Dynamics of the interaction of species in the Keta-Anlo Wetland Ecosystem of Ghana.

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Abstract

This research work seeks to provide a mathematical model to study and analyze the dynamics of the Keta-Anlo wetland of Ghana using the Beddington-DeAngelis functional response. Wetlands across the world are valuable assets for both plants and animals (aquatic and terrestrial) as well as Human. They provide source of food for many animals, habitat for both native and migratory birds and also source of income for surrounding communities.

This wetland is made up of different species of both plants and animals.The ecosystem shall be divided into three compartments, namely: Good biomass (comprising of plants, micro organism in the soil and floating vegetation), Fish population and Bird (including other animals) population. We therefore propose a mathematical model to study the interaction between the good biomass, the fish and birds/other animals population.

The interaction shall be a prey-predator type with the good biomass being the prey and the fish and birds the predator. The fish population is both a predator to the good biomass and a prey to the bird population. ie the bird population feeds on both the good biomass and the fish as well. Hence the feeding process is a food web. The predator feeds on the prey by the *Beddington de Angelis* density dependent functional response.

We shall then analyze the dynamical behavior of the interacting species by finding the ecologically feasible equilibrium points, the local and global stability of the system. We also discuss the state of coexistence among the species and finally perform numerical simulation to support the continuous sustenance of the ecosystem.

keywords: Lyapunov stability, Beddington-DeAgelis functional response, Stability.

Introduction:

Wetland ecosystems are valuable assets across the world over. To manage and maintain these wetlands effectively and efficiently requires the understanding of the dynamics of activities and environmental characteristics around it.

The Wetlands of Ghana are an ecological resource which provide food and habitat for thousands of both migratory and native birds, marine turtles, different species of fish and plants[2]. The wetland ecosystem is distributed all over the entire country and represents approximately ten percent of the total land size of the country. Wetlands are closely associated with the numerous inland fresh water and coastal marine systems in the country. The interface between the fresh water and marine ecosystem is the coastal lagoon systems and mangroves. There are also upland wetlands which are rather rare and limited in their occurrence[7].

There are numerous marine and brackish wetlands along the entire 500km coastline of Ghana. These include the Keta Lagoon Complex, Songhor, Sakumono, Muni lagoons and the Densu Delta which have been designated as Ramsar Sites in view of their national and international importance. Some useful wetlands also include the Amanzuri, Ehulu, Korle, Kpeshie, Butre, Fosu lagoons as well as Ankobra and Whin estuaries[7].

Freshwater wetlands in Ghana are important for agriculture, water supply, recreational and other ecosystem services. This notwithstanding, they have not been adequately documented. The Owabi Wildlife Sanctuary is the only inland Ramsar Site in Ghana. Upland wetlands are rather rare ecosystem with very limited occurrence in Ghana. This is usually a marshy area at high altitudes and usually constitutes the source of rivers and streams[7].

The Keta Lagoon is the largest wetland in Ghana and it is one of the major source of inflows into the Volta River which discharges into the ocean. The Keta wetland provide the most valuable wetland habitat for water birds along Ghana's coast and covers a land size of 101 hectares and stretches 40km along the coast [7]. It is characterized by brackish water as well as mud and salt flats which is economically exploited for the production of salt. Since the construction of the Akosombo dam in 1964, large amounts of water from the Volta River are retained upstream and freshwater as well as sediment inflow into the wet-land has been reduced [7]

The Keta lagoon is the most important wetland on the Ghana coast for water-birds and together with the Songor Lagoon constitutes the fourth most important water-birds site on the

Gulf of Guinea coast. The site supports 76 resident and migratory water-bird species with an estimated population of 110,000, including globally significant numbers of 21 species[7]. The Keta Lagoon and its surrounding wetlands qualifies as an internationally important wetland under the Ramsar criteria by virtue of its total bird population. The site is particularly important for waders, supporting almost one third of the estimated East Atlantic Flyway population of Tringa erythropus.

According to Junk[6], natural wetland are found all over the world and provide important useful functions for both hydrological and biogeochemical cycles. Wetlands are also amongst the world's most productive environments in terms of bio-diversity and primary productivity and hence delivers natural resources that are often directly or indirectly exploited by humans for economic benefits[5][4]. Important ecosystem services provided by wetlands include storm prevention, flood control, continuous water supply, maintenance of the water table, groundwater recharge as well as nutrient and pollution retention in flood-plains[2][8]. Wetland specific assets also comprise the support of rich wildlife, fisheries, fertile soil for agriculture, timber and energy supply in addition to recreational and tourist opportunities. In the context of climate change, natural wetlands may act as important carbon sink, thereby reducing greenhouse emissions and global warming[12].

Early studies of the ecological modeling focused mainly on the predation role, neglecting the role of mutual interference among predators. Jianfeng Fenga et al [5] therefore examined the stability of ecosystem induced by mutual interference between predators. A three-species predator-prey model with Beddington-DeAngelis functional response was established with a food chain including two species of zooplanktons and one of phytoplankton. Results from the research showed that, in a certain range, increasing competition and saturation constant factor could stabilize the ecosystem[11].

Mathematical model

We shall consider a wetland comprising of a good biomass density (Prey) $G(t)$, fish population $F(t)$ and the population of birds and other animals $P(t)$ in the ecosystem at any time t. The good biomass $G(t)$ is the accumulation of floating vegetation, crawling and flying insects and any other species which is consumable by both fish and birds/other animals.

We shall assume that the dynamics of the good biomass is governed by the logistics equation

whiles both fish and bird population interferes with the good biomass by the Beddington type functional response. The fish is also interfered by the birds and other animals. The ratio-dependent functional response is just one example of the more general predator dependent functional responses[1]. For cases where predators compete directly for catching prey the most famous functional response is the function proposed by Beddington (1975)[11].

r is the growth rate of good biomass G, K is the carrying capacity, a, b are the rate of capture of good biomass by fish population and bird and other animals respectively, f is the rate of capture of the fish population by the bird and other animals population. c is the rate of competition between good biomass before being fed on, d is the competition between fish to hunt for food, e is the rate of competition between birds and other animals, α, β are natural rate of death of fish and birds respectively, m, n, s measures the predator's efficiency to convert prey biomass into fertility (reproductivity).

