Doubly Truncated Generalized Gaussian Distribution and its Distributional Properties

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Abstract

The generalized Gaussian distribution is useful in analyzing data sets arising in Image processing, Signal processing, Speech recognition, Statistical Quality Control, Industrial experimentation, and Biological experiments. In this paper, a Doubly Truncated generalized Gaussian distribution is introduced. The various distributional properties such as the distribution function, the four moments, skewness, kurtosis, hazard function, survival function are derived.

Keywords: Doubly Truncated, Generalized Gaussian distribution, distributional properties.

Introduction

In the earlier papers, we developed and analyzed left truncated generalized Gaussian distribution and right truncated generalized Gaussian distribution, assuming the variate under study is truncated either left or right of the range. But in many practical situations arising at places like industrial experiments, agricultural experiments, financial modeling, warranty studies the variate under consideration may have a finite range. For example, in inventory modeling for deteriorating item, life time of commodity is random and may have a finite range. The lower bound may be zero because there is no negative life time and the upper bound is constrained with a finite value since, the product is perishable. Approximating the finite range with the infinite range will provide the results inaccurate. So, to have an accurate analysis of the data set it is needed to consider a doubly truncated distribution. Hence, in this paper, we develop and analyze a doubly truncated generalized Gaussian distribution with the assumption that the range of the variate under study is having both upper and lower bounds. The various distributional properties such as the probability density function, the distribution function, the four moments, the skewness, the kurtosis, the hazard function and survival function are derived.

Doubly Truncated Three Parameter Generalized Gaussian Distribution

A Continuous random variable X is said to have a three parameter generalized Gaussian distribution if its probability density function (p.d.f) is of the form

$$f(x)=\frac{g(x;\mu,\alpha,\beta)}{\Phi(B)-\Phi(A)}; \qquad A<x<B; \qquad A<\mu<B; \qquad \alpha>0; \qquad \beta>0$$

Consider that the range variable is finite say (A, B). Then the probability density function (p.d.f) of the doubly truncated three parameter generalized Gaussian distribution is

$$f(x)=\frac{g(x;\mu,\alpha,\beta)}{\Phi(B)-\Phi(A)}; \qquad A<x<B; \qquad A<\mu<B; \qquad \alpha>0; \qquad \beta>0 \tag{1}$$

where $\Phi(A)=\int_{-\infty}^{A}\frac{\beta}{2\alpha\Gamma\left(\frac{1}{\beta}\right)}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}dx$ $\qquad$ $\Phi(B)=\int_{-\infty}^{B}\frac{\beta}{2\alpha\Gamma\left(\frac{1}{\beta}\right)}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}dx$

The lower and upper truncation points are A and B respectively. The degree of truncation are $\Phi\left(\frac{A-\mu}{\beta}\right)$ and $1-\Phi\left(\frac{B-\mu}{\beta}\right)$. If A is replaced by $-\infty$ or B is replaced by ∞, the distribution is singly truncated from below or above respectively.

Hence, the probability density function of doubly truncated three parameter generalized Gaussian distribution is

$$f(x)=\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \qquad for \quad A<\mu<B \tag{2}$$

$$f(x)=\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \qquad for \quad \mu<A<B \tag{3}$$

$$f(x)=\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} \qquad for \quad A<B<\mu \tag{4}$$

Distributional Properties

The various distributional properties of the doubly truncated three parameter generalized Gaussian distribution are discussed in this section. Different shapes of the frequency curves for given values of the parameter are shown in figure 1

