runge-Kutta Method (RKM) is established. A rigorous derivation was carried out through the process of binomial and Taylor series expansion to obtain the values of the parameters. Solution of some ivps were obtained using the method and was tested for efficiency, convergence and stability The results show that the formula has minimal errors, is consistent, converges, stable and it compares favourably well with other RKMs with relatively smaller time.

**Key words:** *parametric factorization, geometric approach, binomial expansion and cubic 4th order*.

**2010 Mathematical Subject Classification: Primary 65L20; Secondary 65L06**

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**1** **INTRODUCTION**

Physical systems give rise to system of Ordinary Differential Equations (ODEs) with widely varying eigenvalues resulting precisely in the concept of singular initial value and stiffness problems in ordinary differential equations. This paper seeks to derive a new formula through the process of binomial and Taylor series expansion, implement the new method using some well known Initial Value Problems (IVPs) in ODEs and investigate its level of compliance with numerical solution in ODEs.The scheme developed here is also adaptable to mildly stiff ordinary differential equations. However, the focus here is on the solution of singular and non-stiff initial-value problems. Sometimes step size is restricted by stability rather than by accuracy; just as solutions to differential equations evolve with time, so do numerical approximations progress in small time steps. In each step, an error is made and it is important to keep these errors small. But the error caused in one time-step may have an effect on the accuracy of later steps. It may be more important to control the buildup of errors than to control the size of the errors themselves. According Butcher (2010), in numerical analysis the smallness of the individual errors is called accuracy and the ability to keep the effect of errors under control is called stability. In this paper, the new cubic Rung-Kutta algorithm is expected to solve singular ivps in ordinary differential equations.

According to Lambert (1995), there are various one-step schemes in existence, as established in Jain (1983), Jackiewiez et al (1991), Butcher (1987), Agbeboh et al (2007) Fatunla (1986) Agbeboh and Ehiemua (2012), Agbeboh and Omokaro (2012) and Agbeboh (2013), but a method becomes useful only when it has properties like consistency, convergence and error inherent in it. Also, Ababneh, et al (2009), also acknowledged that a one-step method is said to be consistent, if the difference equation of the computation formula exactly approximates the differential equation it intends to solve. In this section, we establish the consistency, convergence and error properties of our new algorithm. It is important to recall that a first – order differential equation  may possess an infinite number of solutions. For example, the function  is, for any value of the constant C, a solution of the differential equation, where  is a given constant. Lambert (1977), noted that we can pick out any particular solution by prescribing an *initial condition,.* For the above example, the particular solution satisfying this initial condition is easily found to be  (1.1a) We say that the differential equation together with an initial condition constitute an initial value problem,  (1.1b) The absolute stability property of a one-step numerical process is normally investigated by applying the test equation  (1.2)

Where  is a complex constant with negative real party the resultant equation is the first order differential equation ,  (1.3) The function  is called the stability function which is either a polynomial or a rational function in . The parameters in the one-step method are sometimes chosen as to ensure that is an approximation to.

To get a clearer picture of our mission, let us examine some relevant definitions in line with the objective of this paper.

1.2 **Definition of terms**

Definition 1**:** A one-step scheme is said to be absolutely stable at a point  in the complex plane provided the stability function  satisfies the following condition  (1.4) and the corresponding region of absolute stability is  (1.5) Definition 2**:** The numerical integration scheme is said to be A-stable provided that the region of absolute stability include the entire left hand of the complex plane.

A-stability concept which was introduced by Dalhquist (1963) is a very desirable property for any numerical integration algorithm particularly if the IVP were to be stiff and highly oscillatory. As A-stability requirement is rather too stringent, weaker but less desirable stability criteria which accommodates higher orders has since been proposed. These include A()-stability introduce by Widlund (1967) stiff- Stability Gear (1967), and A(0)-stability Cryer (1973).