With the above assumptions, we develop the underlying differential equations as follows:

We shall setup the associated model to the prey-predator interaction, by making the following assumptions:

i. The prey population (good biomass) grows logistically

ii. The fish population will grow depending on the conversion rate from predating on Good Biomass and decrease due to predation of birds and animals and also natural death rate.

iii. The Birds and Animals population will increase based on the conversion rate from predating on both good biomass and fish, and decrease due to natural death rate only.

iv. No harvesting is allowed in the ecosystem and each predator has equal chances of capturing a prey.

The model equation is

$$\frac{dG}{dt} = rG\left(1 - \frac{G}{K}\right) - \frac{aGF + bGP}{1 + cG + dF + eP} \tag{1}$$

$$\frac{dF}{dt} = -\alpha F + \frac{maGF - fFP}{1 + cG + dF + eP} \tag{2}$$

$$\frac{dP}{dt} = -\beta P + \frac{nfFP + sbFP}{1 + cG + dF + eP} \tag{3}$$

where $G(t) \geq 0$, $F(t) \geq 0$, $P(t) \geq 0$ and $t \geq 0$.

Existence and Boundedness of model

Theorem 1: The solution to the system of equations 1, 2 and 3 exist in the positive octant, R_+^3, for all values of $t \geq 0$ and is bounded.

proof: From 1, we have $\dfrac{dG}{dt} \leq rG\left(1 - \dfrac{G}{K}\right)$. Solving for $G(t)$, we obtain:

$G(t) \leq \dfrac{K}{1 - (1 - \dfrac{K}{G})\exp^{-rt}}$. For this to hold, $1 - \dfrac{G}{K} \geq 0. \implies G \leq K$.

Hence $0 \leq G(t) \leq K$ for all $t \geq 0$. The boundedness of F and P follows from the fact that G is bounded since the existence of F and P depends on G.

Existence of Equilibrium

The ecologically reasonable equilibrium points corresponding to 1, 2 and 3 are $E_0(0,0,0)$, $E_1(G^*,0,0)$, $E_2(G^*,0,P^*)$, $E_3(G^*,F^*,0)$ and $E_4(G^*,F^*,P^*)$. E_0 and E_1 will always exist. We therefore show the existence of E_2, E_3 and E_4.

Existence of $E_2(G^*,0,P^*)$:

From 1 and equation 3, we have

$$r\left(1 - \frac{G}{K}\right) = \frac{bP}{1 + cF + eP} \tag{4}$$

$$\beta = \frac{sbG}{1 + cG + eP} \tag{5}$$

From 5

$$P^* = \frac{(sb - \beta c)G - \beta}{sb} \tag{6}$$

$P^* > 0$ if $s > \dfrac{\beta c}{b}$. Substituting P^* into 4, we have $G^* = \dfrac{-B + \sqrt{B^2 - 4AC}}{2A}$. where,

$A = bcekr + bers + \beta cer$, $B = \beta cekr + \beta er + \beta bck + brs + b^2ks - bckrs - bekrs$ and $C = \beta ekr + \beta bk - bkrs$.

G^* is positive if $C < 0$. ie. $\beta < \dfrac{brs}{b + er}$ and hence exist. Knowing the positive value of G^*, P^* can be found from 6

Existence of $E_3(G^*, F^*, 0)$

From 1 and 2 we obtain

$$r\left(1 - \frac{G}{K}\right) = \frac{aF}{1 + cG + dF} \tag{7}$$

$$\alpha = \frac{maG}{1 + cG + dF} \tag{8}$$

Hence from 8,

$$F^* = \frac{(am - \alpha c)G - \alpha}{\alpha d} \tag{9}$$

$F > 0$ if $G > \dfrac{\alpha}{am - \alpha c}$ implying $\alpha < \dfrac{am}{c}$

Substituting 9 into 7 and rearranging, we have

$$G^2 + \left(\frac{aK}{rd} + \frac{\alpha c}{dmr} - K\right)G - \frac{\alpha K}{dmr} = 0$$

and

$$G^* = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

Where $A = 1$, $B = \left(\dfrac{aK}{rd} + \dfrac{\alpha c}{dmr} - K\right)$ and $C = -\dfrac{\alpha K}{dmr}$.

$G^* > 0$ since $C < 0$.

Hence knowing the positive value of G^*, F^* can be calculated from 9

Coexistence Equilibrium $E_4(G^*, F^*, P^*)$

We solve 1, 2 and 3 for the coexistence equilibrium $E_3(G^*, F^*, P^*)$. At equilibrium,

$$F = -\frac{amrG^2 - akmrG + (\alpha b - fr)GP + fkrP}{cK} \tag{10}$$

$$F = \frac{(\beta am - \alpha bs)G - bfP}{\alpha fn} \tag{11}$$

$$F = \frac{brsG^2 - bkrsG + \beta bK}{fknr - \beta ak - fnrG} \tag{12}$$

Equating 10, 11 and 11, 12, we respectively obtain

$$P = \frac{(\alpha bfKnrs - 2\alpha bfnrs - \beta afmnr)G^2 + (\beta afKmnr + \alpha\beta abKs - \beta^2 a^2 Km - \alpha bfKnrs)G}{\alpha\beta bfKn + \beta f^2 nr - \beta^2 afK - \beta f^2 nrG}$$

and

$$P = \frac{\alpha afmnrG^2 + (\alpha\beta a^2 Km - \alpha^2 abKs - \alpha afKmnr)G}{(\alpha f^2 nr - \alpha^2 bfn)G + \alpha f^2 Knr + \alpha\beta afK}.$$

The last two equations will yield $G^* = \dfrac{-B + \sqrt{B^2 - 4AC}}{2A}$ where $A \neq 0$ and