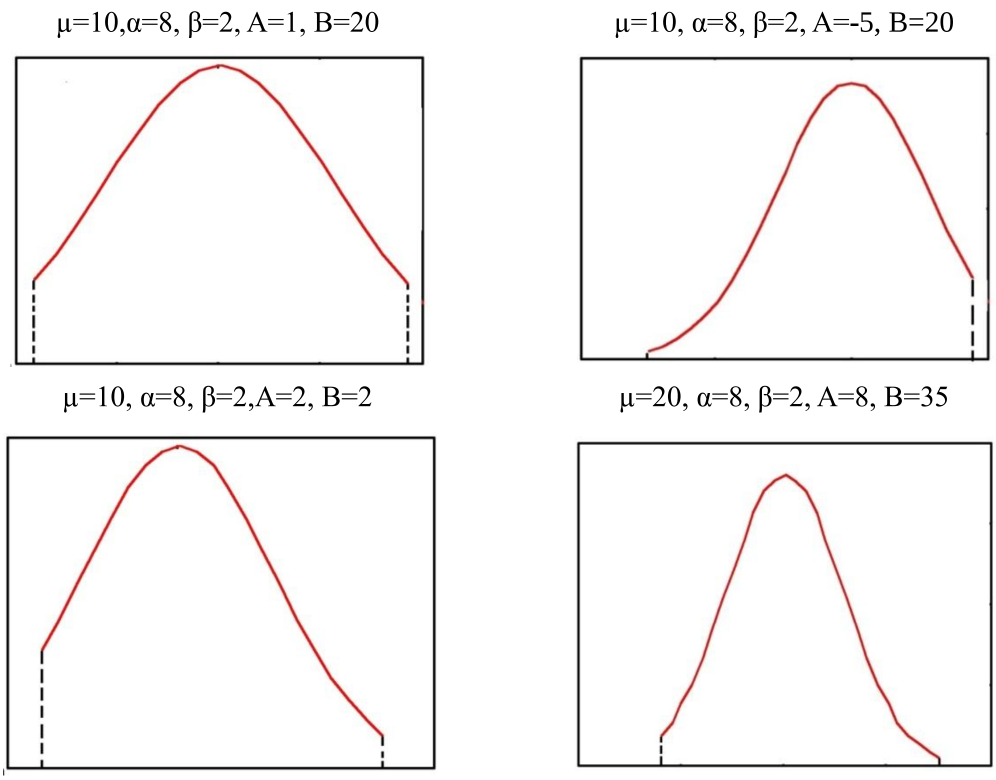

Figure 1: The frequency curves of the doubly truncated three parameter generalized Gaussian distribution.

The distribution function of X is given by

$$F(x)=\int_A^x \frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}\,dx$$

On simplification, we get

$$F(x)=\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^\beta\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)} \qquad \text{for} \quad A<\mu<B \tag{5}$$

Similarly, we get

$$F(x)=\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^\beta\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^\beta\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^\beta\right)} \qquad \text{for} \quad \mu<A<B \tag{6}$$

$$F(x)=\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^\beta\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^\beta\right)} \qquad \text{for} \quad A<B<\mu \tag{7}$$

where $\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^\beta\right),\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right),\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)$ are incomplete gamma functions.

The mean of the distribution is

$$E(X)=\int_A^B x\frac{\beta}{\alpha}\frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^\beta\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^\beta\right)}\,dx$$

On simplification, we get

$$E(X)=\mu+\alpha\left(\frac{\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\right) \qquad for \quad A<\mu<B \tag{8}$$

Similarly, we get

$$E(X)=\mu+\alpha\left(\frac{\gamma\left(\frac{2}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{2}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right) \qquad for \quad \mu<A<B \tag{9}$$

$$E(X)=\mu+\alpha\left(\frac{\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right) \qquad for \quad A<B<\mu \tag{10}$$

Let M be the median of the distribution, then we have

$$\int_A^M \frac{\beta}{\alpha}\frac{e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}\,dx=\frac{1}{2} \tag{11}$$

On simplification, we get

$$\frac{\gamma\left(\frac{1}{\beta},\left(\frac{M-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\left(\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)\right)}=\frac{1}{2} \qquad for \ A<\mu<B \tag{12}$$

$$\frac{\gamma\left(\frac{1}{\beta},\left(\frac{M-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\left(\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)\right)}=\frac{1}{2} \qquad for \ \mu<A<B \tag{13}$$

$$\frac{\gamma\left(\frac{1}{\beta},\left(\frac{M-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\left(\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}=\frac{1}{2} \qquad for\ A<B<\mu \tag{14}$$

The median M of the distribution can be obtained by solving the equations (12), (13) and (14).
For obtaining the mode of the distribution consider the probability density function of the distribution.

$$f(x)=K_1\, e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}} \qquad for \quad A\le\mu\le B$$

where $K_1=\dfrac{\frac{\beta}{\alpha}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}$