Definition 3**:** A numerical process is said to be A()-stable for x  (0,) if its solution set tend to zero as n 🡪  when this process is applied with fixed position into the test problem in (1.2) where in this case or, If its region of absolute stability is continuous at infinite wedge See Agbeboh and Omokaro (2010).

A numerical process is said to be A(0)-stable if it is A()-stable for all (some)

(0,/2),

such that o < < /2.

Definition 4: A numerical method is said to be stiffly stable if (i) Its region of absolute stability contains  and  (ii) It is accurate for all q when applied to the scalar test equation

where ,  (1.6)

and a, b and c are positive constants. See Agbeboh and Omokaro (2010).

The reason for this definition is to represent eigenvalue with rapidly decaying terms in the transient solution by corresponding  in.

Since, we shall be dealing with one step method that may result in rational stability function, which can definitely be investigated as , then we need to examine L-stability property otherwise known as stiff A-stability proposed by Hall (1986) and shall comprise the behaviour of stability function as .

Definition 5: A one-step method is said to be L-stable if it is A-stable and, in addition when applied to the stable test equation  is a complex constant with  it yields

when  (1.7)

The above definitions highlight the fact that this paper is mainly constructed to establish the stability of the new method as will be seen in section 4.

**1.2.** S**tabilizing condition:** Generalizing, Lambert (1990) postulated as follows. “Let  be the principal root, and  be the spurious roots of the first characteristic polynomial

 (1.8)

Let, then in the case when the spurious roots of P are distinct, the stabilizing condition requires that

 (1.9)

 (1.10)

A method is said to be stabilized if it satisfies the stabilizing condition and the strict root condition 

This condition remains an instrument in establishing the stability of this scheme.

**2. The Principles of Derivation for the New Method**

The procedure for obtaining the new Runge-Kutta method is basically the same with that of classical method except that the binomial expansion will be applied in combination with Taylor series expansion in a more rigorous manner to derive the k functions.

According to Fatunla (1988), “Runge-Kutta methods can be classified into three categories called an s-stage Runge-Kutta process given by:

 (2.1)

where 

In tabular form the three types of Runge-Kutta methods are as shown below:

Table 1.1 Type of Runge-Kutta methods

|  |  |  |
| --- | --- | --- |
| Type | Status of Coefficients | Number of Coefficients |
| Explicit | (i.e. A us lower triangle with zero diagonal elements) |  |
| Semi-Explicit | (i.e. A is lower triangular with non-zero diagonal elements) |  |
| Implicit | (ie A is not a lower triangular matrix) |  |

The numerical values of the unknown coefficients are obtained from a set of nonlinear equations, derived as follows:

Step 1 Obtain the Taylor Series expansion of  at the point  for j = 1 (1) s

Step 2: Insert these expansion in equation (1.3) in equation (1.2)

Step 3: Compare these coefficients in power of h for both the increment function  of the Runge-Kutta method with the increment function for the Taylor expansion method specified in (3.1a) to (3.1c) in the next section.

All of the unknown coefficients normally exceed the number of equations, so some can be chosen so as to attain some desired goal. Some of these goals, according to Fatunla (1988) are

to minimize a bound of local truncation error (c.f Ralston 1965, 1962b);

to maximize the attainable order of the scheme (king 1966) achieved this for the

 (2.3)

Thus, we have the approximation

 (2.4)

Equation (2.4) is the same as

(Average slope) (2.5)

This is the underlying principle of the Runge-Kutta method. In the above analysis the average slope are represented by ki which can be arithmetic mean,(Classical Runge-Kutta Method), geometric mean constructed by Evans and Sangui (1986) and developed by Agbeboh (2006) is being extended to cubic root mean in this paper .In general, we find the slope at  and at several other points; take the average of these slopes, multiply by h, and add the result to yn. Thus, the Runge-Kutta method with v-shape can be written as

 (2.6)

 (2.7)

where the parameters ci, aji and wi are arbitrary.