$A = \alpha^2\beta abf^2 mn^2 r + 2\alpha^3 b^2 f^2 n^2 rs + \alpha^2 bf^3 Kn^2 r^2 s + \alpha\beta af^3 mn^2 r^2 - \alpha\beta af^3 mn^2 r^2 - 2\alpha^2 bf^3 n^2 r^2 s - \alpha^3 b^2 f^2 Kn^2 rs$

$B = \alpha\beta af^3 Kmn^2 r^2 - \alpha^2\beta a^2 bf^2 Kmn^2 r^2 - \alpha\beta^2 a^2 f^2 Kmnr + \alpha^2\beta^2 a^2 bfKmn - \alpha\beta af^3 Kmn^2 r^2 - \alpha\beta^2 a^2 fKmnr^2 + \alpha^2\beta abf^2 Knrs - \alpha^3\beta ab^2 fKns - \alpha^2 bf^3 Kn^2 r^2 s + \alpha^2\beta^2 abf^2 Knrs - \alpha^3\beta ab^2 fns - 2\alpha^2 bf^3 Kn^2 r^2 s - 2\alpha^2\beta abf^2 Knrs + \alpha^2 bf^3 K^2 n^2 r^2 s + \alpha^2\beta abf^2 K^2 nrs - \alpha\beta af^3 mn^2 r^2 + \alpha\beta^2 af^2 Kmnr - \alpha^2\beta abf^2 Kmn^2 r + \alpha\beta^2 a^2 f^2 Kmnr - \alpha^2\beta abf^2 Knrs - \alpha\beta af^3 Kmn^2 r^2$

$C = \alpha\beta af^3 K^2 mn^2 r^2 - \alpha\beta^3 a^3 fK^2 m + 2\alpha^2\beta abf^2 K^2 nr - \alpha\beta^2 a^2 f^2 Kmnr + \alpha\beta^3 a^3 fK^2 m - \alpha^2\beta^2 a^2 bfK^2 mn + \alpha^2\beta abf^2 Knrs + \alpha^3\beta ab^2 fK^2 ns + \alpha\beta af^3 Kmn^3 r^2 - \alpha\beta^2 a^2 f^2 K^2 mnr + \alpha^2\beta abf^2 K^2 mn^2 r.$

Knowing the positive value of G^*, the positive values of F^* and P^* can be calculated from previous equations. We therefore state the following theorem:

Theorem 1: i. The model equation exhibits unique positive equilibrium in the positive

quadrant if $C < 0$

    ii. The model equation has no equilibrium point if $C = 0$.

Local Stability

The local stability of the equilibrium points of the model is studied by finding the eigenvalues of the Jacobian Matrix evaluated at each of the equilibrium points. For the system to be stable, the real parts of the eigen values of the Jacobian matrix must be negative. Let the entries of the Jacobian matrix at equilibrium be $j_{i,j}$ $\forall i = 1, 2, 3$ and $j = 1, 2, 3$. where

$$j_{11} = r - \frac{2rG^*}{K} - \frac{aF^* + bP^* + asF^{*2} + bdF^*P^* + aeF^*P^* + beP^{*2}}{(1 + cG^* + dF^* + eP^*)^2}, \quad j_{12} = -\frac{aG^* + acF^{*2} + aeG^*P^* - bdG^*}{(1 + cG^* + dF^* + eP^*)^2}$$

$$j_{13} = -\frac{bG^* + bcG^{*2} + bdG^*F^* - aeG^*F^*}{(1 + cG^* + dF^* + eP^*)^2}, \quad j_{21} = \frac{admF^{*2} + amF^* + (aem + cf)F^*P^*}{(1 + cG^* + dF^* + eP^*)^2}, \quad j_{22} =$$

$$-\alpha + \frac{acmG^{*2} + amG^* + (aem - cf)G^*P^* + dfF^{*2} - dfF^*P^* - efP^{*2}}{(1 + cG^* + dF^* + eP^*)^2}, \quad j_{23} = -\frac{(aem + cf)G^*F^* + fF^* + c}{(1 + cG^* + dF^* + eP^*)}$$

$$j_{31} = \frac{bsP^* + bdsF^*P^* + besP^{*2} + cnfF^*P^*}{(1 + cG^* + dF^* + eP^*)^2}, \quad j_{32} = \frac{nfP^* + enfP^{*2} + cnfG^*P^* - bdsG^*P^*}{(1 + cG^* + dF^* + eP^*)^2}$$

and $j_{33} = -\beta + \dfrac{nfF^* + cnfG^*F^* + dnfF^{*2} + bsG^* + bcsG^{*2} + bdsG^*F^*}{(1 + cG^* + dF^* + eP^*)^2}$.

    1. $E_0(0, 0, 0)$: The eigen values at E_0 are r, $-\alpha$ and $-\beta$. For $r > 0$, E_0 exhibits an unstable manifold in the $G-$ direction and a stable manifold in F, P direction. Hence E_0 is unstable.

    2. $E_1(G, 0, 0)$: E_1 has eigen values $r - \dfrac{2rG^*}{K}$, $-\alpha + \dfrac{amG^*}{1 + cG^*}$ and $-\beta + \dfrac{bsG^*}{1 + cG^*}$. E_1 is locally asymptotically stable if the following conditions are satisfied   i). $G^* > \dfrac{K}{2}$ ii). $\alpha > \dfrac{amG^*}{1 + cG^*}$ iii). $\beta > \dfrac{bsG^*}{1 + cG^*}$

    3. $E_2(G, 0, P)$: The characteristic equation associated with the Jacobian matrix evaluated at E_2 (with $F = 0$) is $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$ where $a_1 = -(j_{11} + j_{22} + j_{33})$, $a_2 = (J_{11}J_{22} + J_{11}J_{33} + J_{22}J_{33} - J_{13}J_{31})$ and $a_3 = (J_{11}J_{22}J_{33} - J_{13}J_{31}J_{22})$. Applying the Routh criteria, the condition for which the system is stable is $a_1a_2 - a_3 > $ since a_1, a_2 and a_3 are positive. But $a_1a_2 - a_3 = (J_{11}J_{22} + J_{11}J_{33} + J_{22}J_{33} + J_{22}^2 - J_{13}J_{22}J_{31}) > 0$ since $J_{13} < 0$.

