ϵ-pseudospectrum of a matrix and the numerical range

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Abstract

Numerical range and pseudospectra of a matrix play a crucial role in different area and have several applications. The numerical range of a matrix A is determined by the behavior of the pseudospectra $\Lambda_\epsilon(A)$ in the limit $\epsilon \to \infty$. In this paper, we give some properties of the pseudospectra of a matrix. We estimate $\|f(A)\|$ where $f(A)$ is the Cauchy's integral formula for matrices. Some results concerning particular quantities which are important in the study of time-dependent dynamical systems are proposed, too.

Key words. Pseudospectra; numerical range; numerical positive abscissa; numerical negative abscissa.
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1 Introduction

Pseudospectra were introduced by H. J. Landau (1975), who used the term ϵ-spectrum [9]. In 1979 J. M. Varah published a paper entitled " On the separation of two matrices, " in which he defined the 2-norm ϵ- pseudospectrum using the smallest singular value $\sigma_{min}(A - \lambda I)$, giving it the name ϵ- spectrum and the notation $S_\epsilon(A)$, see [**?**]. Also pseudospectra were investigated in several papers by L. N. Trefethen in the 1990s, [3, 10, 11, 12, 13, 14, **?**]. In recent years, the study of pseudospectra has been very active,wq numerous contributions related to pseudospctra were made by various people, including J. S. Baggett [1], A. Böttcher [2], T. A. Driscoll [3], M. Embree [**?**], N. Higham [6], S. C. Reddy [10], L. Reichel [11].
It is known [12], [**?**] that, the ϵ-pseudospectrum of a *normal* matrix A (A is

a normal matrix if and only if $AA^* = A^*A$ where A^* is the conjugate transpose of A) is consisted of circles of radius ϵ about each eigenvalue of A. For a nonnormal matrix, ϵ-pseudospectrum takes different shapes in the complex plane. In [13], the pseudospectra of thirteen highly nonnormal matrices are shown. In this paper, we characterize the pseudospectra of a matrix using the conjugate transpose of the ϵ-pseudo-eigenvector. We estimate $\|f(A)\|$ where $f(A)$ is the Cauchy's integral formula for matrices. Here $\mathbb{C}^n$ and $\mathbb{C}^{n\times n}$ stand for complex $n-$vectors and $n \times n$ matrices respectively. $\langle x, y\rangle = x^*y$ is the inner product of the vectors x and y.

2 Pseudospectra of a matrix

Four definitions of pseudospectra are given in [14], [**?**]. The first deals with the perturbation, the second is related to the resolvent, the third is given with the use of a normalized ϵ-pseudo-eigenvectors and ϵ-pseudo-eigenvalues, finally the fourth use the smallest singular value. Other definitions are presented in [8].

Definition 2.1. *Let $A \in \mathbb{C}^{n\times n}$ and $\epsilon \geq 0$ be arbitrary. The ϵ-pseudospectrum $\Lambda_\epsilon(A)$ of A is the set of $z \in \mathbb{C}$ such that*

$$z \in \Lambda(A + E) \tag{1}$$

for some $E \in \mathbb{C}^{n\times n}$ with $\|E\| \leq \epsilon$.
$\Lambda(A + E)$ denotes the spectrum of the matrix $(A + E)$.

The 0-pseudospectrum of A is just the spectrum of A i.e., $\Lambda_0(A) = \Lambda(A)$.

Definition 2.2. *The ϵ-pseudospectrum $\Lambda_\epsilon(A)$ of A is the set of $z \in \mathbb{C}$ such that*

$$\|(zI - A)^{-1}\| \geq \epsilon^{-1}. \tag{2}$$

I is the identity matrix and $(zI - A)^{-1}$ is the resolvent of A at z.

Definition 2.3. *The ϵ-pseudospectrum $\Lambda_\epsilon(A)$ of A is the set of $z \in \mathbb{C}$ such that*

$$\|(zI - A)v\| \leq \epsilon \tag{3}$$

for some $v \in \mathbb{C}^n$ with $\|v\| = 1$.
z is an ϵ-pseudo-eigenvalue *of A, and v is a corresponding* ϵ-pseudo-eigenvector.

Definition 2.4. *(Assuming that the norm is $\|\cdot\|_2$)*
The ϵ-pseudospectrum $\Lambda_\epsilon(A)$ of A is the set of $z \in \mathbb{C}$ such that

$$\sigma_{min}(zI - A) \leq \epsilon. \tag{4}$$

$\sigma_{min}(zI - A)$ denotes the smallest singular value of the matrix $(zI - A)$.