Taking logarithms on both sides, we get

$$\log\, f(x)=\log(K_1)-\left|\frac{x-\mu}{\alpha}\right|^{\beta}$$

Differentiating both sides w.r.to *x*, we get

$$\frac{d}{dx}\log\, f(x)=\frac{-\frac{\beta}{\alpha}\left|\frac{x-\mu}{\alpha}\right|^{\beta-1}\left|\frac{x-\mu}{\alpha}\right|}{\left(\frac{x-\mu}{\alpha}\right)}$$

$$\frac{d}{dx}\log\, f(x)=0$$

$$\Rightarrow\frac{-\frac{\beta}{\alpha}\left|\frac{x-\mu}{\alpha}\right|^{\beta-1}\left|\frac{x-\mu}{\alpha}\right|}{\left(\frac{x-\mu}{\alpha}\right)}=0 \tag{15}$$

Solving equation (15), we get x = μ.
Thus, $x=\mu$ is the unique solution which indicates this distribution is uni-model.

$$\frac{d^2}{dx^2}\log\, f(x)=-\frac{\beta}{\alpha^2}\left(\frac{\beta\left|\frac{x-\mu}{\alpha}\right|^{\beta}-\left(\frac{x-\mu}{\alpha}\right)}{\left(\frac{x-\mu}{\alpha}\right)^{2}}\right)<0$$

This distribution reaches its maximum of the point $x=\mu$

The raw moments of the distribution are

$$\mu_r' = \int_A^B x^r \frac{\frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^\beta\right)} dx$$

On simplification, we get

$$\mu_r' = \sum_{j=0}^{r} \binom{r}{j} \alpha^j \mu^{r-j} \left(\frac{\gamma\left(\frac{j+1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) + \gamma\left(\frac{j+1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^\beta\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^\beta\right)} \right) \quad \text{for } A<\mu<B \tag{16}$$

Similarly for $\mu<A<B$, the r^th non central moment is

$$\mu_r' = \sum_{j=0}^{r} \binom{r}{j} \alpha^j \mu^{r-j} \left(\frac{\gamma\left(\frac{j+1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) - \gamma\left(\frac{j+1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^\beta\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^\beta\right)} \right) \tag{17}$$

Similarly for $A<B<\mu$, the r^th non central moment is

$$\mu_r' = \sum_{j=0}^{r} \binom{r}{j} \alpha^j \mu^{r-j} \left(\frac{\gamma\left(\frac{j+1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^\beta\right) - \gamma\left(\frac{j+1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^\beta\right)}{+\gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^\beta\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^\beta\right)} \right) \tag{18}$$

The central moments of this distribution

$$\mu_r = \int_A^B (x-\mu-D)^r \frac{\frac{\beta}{\alpha} e^{-\left|\frac{x-\mu}{\alpha}\right|^\beta}}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^\beta\right)} dx$$

On simplification, we get

$$\mu_r = \sum_{j=0}^{r} \binom{r}{j} \alpha^j (-D)^{r-j} \left(\frac{\gamma\left(\frac{j+1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{j+1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right) \quad for\ A < \mu < B \quad (19)$$

where $D = \alpha \left(\frac{\gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{2}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \right)$

Similarly for $\mu < A < B$, the r^th central moment is

$$\mu_r = \sum_{j=0}^{r} \binom{r}{j} \alpha^j (-D)^{r-j} \left(\frac{\gamma\left(\frac{j+1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{j+1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right) \quad (20)$$

where $D = \alpha \left(\frac{\gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)} \right)$

Similarly for $A < B < \mu$, the r^th central moment is

$$\mu_r = \sum_{j=0}^{r} \binom{r}{j} \alpha^j (-D)^{r-j} \left(\frac{\gamma\left(\frac{j+1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{j+1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} \right) \quad (21)$$

where $D = \alpha \left(\frac{\gamma\left(\frac{2}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{2}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^{\beta}\right) - \gamma\left(\frac{1}{\beta}, \left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)} \right)$

The skewness of the distribution is

$$\beta_1 = \frac{\left(2S_1^3 - 3S_1S_2 + S_3\right)^2}{\left(S_2 - S_1^2\right)^3} \qquad for \quad A < \mu < B$$

$$\beta_1 = \frac{\left(2P_1^3 - 3P_1P_2 + P_3\right)^2}{\left(P_2 - P_1^2\right)^3} \qquad for \quad \mu < A < B$$

$$\beta_1 = \frac{\left(2Q_1^3 - 3Q_1Q_2 + Q_3\right)^2}{\left(Q_2 - Q_1^2\right)^3} \qquad for \quad A < B < \mu$$