    We therefore state the following theorem:

    Theorem: If the following inequalities

i). $r < \dfrac{bK(1 + eP^*)P^*}{(K - 2G^*)(1 + cG^* + eP^*)^2}$,   ii). $\alpha > \dfrac{acmG^{*2} + amG^* + (aem - cf)G^* - efP^{*2}}{(1 + cG^* + eP^*)^2}$

and

    iii). $\beta > \dfrac{bs(1 + cG^*)G^*}{(1 + cG^* + eP^*)^2}$ hold,

then E_2 is locally asymptotically stable.

Theorem. Let the following inequalities hold, i). $\quad r > \dfrac{bK(1+eP^*)P^*}{(K-2G^*)(1+cG^*+eP^*)^2}$,

ii). $\quad \alpha > \dfrac{acmG^{*2}+amG^*+(aem-cf)G^*-efP^{*2}}{(1+cG^*+eP^*)^2}$ and iii). $\quad \beta > \dfrac{bs(1+cG^*)G^*}{(1+cG^*+eP^*)^2}$

Then E_2 is a saddle with a local stable manifold in the F, P plane and an unstable manifold locally in the G-direction.

4. $E_3(G^*, F^*, 0)$: The corresponding characteristic equation associated with the Jacobian of E_3 is defined by $\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$ where

$$b_1 = -(J_{11}+J_{22}+J_{33}), \quad b_2 = (J_{11}J_{22}+J_{11}J_{33}+J_{22}J_{33}-J_{12}J_{21}) \text{ and } b_3 = J_{12}J_{21}J_{33}-J_{11}J_{22}J_{33}.$$

By Routh-Hurwitz criterion, the eigen values of the variational matrix are negative and hence the system stable if $b_1 > 0$, $b_2 > 0$ and $b_1b_2 - b_3 > 0$. From previous calculations, $b_1 > 0$ and so $b_1b_2 - b_3 > 0$ since $j_{12} < 0$.

Theorem: Let the following inequalities hold

i). $\quad r < \dfrac{aK(1+sF^*)F^*}{(K-2G^*)(1+cG^*+dF^*)^2}$, $\quad ii$). $\quad \alpha > \dfrac{acmG^{*2}+amG^*+dfF^{*2}}{(1+cG^*+dF^*)^2}$,

iii). $\quad \beta > \dfrac{fn(1+dF^*)F^*+bs(1+cG^*)G^*+(bds+cfn)G^*F^*}{(1+cG^*+eP^*)^2}$ $\quad iv$). $\quad \dfrac{a(1+cG^*)G^*}{(1+cG^*+dF^*)^2} < 0$,

Then E_3 is locally asymptotically stable. On the other hand is any of the conditions above is reversed, then E_3 is an unstable saddle.

5. **Stability of coexistence Equilibrium** $E_4(G^*, F^*, P^*)$:

$$\lambda^3 - (J_{11}+J_{22}+J_{33})\lambda^2 - (J_{11}J_{22}+J_{11}J_{33}+J_{22}J_{33}+J_{12}J_{21}-J_{23}J_{32}-J_{13}J_{31})\lambda + (J_{11}(J_{22}J_{33}-J_{23}J_{32}) + J_{12}(J_{23}J_{31}-J_{21}J_{33}) + J_{13}(J_{21}J_{32}-J_{31}J_{22})) = 0$$

$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$, where a_1, a_2 and a_3 corresponds to the respective coefficients of λ. Using the Routh-Hurwitz criterion, the eigen values are always negative when ever $a_1 > 0$, $a_3 > 0$ and $a_1a_2 - a_3 > 0$.

Global Stability

We study the global stability of each equilibrium point E_1, E_2, E_3 and E_4 by constructing a suitable Lyapunov function. We prove that under some suitable assumptions, the equilibrium points are globally stable.

Theorem 2: The equilibrium point E_1 is globally asymptotically stable if $r > 0$.

Proof: We define a lyapunov function v_1, which is continuous on R_+^2. Consider the positive definite function, $v_1(G)$ about E_1:

$v_1(G) = (G - G^* - G^* ln(\frac{G}{G^*}))$. It is obvious to show that $v_1(0) = 0$. Next we show that $\frac{dv_1}{dt} < 0$.

$$\frac{dv_1}{dt} = (1 - \frac{G^*}{G})\frac{dG}{dt} = (G - G^*)\left(r(1 - \frac{G}{K})\right)$$

$$\implies \frac{dv_1}{dt} = (G - G^*)\left(r(1 - \frac{G^*}{K}) - r(1 - \frac{G}{K})\right) = \frac{r}{K}(G - G^*)(-G^* + G)$$

$\therefore \frac{dv_1}{dt} = -\frac{r}{K}(G - G^*)^2 < 0$. Hence $\frac{dv_1}{dt}$ is negative definite on condition that $r > 0$. This concludes the prove.

2. **Theorem 3**: If $r > ac_1$ and $l > 1$, then the equilibrium point $E_2(G^*, F^*, 0)$ is globally asymptotically stable.

Proof: We define a function $v_2 = (G - G^* - G^* ln(\frac{G}{G^*})) + l(F - F^* - F^* ln(\frac{F}{F^*}))$ where l is a suitable positive constant. Differentiating V_2 with respect to t, along the solution set of 1 and 2

$\frac{dv_2}{dt} = (\frac{G - G^*}{G})(\frac{dG}{dt}) + l(\frac{F - F^*}{F})(\frac{dF}{dt}).$

$\frac{dv_2}{dt} = (\frac{G - G^*}{G})\left(rG(1 - \frac{G}{K}) - \frac{aGF}{1 + cG + dF}\right) + l(\frac{F - F^*}{F})(-\alpha F + \frac{amGF}{1 + cG + dF}$

$= (G - G^*)\left(r(1 - \frac{G}{K}) - \frac{aF}{1 + cG + dF}\right) + l(F - F^*)(-\alpha + \frac{amG}{1 + cG + dF})$

$= (G - G^*)\left(-\frac{rG}{K} - \frac{aF}{1 + cG + dF} + \frac{rG^*}{K} + \frac{aF^*}{1 + cG^* + dF^*}\right) + l(F - F^*)\left(-\alpha + \frac{amG}{1 + cG + dF} + \alpha - \frac{amG^*}{1 + cG^* + dF^*}\right)$

Simplifying,

$\frac{dv_2}{dt} = (acF^* - \frac{r}{K})(G - G^*)^2 - \frac{am(l - 1)}{(1 + cG + dF)(1 + cG^* + dF^*)}(G - G^*)(F - F^*) -$

$\frac{admlG^*}{(1 + cG + dF)(1 + cG^* + dF^*)}(F - F^*)^2.$

It is clear to see that $\frac{dv_2}{dt} < 0$ when ever $r > ac$ and $l > 1$. Let $l = \frac{aF}{c(1 + cG)}$ which completes the prove.

