# PRODUCTS OF CONJUGATE SECONDARY NORMAL MATRICES

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**Abstract:**

In this paper, the properties of the products of conjugate secondary normal (con-s-normal) matrices are developed, their relation, in a sense, to s-normal matrices is considered and further results concerning s-normal products are obtained.

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## 1. Introduction

Let $C_{nxn}$ be the space of $n$x$n$ complex matrices of order $n$. For $A\in C_{n\times n}$, let $A^T$, $\overline{A}$, $A^*$, $A^S$, $A^\theta$ and $A^{-1}$ denote the transpose, conjugate, conjugate transpose, secondary transpose, conjugate secondary transpose and inverse of matrix $A$ respectively. The conjugate secondary transpose of $A$ satisfies the following properties such as $\left(A^\theta\right)^\theta = A$, $\left(A+B\right)^\theta = A^\theta + B^\theta$, $\left(AB\right)^\theta = B^\theta A^\theta$ . etc

**Definition 1**

A matrix $A\in C_{n\times n}$ is said to be normal if $AA^* = A^*A.$

**Definition 2**

A Matrix $A\in C_{n\times n}$ is said to be conjugate normal (con-normal) if $AA^* = \overline{A^*A}.$

**Definition 3**

A matrix $A\in C_{n\times n}$ is said to be secondary normal (*s*-normal) if $AA^\theta = A^\theta A.$

**Definition 4**

A matrix $A\in C_{n\times n}$ is said to be unitary if $AA^* = A^*A = I.$

**Definition 5**

A matrix $A\in C_{n\times n}$ is said to be *s*-unitary if $AA^\theta = A^\theta A = I.$

**Definition 6 [2]**

A matrix $A\in C_{n\times n}$ is said to be a conjugate secondary normal matrix (con-*s*-normal) if $AA^\theta = \overline{A^\theta A}$ where $A^\theta = \overline{A}^S$. **. . . (1)**

## 2. Properties of Con-s-Normal Matrices

**Theorem 1**

A matrix $A$ is con-s-normal iff there exists an s-unitary matrix $U$ such that $UAU^S$ is a direct sum of non-negative real numbers and of $2x2$ matrices of the form: $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ where a and b are non negative real numbers.

**Proof**

Let $A$ be con-s-normal where $A = P+Q$ where $P = P^S$ and $Q = -Q^S$. Then $A\overline{A}^S = A^S\overline{A}$ gives $(P+Q)\left(\overline{P}^S + \overline{Q}^S\right) = \left(P^S + Q^S\right)\left(\overline{P}+\overline{Q}\right)$ *or* $(P+Q)\left(\overline{P}+\overline{Q}\right) = (P-Q)\left(\overline{P}+\overline{Q}\right)$

and so: $P\overline{P} + Q\overline{P} - P\overline{Q} - Q\overline{Q} = P\overline{P} - Q\overline{P} + P\overline{Q} - Q\overline{Q}$ or $Q\overline{P} - P\overline{Q}$. There exists a s-unitary $U$ such that $USU^S = D$ is a secondary diagonal matrix with real, non–negative elements. Therefore $UQU^S\overline{U}\ \overline{P}\ \overline{U}^S = U\,PU^S\overline{U}\ \overline{Q}\overline{U}^S$ or $WD = D\overline{W}$ where $W = -W^S$. Let $U$ be chosen so that $D$ is such that $d_i \geqq d_j \geqq 0$ for $i<j$ where $d_i$ is the $i^{th}$ secondary diagonal element of $D$. $W = (t_{ij})$, where $t_{ji} = -t_{ij}$ then $t_{ij}\,d_j = d_i\,\bar{t}_{ij}$, for $j>i$, and 3 possibilities may occur : if $d_j = d_i \neq 0$, then $t_{ij}$ is real; if $d_j = d_i = 0$, $t_{ij}$ is arbitrary (through $W = -W^S$ still holds); and if $d_j \neq d_i$, then $t_{ij} = 0$ for if $t_{ij} = a+ib$ then $(a+ib)\,d_j = d_i(a-ib)$ and $a(d_j - d_i) = 0$ implies $a=0$ and $b\left(d_i + d_j\right) = 0$ implies $d_i = -d_j$ (which is not possible since the $d_i$ are real and non–negative and $d_j \neq d_i$) or $b=0$ so $t_{ij} = 0$. So if $UPU^S = d_1 I_1 \oplus d_2 I_2 \oplus \ldots \oplus d_k I_k$ where $\oplus$ denotes direct sum, then $UQU^S = T_1 \oplus T_2 \oplus \ldots. \oplus T_k$ where $Q_i = -Q_i^S$ is real and $Q_K = -Q_K^S$ is complex iff $d_k = 0$. For each real $Q_i$ there exists a real-s-orthogonal matrix $V_i$ so that $V_i T_i V_i^S$ is direct sum of zero matrices and matrices of the form $\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$ where $b$ is real **[1]**. If $Q_K = -Q_K^S$ is complex, there exists a complex s-unitary matrix $V_k$ such that $V_k Q_k V_k Q$ is a direct sum of matrices of the form **[3]** so that if $V = V_1 \oplus V_2 \oplus \ldots \oplus V_k$ then $VUPU^S V^S = D$ and $VUQ^S U^S = F$ the direct sum. Therefore $V\,UAU^S\,V^S = D + F$ this is the desired form.