For the equivalence of these definitions, see [14].

Theorem 2.5. *Let $A \in \mathbb{C}^{n\times n}$ and $\epsilon \geq 0$ be arbitrary. The ϵ-pseudospectrum $\Lambda_\epsilon(A)$ of A is the set of $z \in \mathbb{C}$ such that*

$$\|u^*(zI - A)\| \leq \epsilon \tag{5}$$

for some $u \in \mathbb{C}^n$ with $\|u\| = 1$.
u^* is the conjugate transpose of u.

Proof. Let $z \in \Lambda_\epsilon(A)$ and let v be its corresponding left eigenvector of the matrix $(A + E)$ with $\|E\| \leq \epsilon$. Thus $v^*(A + E) = zv^*$, then $\dfrac{v^*}{\|v\|}(zI - A) = \dfrac{v^*}{\|v\|}E$. Hence $\|u^*(zI - A)\| \leq \epsilon$ with $u = \dfrac{v^*}{\|v\|}$ and $\|u\| = 1$.
Now let $\|u^*(zI - A)\| \leq \epsilon$, then there exist η with $0 < \eta \leq \epsilon$ and $\phi \in \mathbb{C}^n$ where $\|\phi\| = 1$, such that $u^*(zI - A) = \eta\phi^*$. Choosing $E = \eta u\phi^*$, it follows $E \in \mathbb{C}^{n\times n}$, $\|E\| = \|\eta u\phi^*\| \leq \eta\|u\|\|\phi^*\| \leq \eta \leq \epsilon$ and $u^*E = u^*(zI - A)$. Hence $z \in \Lambda_\epsilon(A)$. □

Here we give some useful functional properties of the pseudospectra of a matrix. The first one can be found in [14].

Properties 2.6. *Let $A \in \mathbb{C}^{n\times n}$, $F = (f_{ij}) \in \mathbb{C}^{n\times n}$ then*

1. $\Lambda_{\epsilon_1}(A) \subseteq \Lambda_{\epsilon_2}(A), \qquad 0 \leq \epsilon_1 \leq \epsilon_2.$
2. *If $|f_{ij}| \leq 1$ then $\Lambda_\epsilon(A + F) \subseteq \Lambda_{n+\epsilon}(A)$.*
3. $\Lambda_{\epsilon_1}(A) + \Lambda_{\epsilon_2}(A) \subseteq \Lambda_{2\epsilon_1+2\epsilon_2}(2A).$

In the third property a sum of sets has the usual meaning
$\Lambda_{\epsilon_1}(A) + \Lambda_{\epsilon_2}(A) = \{z : z = z_1 + z_2,\ z_1 \in \Lambda_{\epsilon_1}(A), z_2 \in \Lambda_{\epsilon_2}(A)\}$.

Proof. 1. Let $z \in \Lambda_{\epsilon_1}(A)$, there exists $E \in \mathbb{C}^{n\times n}$ where $\|E\| \leq \epsilon_1$ such that $z \in \Lambda(A+E)$. Since $\epsilon_1 \leq \epsilon_2$, it follows $z \in \Lambda(A+E)$ where $\|E\| \leq \epsilon_2$. Hence $z \in \Lambda_{\epsilon_2}(A)$.
2. Let $z \in \Lambda_\epsilon(A + F)$, then $z \in \Lambda(A + F + E)$ where $\|E\| \leq \epsilon$. We have $\|F + E\| \leq \epsilon + \|F\|$, then $\|F + E\| \leq n + \epsilon$ hence $z \in \Lambda_{n+\epsilon}(A)$.
3. Let $z \in \Lambda_{\epsilon_1}(A) + \Lambda_{\epsilon_2}(A)$, then $z = z_1 + z_2$ with $z_1 \in \Lambda_{\epsilon_1}(A)$ and $z_2 \in \Lambda_{\epsilon_2}(A)$. Assume that u_1 is the normalized ϵ-pseudo-eigenvector of A corresponding to z_1, so $(A + E_1)u_1 = z_1u_1$, $\|E_1\| \leq \epsilon_1$ and $(A + E_2)u_1 = z_2u_1 + w_2$, $\|E_2\| \leq \epsilon_2$, $w_2 \in \mathbb{C}^n$. Thus $zu_1 = z_1u_1 + z_2u_1$, then $zu_1 = (2A+E_1+E_2-w_2u_1^*)u_1$. On the other hand, $\|E_1+E_2-w_2u_1^*\| \leq \epsilon_1+\epsilon_2+\|w_2\|$ with $\|w_2\| = \|(A + E_2)u_1 - z_2u_1\| \leq \|(z_2 - A)u_1\| + \|E_2\| \leq \|(z_1 - A)u_1\| + |z_1 - z_2| + \epsilon_2$. Hence $\|w_2\| \leq \epsilon_1 + \epsilon_2 + |z_1 - z_2|$, therefore, $\|E_1 + E_2 - w_2u_1^*\| \leq 2\epsilon_1 + 2\epsilon_2 + |z_1 - z_2|$. Taking $z_1 = z_2$ it follows $z \in \Lambda_{2\epsilon_1+2\epsilon_2}(2A)$. □