From (2.5) we may interpret the increment function as the linear combination of the slopes at xn and other points between xn and xn+1. The expansion of (2.6) and (2.7) helps in obtaining specific values for the parameters; we expand yn+1 in powers of h such that it agrees with the Taylor series expansion to a specific number of terms.

Using the above principle, the new Rung-Kutta method as stated in section 3 was derived.

**3. The Analysis of the Cubic Order Runge-Kutta Formula.**

According to Lambert (1995), and established in Agbeboh and Ehiemua (2012), Agbeboh and Omokaro (2012) and Agbeboh (2013), there are various one-step schemes in existence, but a method becomes useful only when it has properties like consistency, convergence and stability inherent in it. Also, Ababneh, Ahmed and Ismail (2009), also acknowledged that a one-step method is said to be consistent, if the difference equation of the computation formula exactly approximates the differential equation it intends to solve. In this section, the consistency, convergence and stability properties of our new algorithm carved out of the Kutta one- step method is established.

The stability properties of the method are analyzed by adopting the above definitions and the derivation below is used.

, (3.1a) , for j=i=2,3,4. (3.1b)

 (3.1c)  (3.2)  (3.3)

where 4th order accuracy is obtained by choosing a1 = ½, a2 = 0, a3 = ½, a4 = 0

a5 = 0 and a6 = 1, the cubic order formula as given below is developed by setting.

To get the new formula, we set equation (3.3) as follows:

 (3.4)

By replacing (3.4) with their root means parameters, equation (3.3) becoming.

 (3.5a)

Where , (3.5b)

by the same process of experimentation and using parameters  leads to a low accuracy formula of order 4. We therefore, hold that these parameters will not be suitable for use in equation (3.4).

It is our desire therefore, to find values for parameters ai, i = 1(1)6 that will cause higher accuracy of order 4. Equation (3.5) was derived using the process of binomial expansion, arising from the fact that (3.5) can be expanded to give

 (3.6)

such that  (3.7)

Using the binomial expansion of rational index 

 (3.8)

where  (3.9)

we can obtain the expansion of (3.7) in terms of x in (3.8) to get

 (3.10a)

and

 (3.10b)

Using Taylor series expansion for function of two variable, we adopt the expansion of y functions only and reduce all ki’s to, such that ki  i = 1,2,3,4 will become functions of k1 and y only. Using equations (1.3) we have

 and  (3.11)  (3.12)

In summary, the expansion of k2, k3, k4 and their powers are:

 (3.13)

Similarly, by setting 

 (3.14)

 (3.15)  (3.16)

 (3.17) to obtain values for, we set, for, r = 1, 2 and 3. Such that (3.18)   (3.19)  (3.20)

Hence

 (3.21)

   (3.22) Using the same method of expanding we obtain, by setting  and  (3.23)     (3.24)    (3.25) Substituting equations (2.24)-(3.26) into

 , (3.26) where,  (3.27)

We have , (3.28) Substituting equations (3.23) and (3.28) into equation (3.4), we have:

  (3.29)

we get  (3.30) Comparing with the Taylor Series expansion given as  We have the following equations:

 (3.32a)  (3.32b)  (3.32c)  (3.32d)   (3.32e)  (3.32f)  (3.32g) By setting A=1/2 and B=1, we obtain the following values for the parameters.