Theorem 4: If $E_3(G^*, F^*, P^*)$ of 1 2 and 3 exist, then it is globally asymptotically stable

Proof: Consider the positive definite function about E_3 defined by:

$$V_3 = (G - G^* - G^* ln(\frac{G}{G^*})) + l_1(F - F^* - F^* ln(\frac{F}{F^*})) + l_2(P - P^* - P^* ln(\frac{P}{P^*}))$$

where l_1 and l_2 are suitably selected positive constants. The time derivative of V_3 along the solution set of system 3 is: $\frac{dV_3}{dt} = (\frac{aeF^* + bdP^*}{de(1 + cG + dF + eP)(1 + cG^* + dF^* + eP^*)} - \frac{r}{K})(G - G^*)^2$

$$-\frac{ac+adml_1}{cd(1+cG+dF+eP)(1+cG^*+dF^*+eP^*)}(G-G^*)(F-F^*)+\frac{l_2bes-bc}{ce(1+cG+dF+eP)(1+cG^*+dF}$$

$$(G-G^*)(P-P^*)+\frac{efnl_2+l_1df}{de(1+cG+dF+eP)(1+cG^*+dF^*+eP^*)}(F-F^*)(P-P^*)$$

$$+\frac{l_1(cfP^*-ameG^*)}{ce(1+cG+dF+eP)(1+cG^*dF^*eP^*)}(F-F^*)^2-\frac{l_2cfnF^*+l_1bdsG^*}{cd(1+cG+dF+eP)(1+cG^*+dF^*+eP^*)}(P-$$

$$P^*)^2$$

The quadratic function $\dot{V}_3$ can be expressed as a matrix in the form $\dot{V}=-X^TAX$ where A is a symmetric matrix defined by

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where $a_{11}=\dfrac{aeF^*+bdP^*}{de(1+cG+dF+eP)(1+cG^*+dF^*+eP^*)}-\dfrac{r}{K}$, $a_{12}=0$, $a_{13}=0$, $a_{23}=0$

$$a_{21}=-\frac{ac+adml_1}{(1+cG+dF+eP)(1+cG^*+dF^*+eP^*)},\ a_{22}=\frac{l_1(cfP^*-ameG^*)}{ce(1+cG+dF+eP)(1+cG^*+dF^*+eP}$$

$$a_{31}=\frac{l_2bes-bc}{ce(1+cG+dF+eP)(1+cG^*+dF^*+eP^*)},\ a_{32}=\frac{l_2efn+l_1df}{de(1+cG+dF+eP)(1+cG^*+dF^*+eP^*)}$$

and

$$a_{33}=\frac{l_2cfnF^*+l_1bdsG^*}{dc(1+cG+dF+eP)(1+cG^*+dF^*+eP^*)}.$$

Thus we have: $(G-G^*,F-F^*,P-P^*)\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} G-G^* \\ F-F^* \\ P-P^* \end{pmatrix}$

We note that sufficient conditions required for $\dot{V}_3$ to be negative definite are that: $a_{11}>0$, $a_{22}>0$, $a_{33}>0$, $a_{31}=0$, $a_{11}a_{22}>a_{21}^2$ which are easily verifiable.

We therefore conclude that $E_3(G^*,F^*,P^*)$ is globally asymptotically stable if $l_1>0$ and $l_2>0$.

Numerical Simulation

We present in this section numerical simulation to support our results obtained from the theoretical analysis. This is carried out with the help of MATLAB-R2018a and MAPLE-2018. We assume the following parameter values $\alpha = 0.42, \beta = 0.61, a = 0.72, K = 5, b = 0.8, c = 0.3, c1 = 0.5, c2 = 0.5, e = 0.5, d = 0.03, r = 3, f = 1.8, m = 5, n = 0.643, s = 0.633, d1 = 0.02$. with initial conditions $G(0) = 1$, $F(0) = 0.5$ and $P(0) = 0.5$

Stability of $E^*(G, F, 0)$.

We discuss the dynamical behavior of the good biomass with the fish population. We shall examine the effect of varying the growth rate r of the good biomass and also the effect of varying the death rate α of the fish population. For model systems 4 and 5 and the above simulation parameters, we discuss the dynamics of the equilibrium point $E^*(G, F, 0)$. The dynamics of $E^*(G, F, 0)$ will also hold similarly arguments for $E^*(G, 0, P)$ and hence we omit the discussion on that.

The simulation results from the above parameter values shows that the system is ecologically feasible for some range of values of parameters and ecologically non-feasible for other range of values. Figure 1(a) shows an oscillatory stability for both G and F. We see that there exist a unique positive equilibrium point $E^*(5.89, 233.57)$ onto which all other equilibrium points converge. Fig 1(b) is a plot showing continuous stability of both good biomass and fish population with time. The effect of varying the growth rate r, the death rate α and the rate of reproduction m of good biomass and fish population, are shown in fig $4(b), 2(a)$ and $5(a)$ respectively.

## Stability of Coexistence Equilibrium $E^* = (G, F, P)$

Figure $3a$ shows the trajectories of the species of the ecosystem with respect to time t. We observe that the good biomass G increases with time then exhibits a state of oscillation slightly before achieving a stable steady state. The fish population F grows sharply with time but suddenly drops in growth owing to both natural death and death due to predation from the bird population. The bird population on the other hand rises quickly and drops due natural death β but stays above both the good biomass and the fish population due the combined

reproductive rate from the good biomass and the fish population.