Kurtosis of the distribution is

$$\beta_2 = \frac{3S_1^2\left(2S_2 - S_1^2\right) + S_4 - 4S_1S_3}{\left(S_2 - S_1^2\right)^2} \qquad for \ A < \mu < B$$

$$\beta_2 = \frac{3P_1^2\left(2P_2 - P_1^2\right) + P_4 - 4P_1P_3}{\left(P_2 - P_1^2\right)^2} \qquad for \ \mu < A < B$$

$$\beta_2 = \frac{3Q_1^2\left(2Q_2 - Q_1^2\right) + Q_4 - 4Q_1Q_3}{\left(Q_2 - Q_1^2\right)^2} \qquad for \ A < \mu < B$$

where $S_1 = \left(\frac{\gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) + \gamma\left(\frac{2}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^\beta\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^\beta\right)} \right)$; $\quad S_2 = \left(\frac{\gamma\left(\frac{3}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) + \gamma\left(\frac{3}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^\beta\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^\beta\right)} \right)$;

$$S_3 = \left(\frac{\gamma\left(\frac{4}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) + \gamma\left(\frac{4}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^\beta\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^\beta\right)} \right); \qquad S_4 = \left(\frac{\gamma\left(\frac{5}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) + \gamma\left(\frac{5}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^\beta\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) + \gamma\left(\frac{1}{\beta}, \left|\frac{A-\mu}{\alpha}\right|^\beta\right)} \right).$$

$$P_1 = \left(\frac{\gamma\left(\frac{2}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) - \gamma\left(\frac{2}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^\beta\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^\beta\right)} \right); \qquad P_2 = \left(\frac{\gamma\left(\frac{3}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) - \gamma\left(\frac{3}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^\beta\right)}{\gamma\left(\frac{1}{\beta}, \left(\frac{B-\mu}{\alpha}\right)^\beta\right) - \gamma\left(\frac{1}{\beta}, \left(\frac{A-\mu}{\alpha}\right)^\beta\right)} \right)$$

$$P_3=\left(\frac{\gamma\left(\frac{4}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{4}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right);\qquad P_4=\left(\frac{\gamma\left(\frac{5}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{5}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}\right).$$

and

$$Q_1=\left(\frac{\gamma\left(\frac{2}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{2}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right);\qquad Q_2=\left(\frac{\gamma\left(\frac{3}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{3}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right);$$

$$Q_3=\left(\frac{\gamma\left(\frac{4}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{4}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right);\qquad Q_4=\left(\frac{\gamma\left(\frac{5}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{5}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}\right).$$

The hazard rate function of the distribution is $h(x)=\dfrac{f(x)}{1-F(x)}$

$$h(x)=\left(\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+2\,\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}\right)\qquad \textit{for}\ \ A<\mu<B$$

Similarly for $\mu<A<B$

$$h(x)=\left(\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}\right)$$

Similarly for $A<\mu<B$

$$h(x)=\left(\frac{\frac{\beta}{\alpha}e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}}}{2\ \gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)}\right)$$

The survival rate function S(x) is $S(x)=1-F(x)$

$$S(x)=1-\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)} \qquad for\ \ A<\mu<B$$

Similarly for $\mu<A<B$

$$S(x)=1-\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left(\frac{B-\mu}{\alpha}\right)^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left(\frac{A-\mu}{\alpha}\right)^{\beta}\right)}$$

Similarly for $A<B<\mu$

$$S(x)=1-\frac{\gamma\left(\frac{1}{\beta},\left(\frac{x-\mu}{\alpha}\right)^{\beta}\right)+\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)}{\gamma\left(\frac{1}{\beta},\left|\frac{A-\mu}{\alpha}\right|^{\beta}\right)-\gamma\left(\frac{1}{\beta},\left|\frac{B-\mu}{\alpha}\right|^{\beta}\right)}$$

Conclusions

In this paper, we introduced a doubly truncated Generalized Gaussian Distribution (GGD). Generalized Gaussian Distribution (GGD) got lot of applications in analyzing several data sets as an alternative to the Gaussian distribution where the variable under study is lepty or platy or meso kurtic and symmetric. A doubly truncated GGD includes GGD as limiting case where the truncated point tends to infinite. The various distributional properties such as distribution function, moments, hazard function, and survival function are derived. It is observed that the hazard function is sometimes increasing and decreasing depending upon the truncation parameter. It also includes a J-type distribution when the truncation point is greater than its location parameter.

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