, , , , ,  (3.33) And the resulting cubic 4th order Runge-Kutta formula emerged as

 (3.34a)  (3.34b)  (3.35a)  (3.35b)  , (3.35c) **** (3.35d)

With the Butcher Array as shown in The table I

Table 1 Butcher Array for the Cubic Root Mean 4th order RKF

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **1** | **0** |  |  |  |  |
| **1** | **0** | **½** |  |  |  |
| **1/12** | **0** | **-1/2** | **7/12** |  |  |
| **1/42** | **0** | **-1/6** | **-1/6** | **-1/14** | **26/21** |
|  | **½** | **1/3** | **1/3** | **1/3** | **1/3** |

**4. Implementation of the Method on Some Initial Value Problems:** We implement here, the above derived formula on some selected initial value problems in ordinary differential equations using an appropriate FORTRAN program and compare with some other well known Runge-Kutta Formulae. The results are as follow

(a). (ONE-THIRD RUNGE-KUTTA METHOD





(b). MODIFIED KUTTA’S ALGORITHM



 

 

(c). CLASSICAL RUNGE-KUTTA METHOD





In order to compare the performance of formula (a) with those of (b) and (c), we solve the following examples.

Example 4.1

. Whose exact solution is . See table 4.1 below for the computed results.

Example 4.2

 , Whose exact solution is where c is any constant.

See table 4.2 below for the computed results.

Example 4.3

 , Whose exact solution is  , where c is any constant. See table 4.3 below for the computed results.

Table 4.3: Numerical Results for example 4.3

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| XN | TSOL | CRK4 | ERROR | CUKA | ERROR | 1/3RK4 | ERROR |  |
| .1E+00 | 0.1223E+01 | 0.1223E+01 | 0.3339E-07 | 0.1224E+01 | 0.5884E-03 | 0.1223E+01 | 0.6639E-05 |  |
| .2E+00 | 0.1508E+01 | 0.1508E+01 | 0.1480E-05 | 0.1510E+01 | 0.1669E-02 | 0.1508E+01 | 0.2775E-04 |  |
| .3E+00 | 0.1896E+01 | 0.1896E+01 | 0.1096E-04 | 0.1900E+01 | 0.3782E-02 | 0.1896E+01 | 0.9968E-04 |  |
| .4E+00 | 0.2465E+01 | 0.2465E+01 | 0.6307E-04 | 0.2473E+01 | 0.8294E-02 | 0.2465E+01 | 0.3829E-03 |  |
| .5E+00 | 0.3408E+01 | 0.3408E+01 | 0.4030E-03 | 0.3427E+01 | 0.1910E-01 | 0.3408E+01 | 0.1825E-02 |  |
| .6E+00 | 0.5332E+01 | 0.5328E+01 | 0.3958E-02 | 0.5379E+01 | 0.4735E-01 | 0.5318E+01 | 0.1364E-01 |  |
| .7E+00 | 0.1168E+02 | 0.1155E+02 | 0.1274E+00 | 0.1164E+02 | 0.3892E-01 | 0.1138E+02 | 0.2982E+00 |  |
| .8E+00 | -.6848E+02 | 0.1922E+03 | 0.2606E+03 | 0.1142E+03 | 0.1827E+03 | 0.1233E+03 | 0.2982E+00 |  |
| .9E+00 | -.8688E+01 | 0.3120E+18 | 0.3120E+18 | 0.1225E+10 | 0.1225E+10 | 0.2582E+12 | 0.2582E+12 |  |
| .1E+01 | -.4588E+01 | 0.3278+261 | 0.3278+261 | 0.2376E+75 | 0.2376E+75 | 0.7454+123 | 0.7454+123 |  |