Varying the growth r and carrying capacity K of the good biomass:

We observe that the dynamical behaviour of the system is sensitive to parameter values (more importantly r) and the initial conditions. In lieu with this, the three species can coexist for some range of parameter values of both predator and prey and also the predator can go to extinction for some other parameter values. Under the above admissible values, the system posses a unique equilibrium point $E^*(G, F, P)$ as shown in figure $3(a)$.

We see that the prey growth rate r is a very important parameter in regulating the dynamics of the ecosystem. Varying the growth rate of the good biomass shows differences in the trajectories of the ecosystem. On the other hand if r decreases beyond the critical point r_c, $(r_c = 1)$, the system decreases owing to the decrease in the population of the good biomass which is the primary source of food for both the fishes and birds. The effect of this is an oscillatory behavior of the system for a long time. But this oscillation decreases with increasing time as before eventually attaining stability once again. Since r regulates the state of the good biomass, we observe the changes that occur when r is varied with time. Figure $4(b)$ shows a graph of the good biomass with varying r against time. We observe that below the critical point $r_c < 1$, the amount of good biomass present reduces with increasing time. More over, the good biomass increases with increasing growth rate r. As can be seen from figure $3(b)$, reducing r below 1 shows a state of instability of the entire system. By extension, a drop in the good biomass is a drop in the population of both fish and bird whiles an increase in growth will increase population of fish and birds. Subsequently, r must be kept above the critical point r_c. The carrying capacity of the ecosystem determines the maximum capacity of the ecosystem. Numerical results shows that the system is unstable for $K < 0.2$. For values of K between 0.2 and 0.5, the system dynamics is oscillatory. See figure $7(b)$. This means that the fish and birds do not have enough to feed on leading to a drop in their reproductive rates.
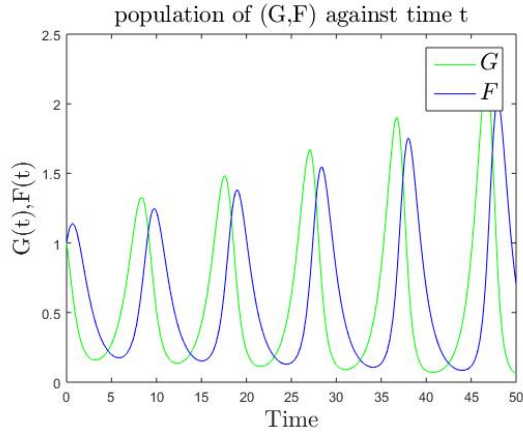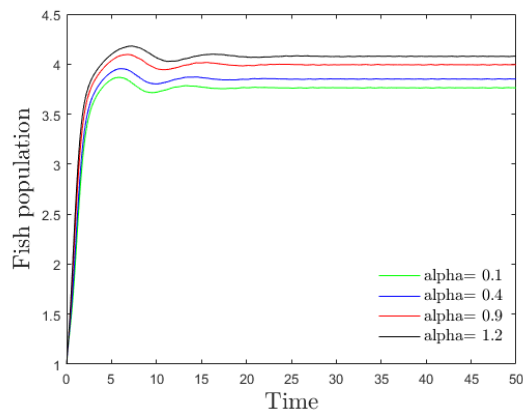
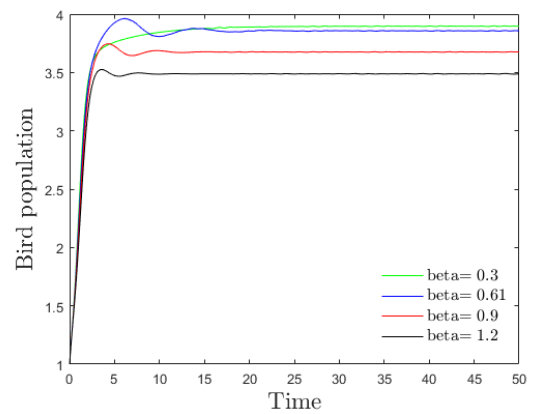(a) Oscillatory solution of good biomass and fish against time.
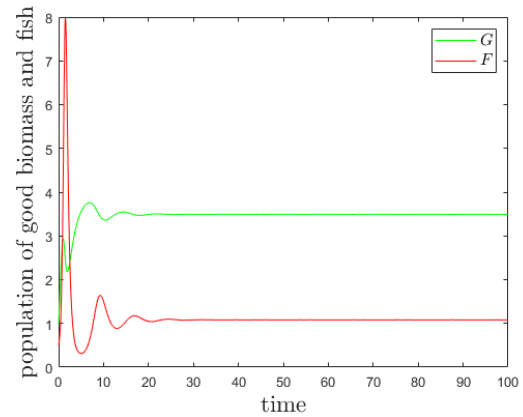
(b) Plot of good biomass and fish population against time showing stability of the system.

Figure 1

(a) Plot of Fish population against time with different natural death.

(b) Graph showing Bird population against time for different death rate r.

Figure 2

### Effect of varying the reproductive rate of fish m and bird n:

The fish population goes to extinction when the reproductive rate $m \leq 1.5$ as shown in fig $6(a)$. This is as a result of the fact that reproductive rate of the predator depends on the amount of prey it converts into food and hence when it feeds less, its population drops. Hence the system is ecologically feasible for $m > 1.5$. Figure $5(a)$ shows the state of the fish population with different values of m. It shows that the fish population will grow as the reproductive rate m increases with time. We see also from figure $4(a)$ that as the reproductive rate n of birds increases, the number of birds also increases. This is as a result of the fact that reproductive rate of the predator depends on the amount of prey it converts into food. We maintain these