To prove the third property, we can also use u_2 the normalized ϵ-pseudo-eigenvector of A corresponding to z_2 instead of u_1. In the general case, this property becomes

$$\sum_{i=1}^{n} \Lambda_{\epsilon_i}(A) \subseteq \Lambda_{n\epsilon_i + \sum_{k=1,k\neq i}^{n} 2\epsilon_k}(nA). \tag{6}$$

To prove (6), we use u_i the normalized ϵ-pseudo-eigenvector of A corresponding to z_i and we follow the same steps as it is shown in the above proof i.e., we give the upper bounds of $\|w_k\|$ where $k \in \{1, 2, \ldots, n\}$, $k \neq i$.
It is shown in [14], [**?**] that, if a matrix A is normal, then $\Lambda_\epsilon(A) = \Lambda(A) + \Delta_\epsilon$ where Δ_ϵ is the closed disk of radius ϵ about the origin. Hence

$$\Lambda_\epsilon(\gamma I) = \overline{D}(\gamma, \epsilon), \quad \gamma \in \mathbb{C}.$$

$\overline{D}(\gamma, \epsilon)$ is the closed disk of radius ϵ and center γ.

Proposition 2.7. *Let $A \in \mathbb{C}^{n\times n}$ and $\epsilon \geq 0$ be arbitrary. Then there exist $\alpha \in \mathbb{C}$ and $r_\epsilon > 0$, such that*

$$\Lambda_\epsilon(A) \subseteq \Lambda_{r_\epsilon}(\alpha I). \tag{7}$$

I is the identity matrix of dimension n.

Proof. Let $z_k \in \partial\Lambda_\epsilon(A)$, $k \in \{1, 2, \ldots, m\}$, where $\partial\Lambda_\epsilon(A)$ is the boundary of $\Lambda_\epsilon(A)$. Choosing α to be the barycenter of $\{(z_k, 1) \text{ with } k \in \{1, 2, \ldots, m\}\}$ and $r_\epsilon = \sup_{z_k\in\partial\Lambda_\epsilon(A)} |\alpha - z_k|$. Since αI is a normal matrix, then it is sufficient to prove that $\Lambda_\epsilon(A) \subseteq \overline{D}(\alpha, r_\epsilon)$ where $\overline{D}(\alpha, r_\epsilon)$ is the closed disk of radius r_ϵ and center α. If $z \in \Lambda_\epsilon(A)$, then $|\alpha - z| \leq r_\epsilon$. Hence $z \in \overline{D}(\alpha, r_\epsilon)$. □

The points of $\Lambda_\epsilon(A)$ lie in the interior and on the boundary of the closed disk $\overline{D}(\alpha, r_\epsilon)$.

Theorem 2.8. *$\overline{D}(\alpha, r_\epsilon)$ is the smallest closed disk which contains $\Lambda_\epsilon(A)$.*

Proof. Since α is the barycenter, then it is unique. Now suppose that, there exists another smallest closed disk $\overline{D'}(\alpha, r'_\epsilon)$ which contains $\Lambda_\epsilon(A)$, therefore, $r'_\epsilon < r_\epsilon$. There exists at least one point z_k on the boundary of $\Lambda_\epsilon(A)$ such that $|\alpha - z_k| = r_\epsilon$. On the other hand, $|\alpha - z_k| < r'_\epsilon$ (by the supposition on $\overline{D'}(\alpha, r'_\epsilon)$), it follows $r_\epsilon < r'_\epsilon$ contradiction. Hence $\overline{D}(\alpha, r_\epsilon)$ is the smallest closed disk which contains $\Lambda_\epsilon(A)$. □

Let $f(A)$ be defined by the operator analogue of the Cauchy integral formula, sometimes called a *Dunford-Taylor integral* :

$$f(A) = \frac{1}{2\pi i} \oint_\Gamma f(z)(zI - A)^{-1} dz.$$

Here $A \in \mathbb{C}^{n\times n}$ and Γ consists of a finite number of simple, closed curves Γ_k with interiors Ω_k such that

1. $f(z)$ is analytic on Γ_k and Ω_k.