Table 4.2: Numerical Results for example 4.2

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| XN | TSOL | CRK4 | ERROR | MKA | ERROR | 1/3RK4 | ERROR |
| .1E+00 | 0.1111E+01 | 0.1111E+01 | 0.6211E-06 | 0.1111E+01 | 0.4154E-04 | 0.1111E+01 | 0.2932E-05 |
| .2E+00 | 0.1250E+01 | 0.1250E+01 | 0.2008E-05 | 0.1250E+01 | 0.1247E-03 | 0.1250E+01 | 0.9354E-05 |
| .3E+00 | 0.1429E+01 | 0.1429E+01 | 0.5242E-05 | 0.1428E+01 | 0.2968E-03 | 0.1429E+01 | 0.2399E-04 |
| .4E+00 | 0.1667E+01 | 0.1667E+01 | 0.1341E-04 | 0.1666E+01 | 0.6758E-03 | 0.1667E+01 | 0.5992E-04 |
| .5E+00 | 0.2000E+01 | 0.2000E+01 | 0.3674E-04 | 0.1998E+01 | 0.1593E-02 | 0.2000E+01 | 0.1587E-03 |
| .6E+00 | 0.2500E+01 | 0.2500E+01 | 0.1171E-03 | 0.2496E+01 | 0.4155E-02 | 0.2500E+01 | 0.4811E-03 |
| .7E+00 | 0.3333E+01 | 0.3333E+01 | 0.4892E-03 | 0.3320E+01 | 0.1310E-01 | 0.3331E+01 | 0.1857E-02 |
| .8E+00 | 0.5000E+01 | 0.4997E+01 | 0.3372E-02 | 0.4941E+01 | 0.5886E-01 | 0.4989E+01 | 0.1116E-01 |

Table 4.3: Numerical Results for example 4.3

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| XN | TSOL | CRK4 | ERROR | CUKA | ERROR | 1/3RK4 | ERROR |
| .1E+00 | 0.2235E+01 | 0.2235E+01 | 0.8351E-05 | 0.2235E+01 | 0.4692E-04 | 0.2235E+01 | 0.3237E-04 |
| .2E+00 | 0.2569E+01 | 0.2569E+01 | 0.3417E-04 | 0.2569E+01 | 0.1899E-03 | 0.2569E+01 | 0.1315E-03 |
| ,3E+00 | 0.3076E+01 | 0.3076E+01 | 0.1266E-03 | 0.3076E+01 | 0.1899E-03 | 0.3076E+01 | 0.4765E-03 |
| .4E+00 | 0.3936E+01 | 0.3935E+01 | 0.5646E-03 | 0.3933E+01 | 0.2880E-02 | 0.3934E+01 | 0.2019E-02 |
| .5E+00 | 0.5693E+01 | 0.5689E+01 | 0.4074E-02 | 0.5675E+01 | 0.1820E-01 | 0.5681E+01 | 0.1297E-01 |
| .6E+00 | 0.1124E+02 | 0.1115E+02 | 0.9555E-01 | 0.1094E+02 | 0.3038E+00 | 0.1102E+02 | 0.2284E+00 |
| .7E+00 | -.1454E+03 | 0.1177E+03 | 0.2632E+03 | 0.7239E+02 | 0.2178E+03 | 0.8493E+02 | 0.2304E+03 |
| .8E+00 | -.8868E+01 | 0.1853E+15 | 0.1853E+15 | 0.2427E+08 | 0.2427E+08 | 0.4039E+10 | 0.4039E+10 |
| .9E+00 | -.4352E+01 | 0.7874+209 | 0.7874+209 | 0.3033E+59 | 0.3033E+59 | 0.1600+102 | 0.1600+102 |
| .1E+01 | -.2784E+01 | \*\*\*\*\*\*\*\*\*\* | \*\*\*\*\*\*\*\*\*\* | \*\*\*\*\*\*\*\*\*\* | \*\*\*\*\*\*\*\*\*\* | \*\*\*\*\*\*\*\*\*\* | \*\*\*\*\*\*\*\*\* |

**4. Stability Analysis of the Method**

Our duty here is to investigate and establish the stability of the method by following Lambert (1973), where it was revealed that “in all computational methods, the use of a scheme for numerical solution of initial value problem (1.1) will generate errors at some stages of the computation due to inaccuracy inherent in the formula and the arithmetic operations adopted during computer implementation. The magnitude of the error determines the degree of accuracy and stability of the method”. Thus, it is important that the numerical solution approximates the exact solution and that the numerical solution tends to the exact solution as the step size tends to zero. Jain et al (2007) observed that if the step length used is too small, excessive computation time and round-off error will result. We should also consider the opposite case, and ask whether there is any upper bound on step length. Often there is such a bound and it is reached when the method becomes numerically unstable, that is the numerical solution produced, no longer corresponds qualitatively with the exact solution.