values within a certain range to avoid extinction or over crowding. For higher reproductive rate, more species of fish and birds are produced and the rippling effect is less growth in the good biomass.
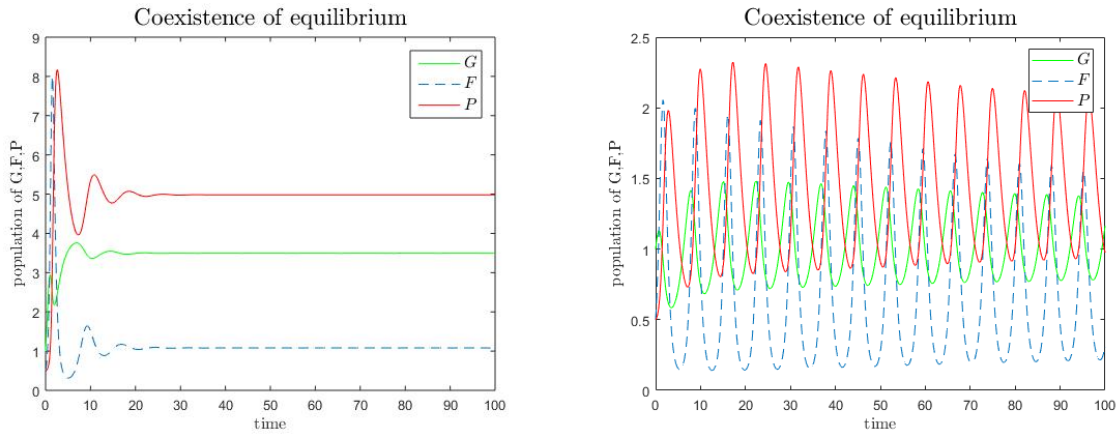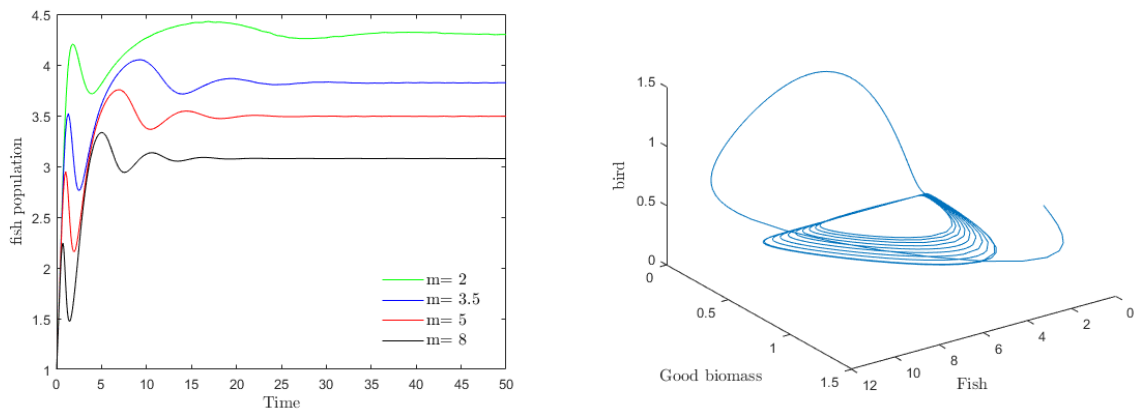

(a) stable coexistence equilibrium of good biomass, fish and bird population for $r = 3$.

(b) Oscillatory behavior of equilibrium when $r = 0.9 < 1$ the critical growth rate.

Figure 3
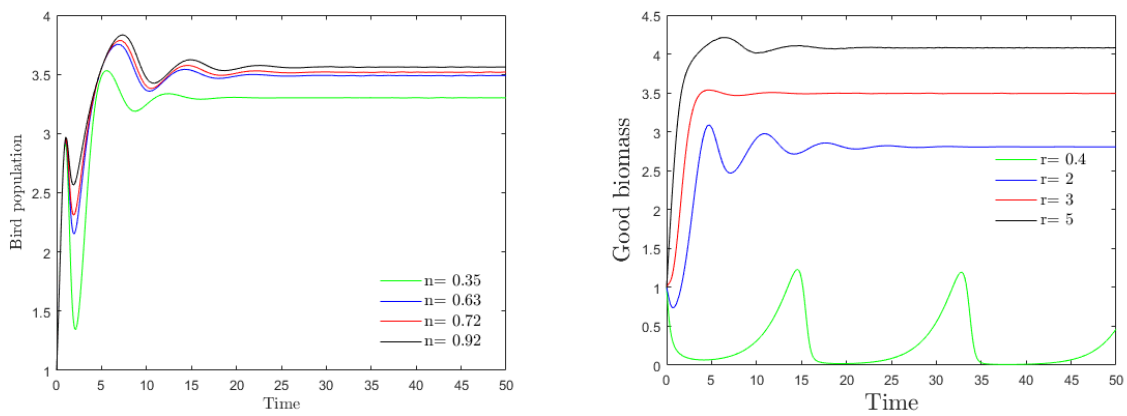
(a) Plot of dynamics of Birds against reproductive rate n from fish.

(b) Graph of good biomass with different growth rate r.
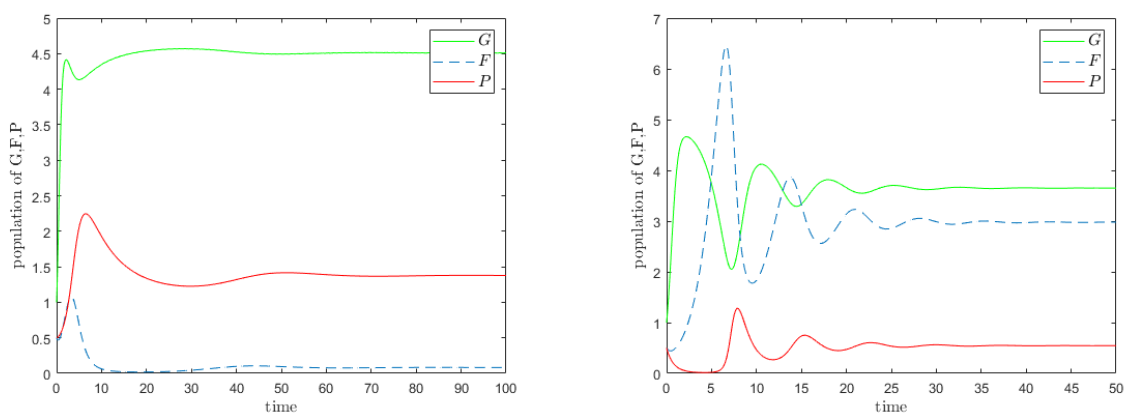
Figure 4

(a) Graph of fish population with time for different values of the reproductive rate m.