2. Each eigenvalue λ_i is contained in some Ω_k.

We refer reader to [**?**] for more details.

Theorem 2.9. *Let f be analytic on $\Lambda_\epsilon(A)$ for some $\epsilon > 0$. Then*

$$\|f(A)\| \leq \frac{r_\epsilon}{\epsilon} \sup_{z\in\Lambda_\epsilon(A)} |f(z)|. \tag{8}$$

Proof. Pick Γ to be the boundary of $\overline{D}(\alpha, r_\epsilon)$

$$\begin{aligned}
\|f(A)\| &= \|\frac{1}{2\pi i}\int_{\partial\overline{D}(\alpha,r_\epsilon)} f(z)(zI-A)^{-1}dz\| \\
&\leq \frac{1}{2\pi}\int_{\partial\overline{D}(\alpha,r_\epsilon)} |f(z)|\|(zI-A)^{-1}\||dz| \\
&= \frac{1}{2\pi\epsilon}\int_{\partial\overline{D}(\alpha,r_\epsilon)} |f(z)||dz| \\
&\leq \frac{1}{2\pi\epsilon}\sup_{z\in\Lambda_\epsilon(A)}|f(z)|\int_{\partial\overline{D}(\alpha,r_\epsilon)}|dz| \\
&\leq \frac{r_\epsilon}{\epsilon}\sup_{z\in\Lambda_\epsilon(A)}|f(z)|.
\end{aligned}$$

□

Definition 2.10. *the ϵ-pseudospectral abscissa is the supremum of the real parts of $z \in \Lambda_\epsilon(A)$ i.e.,*

$$\alpha_\epsilon(A) = \sup_{z\in\Lambda_\epsilon(A)} Re\, z. \tag{9}$$

Theorem 2.11. *For any matrix $A \in \mathbb{C}^{n\times n}$ and $\epsilon > 0$,*

$$\|e^{tA}\| \leq \frac{r_\epsilon}{\epsilon} e^{t\alpha_\epsilon(A)}. \tag{10}$$

Proof. Applying the Cauchy integral formula to e^{tz}, so

$$\begin{aligned}
\|e^{tA}\| &= \|\frac{1}{2\pi i}\int_{\partial\overline{D}(\alpha,r_\epsilon)} e^{tz}(z-A)^{-1}dz\| \\
&\leq \frac{1}{2\pi}\int_{\partial\overline{D}(\alpha,r_\epsilon)} |e^{tz}|\|(z-A)^{-1}\||dz| \\
&= \frac{1}{2\pi\epsilon}\int_{\partial\overline{D}(\alpha,r_\epsilon)} |e^{tz}||dz| \\
&\leq \frac{1}{2\pi\epsilon}\sup_{z\in\Lambda_\epsilon(A)}|e^{tz}|\int_{\partial\overline{D}(\alpha,r_\epsilon)}|dz| \\
&\leq \frac{r_\epsilon}{\epsilon}\sup_{z\in\Lambda_\epsilon(A)} e^{tRe\, z} \\
&\leq \frac{r_\epsilon}{\epsilon} e^{t\alpha_\epsilon(A)}.
\end{aligned}$$

Definition 2.12. *The ϵ-pseudospectral radius is the supremum of magnitudes of points in $\Lambda_\epsilon(A)$ i.e.,*

$$\rho_\epsilon(A) = \sup_{z\in\Lambda_\epsilon(A)} |z|. \tag{11}$$

Theorem 2.13. *For any matrix $A \in \mathbb{C}^{n\times n}$ and $\epsilon > 0$,*

$$\|A^k\| \leq \frac{r_\epsilon}{\epsilon}\rho_\epsilon(A)^k. \tag{12}$$

Proof. Applying the Cauchy integral bound to z^k, so

$$\begin{aligned}
\|A^k\| &= \|\frac{1}{2\pi i}\int_{\partial\overline{D}(\alpha,r_\epsilon)} z^k(z-A)^{-1}dz\| \\
&\leq \frac{1}{2\pi}\int_{\partial\overline{D}(\alpha,r_\epsilon)} |z^k|\|(z-A)^{-1}\||dz| \\
&= \frac{1}{2\pi\epsilon}\int_{\partial\overline{D}(\alpha,r_\epsilon)} |z^k||dz| \\
&\leq \frac{1}{2\pi\epsilon}\sup_{z\in\Lambda_\epsilon(A)}|z^k|\int_{\partial\overline{D}(\alpha,r_\epsilon)}|dz| \\
&\leq \frac{r_\epsilon}{\epsilon}\rho_\epsilon(A)^k.
\end{aligned}$$