According to Lambert (1995), the traditional criterion for ensuring that a numerical method is stable is called “Absolute Stability”, and this analysis will therefore, be carried out to establish the absolute stability of our method by subjecting it to the linear test equation;

 (4.1) where is complex.

Butcher (1987) emphasized that all Runge-Kutta methods including the implicit ones, when applied to the test equation, reduce to an equation of the form;

 (4.2)

where  is called the stability polynomial function. Bearing this in mind we write, so that it produces a linear system for the computation of  which will be solved for, and then inserted into our method to produce

 (4.3) Lambert (1973), says, the key issue for understanding the long term dynamics of Runge-Kutta methods near some fixed points, concerns the region where R ( ≤ 1; that is, the Stability region of the numerical method. The polynomial, for which R( is known as the Stability polynomial of the method, and this method is absolutely stable for a given , if all the roots of the polynomial function lie within the unit circle. The region containing all these points in the complex plane is said to be a region of absolute stability, if the method is stable for all

. (4.4) It is also possible according to Lambert (1995), that applying a method to the test equation (1.1) (where is a scalar) yields

 (4.5)

Now defining  we may then write (4.5) in the form

 and  (4.6)

Where A is the matrix of coefficients.

Solving the first of these for Y and substituting in the second gives

 (4.7)

where is the s x s unit matrix, the stability function is therefore given by

 (4.8)

However, in another approach, Dekker and Verwer (1984) gives an alternative form of, where it was observed that the solution for  by Cramer’s rule is

 (4.9) where;

N=;D= (4.10) Hence

 (4.11)

Where,

 (4.12)

Lambert (1995), noticed that, irrespective of the values given to the parameters in matrix A after satisfying the order requirements, for a given P = 1, 2, 3, 4, all p-stage Runge-Kutta methods of order p have the same interval of absolute stability. These intervals are given in table 1 below, where Rp denote any p-stage Runge-Kutta method of order p.

Table 4.5: INTERVAL OF ABSOLUTE STABILITY FOR ORDER P, FOR P4) 

|  |  |  |
| --- | --- | --- |
| Method | Emerging RKF Polynomial for | Interval of absolute stability |
| R1 | 1+µ | (-2, 0) |
| R2 | 1+ µ+ | (-2, 0) |
| R3 | 1+ µ+ | (-2.51, 0) |
| R4 | 1+ µ+ | (-2.78, 0) |

TABLE 2 (We now analyze the stability of equation (2.36), the new method with a view to establishing its region of absolute stability. To show that the method is stable, we subject it to the test equation ;  (4.13)

 (4.14) Using Binomial Expansion

 (4.15)

 (4.16) If ; and substituting above;

 (4.17) (4.18)

This is the stability polynomial function.

A care look will reveal that this is the same with the classical Runge-Kutta stability polynomial function (R3) as indicated in the table 4.5 above. Therefore, the method is absolutely stable

y

x

**Conclusion**

A careful look at the above numerical results, reveals that our method has relatively low error level and compares favourably well with the results from other existing formulas that we have considered. Lambert (1973) solving example 4.1 with the classical 4th order method stops at step length 3 for which the method was consistent with the theoretical results. The other two examples also show high degree of accuracy as can be seen in the above tables of results. Thus we predict that these parameters (3.33) result in equation (3.34) to have an accuracy of order 4 as proposed.

From the above diagram it is evident that the new scheme is A-stable with all the eigenvalues inside the complex plane as stated in definitions 1and 2. Therefore, from the numerical results we conclude as follows that:

(i) the predicted errors give good estimate for the error values;

(ii) the value of the magnitude error function tends to a constant in some cases as in

example 4.3. This needs to be further investigated.

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