(b) A 3D plot of good biomass, fish and bird populations showing stability of $E^*(G, F, P)$.

Figure 5

Effect of varying death rates α and β of fish and birds populations respectively: Simulation results shows that the system is ecologically feasible or stable for less death rate of birds. For $\beta \geq 2$, the bird population exhibits a state of instability with a possibility of extinction due to the high death rate as can be seen from fig $6(b)$. From figure $2(b)$, the system is stable for the range $0.2 \leq \beta < 2$. We chose the death rate $\beta \geq 0.2$ to avoid overcrowding since a lower death rate means more birds survive, whose long term effect is a possible extinction of the good biomass due to overfeeding. We observe again from figure $7(a)$ that if $\alpha \geq 3$, the fish population approaches 0, the point of extinction. It is therefore important that the death rate be maintained below the critical point. Figure $2(a)$ is the graph of fish population with different death rates.

(a) Trajectory of the system species for $m < 1.5$.

(b) Bird population approaching extinction with $\beta \geq 2$.
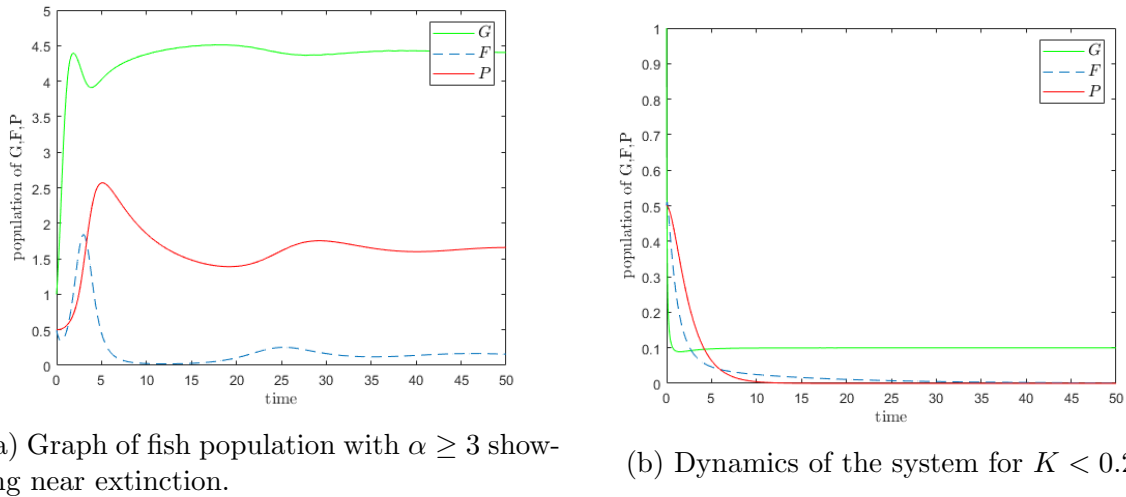
Figure 6

(a) Graph of fish population with $\alpha \geq 3$ showing near extinction.

(b) Dynamics of the system for $K < 0.2$

Figure 7

Conclusion

In this chapter, we propose a mathematical model to study the behavior and interaction of the species of an ecosystem comprising of primary producers (plants and other animals) collectively called good biomass, fish population and birds population. The existence of feasible positive equilibrium points are established and subsequently examined for both local and global stability. The good biomass population serves as the source of primary food production for both fish and bird population in the ecosystem. The good state of the good biomass means the growth and survival of the fish and bird population. We assume that the good biomass grows logistically and the fish and bird population feed on the good biomass by the Beddington deAngelis functional response. The bird population also feeds on the fish by the same functional response. This therefore forms a three specie food chain ecosystem. The continuous existence of the fish and bird population is dependent on the continuous growth of the good biomass. It is observed that both the fish and bird population will coexist as long as the intrinsic growth rate r of the good biomass is greater than the rate c at which both the fish and bird population feed on the good biomass.

One important result is that the good biomass is bounded and as such in the absence of the fish and bird populations (predators), the good biomass (prey) will grow exponentially to the carrying capacity after a long period of time. In effect, the total population of the system is bounded above and that any solution originating from the interior of the positive

octant do not leave it. We showed that the positive equilibrium points are both locally and globally asymptotically stable under some desirable conditions. Due to the boundedness of the good biomass, the growth of the fish and bird populations is limited to the amount of good biomass present. We also observed that a variation in the carrying capacity largely affected the dynamics of the ecosystem. A direct recipient of this effect is the bird population since it has a wider surface area to feed from compared to the fish population with a smaller surface area. It therefore follows that the good biomass, fish and bird population will coexist if the carrying capacity is below the critical point K^*.

To maintain a continues coexistence between the species, the growth rate of the good biomass must be kept within a certain range. From numerical simulation results, we observed that the system enjoyed a state of coexistence if the growth rate $r \geq 1$. This means that r should be maintain at or above the critical point 1. Numerical results revealed that any value of r taken less than 1, renders the system unstable.

In a similar manner, we saw that the reproductive rate of both fish and bird be kept and maintained below the critical point. For instance the reproductive rate of fish, m, if lowered below 1.5 showed an extinction of the fish population numerically. Hence for the fish population to persist, the reproductive rate $m > 1.5$. The numerical computations shows that the dynamical outcomes of the interacting species for the model are sensitive to parameter values and initial values of the species.

In conclusion, the proposed model exposes us to the dynamics of the interacting species in the Keta-Anlo wetland when parameters of the interacting species are varied. Conditions for feasible equilibrium are proposed as well as local and global stability examined for the equilibrium points obtained. We are therefore convinced that the study will help sustain and improve the lives of the birds and fishes for eco-tourism and other purposes the wetlands provides.

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