□

Corollary 2.14. *Let f be analytic on $\Lambda_\epsilon(A)$ for some $\epsilon > 0$. If A is a normal matrix, then*

1. $\|f(A)\| \leq \sup_{z\in\Lambda_\epsilon(A)} |f(z)|$.
2. $\|e^{tA}\| \leq e^{t\alpha_\epsilon(A)}$.
3. $\|A^k\| \leq \rho_\epsilon(A)^k$.

Proof. If A is a normal matrix, then $r_\epsilon = \epsilon$. □

3 Numerical range

The numerical range $W(.)$ is a set of complex numbers associated with a given matrix $A \in \mathbb{C}^{n\times n}$:

$$W(A) = \{x^*Ax : x \in C^n, x^*x = 1\}. \tag{13}$$

Let $A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{n\times n}$, it is shown in [7] that

$$W(A+B) \subset W(A) + W(B).$$

The following proposition shows that $W(A)$ is as robust as one could desire.

Proposition 3.1. *Given $\epsilon \geq 0$, let $A \in \mathbb{C}^{n\times n}$ and $E \in \mathbb{C}^{n\times n}$ such that $\|E\| \leq \epsilon$, then*

$$W(A+E) \subseteq W(A) + \Delta_\epsilon$$

where Δ_ϵ is the closed disk of radius ϵ about 0.

Proof. We have $W(A+E) \subseteq W(A) + W(E)$, assume that $z \in W(E)$, then $z = x^*Ex$ where $\|x\| = 1$, thus $|z| \leq \epsilon$. Hence $W(A+E) \subseteq W(A) + \Delta_\epsilon$. $\square$

Let the numerical positive abscissa of a matrix A be defined by

$$\omega^+(A) = \sup_{z\in W(A)} \operatorname{Re} z. \tag{14}$$

In the Hilbert space case, see [?], the numerical positive abscissa is given by the formula

$$\omega^+(A) = \sup \lambda \,\text{where}\, \lambda \in \Lambda\left(\frac{A+A^*}{2}\right). \tag{15}$$

$\Lambda\left(\frac{A+A^*}{2}\right)$ denotes the spectrum of $\left(\frac{A+A^*}{2}\right)$.
Let the numerical negative abscissa of a matrix A be defined by

$$\omega^-(A) = \inf \lambda \,\text{where}\, \lambda \in \Lambda\left(\frac{A+A^*}{2}\right). \tag{16}$$

Proposition 3.2. *For any matrix $A \in \mathbb{C}^{n\times n}$,*

$$\omega^+(A^*A) \leq \|A\|^2. \tag{17}$$

Proof. $\omega^+(A^*A) = \sup_{\|x\|=1} \operatorname{Re} \langle x, A^*Ax\rangle \leq \sup_{\|x\|=1} |\langle x, A^*Ax\rangle| \leq \sup_{\|x\|=1} |\langle Ax, Ax\rangle| \leq \|A\|^2$. $\square$

Theorem 3.3. *Let $A \in \mathbb{C}^{n\times n}$, $z \in \mathbb{C}$ then*

$$\omega^+(zI - A) = \mathit{Re}\, z - \omega^-(A)$$

Proof. Let $z \in \mathbb{C}$, we assume that $\omega^+(zI - A) = \omega_0$, so $\omega_0 = \sup \lambda$ where $\lambda \in \Lambda\left(\frac{zI - A + \overline{z}I - A^*}{2}\right)$. Then $(\operatorname{Re} z - \omega_0) = \inf \lambda$ where $\lambda \in \Lambda\left(\frac{A+A^*}{2}\right)$, we have $\omega^+(zI - A) = \omega_0$ implies $\omega^-(A) = \operatorname{Re} z - \omega_0$. Hence the desired result is obtained. $\square$

Conclusion: The ϵ-pseudospectrum of a matrix is a subset of $\mathbb{C}$ that can be used to learn something else about the matrix and it can also give information that the spectrum alone cannot give. In general the spectrum is sensitive to perturbation, whereas the numerical range is not.

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