If $A$ and $B$ are two con-s-normal matrices such that $A\overline{B} = B\overline{A}$ then $A$ and $B$ can be simultaneously brought into the above secondary normal form under the same $U$ (with a generalization to a finite number) but not conversely; if $A$ is con-s-normal , $A\overline{A}$ is s-normal in the usual sense, but not conversely; and if $A$ is con-s-normal and $A\overline{A}$ is real, there is a real secondary orthogonal matrix which gives the above form. Among properties of con-s-normal matrices not obtained but of subsequent use are the following:

**(a)** $A$ is con-s-normal iff $A = HU = UH^S$ where $H$ is s-hermitian and $U$ is s-unitary.

For if $A = HU$ is a polar form of $A$, then $\overline{U}^S HU = K$ is such that $A=HU=UK$ and if $A\overline{A}^S = A^S A$, then $H^2 = \left(K^S\right)^2$ and since this is an s-hermitian matrix with non-negative roots, $H = K^S$ and $A = HU = UH^S$. The converse is immediate. This same result may be seen as follows. If $UAU^S = F$ is the s-normal form in **Theorem 1**, $F = D_r V = VD_r$ where $D_r$ is real secondary diagonal and $V$ is a direct sum of 1's block of the form $\left(a^2+b^2\right)^{-1/2}\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ which are s-unitary. Therefore $A = \overline{U}^S D_r U\,\overline{U}^S V\overline{U} = \overline{U}^S V\overline{U}\,U^S D_r \overline{U}$ which exhibits the polar form in another guise.

**(b)** $A$ is both s-normal and con-s-normal iff $A = HU = UH = UH^S$ so $H = H^S = \overline{H}^S$ so that $H$ is real.

**(c)** If $A = HU = UH^S$ is con-s-normal, then $UH$ is con-s-normal iff $HU^2 = U^2H$, that is $HU^2$ is s-normal. For if $UH$ is con-s-normal, $UH = H^S U$ so that $HU^2 = UH^S U = U^2 H$; and if $HU^2 = U^2 H$, then $HUU = UH^S U = UUH$ or $H^S U = UH$.

**(d)** A matrix $A$ is con-s-normal, iff $A$ can be written $A = PW = \overline{W}P$ where $P = P^S$ and $W$ is s-unitary. If $A$ is con-s-normal, form the above $A = \overline{U}^S F\overline{U} = \overline{U}^S D_r \overline{U} U^S V\overline{U} = PW = \overline{U}^S VU\overline{U}^S D_r \overline{U} = \overline{W}P$ where $P = \overline{U}^S D_r \overline{U}$ s-symmetric and $W = U^S V\overline{U}$ is s-unitary. Conversely, if $A = PW = \overline{W}P$, $A\overline{A}^S = PW\overline{W}^S\overline{P}^S = A^S\overline{A} = P^S\overline{W}^S\overline{P}$.

Note that if $B$ is con-s-normal, and if $B=PU$ where $P = P^S$ and $U$ is s-unitary, it does not necessarily follow that $B = \overline{U}P$; but it possible to find on $P_1$ and $U_1$ such that $B = P_1U_1 = \overline{U_1}P_1$ holds. This may be seen as follows. If $B=PU$ is con-s-normal, Let $V$ be s-unitary such that $VPV^S = D$ is secondary diagonal, real and non negative, so that $VBV^S = VPV^S\,\overline{V}UV^S = DW$ is con-s-normal from which $DW\overline{W}^S\overline{D} = W^S D^S\,\overline{D}\overline{W}$ or since $D$ is real, $WD^2 = D^2W$ and $WD=DW$ since $D$ is non-negative. Then $B = \left(\overline{V}^S DV\right)\left(V^S W\overline{V}\right) = PU = \left(\overline{V}^S WV\right)\left(\overline{V}^S D\overline{V}\right)$ which is not necessarily equal to $\overline{U}P = \left(\overline{V}^S\overline{W}V\right)\left(\overline{V}^S D\overline{V}\right)$ However, if $D = r_1I_1 \oplus r_2I_2 \oplus \ldots \oplus r_kI_k$, $r_i > r_j$ for $i > j$, then $W = W_1 \oplus W_2 \oplus \ldots \oplus W_K$. Since each $W_i$ is s-unitary, it is con-s-normal and there exist s-unitary $X_i$ so that $X_iW_iX_i^S = F_i$ is in the real s-normal form of **Theorem 1** if $X = X_1 \oplus X_2 \oplus \ldots \oplus X_k$, then $XVBV^SX^S = XDWX^S = DXWX^S = DF = FD$ where $F = F_1 \oplus F_2 \oplus \ldots \oplus F_k$.

So

$$B = \left(\overline{V}^S\,\overline{X}^S\,D\overline{X}\,\overline{V}\right)\left(V^S X^S F\overline{X}\,\overline{V}\right)$$

$$= \left(\overline{V}^S\,\overline{X}^S FXV\right)\left(\overline{V}^S\,\overline{X}^S D\overline{X}\,\overline{V}\right) = P_1U_1 = \overline{U}_1^S P_1 \; and$$

$$P_1 = \overline{V}^S\,\overline{X}^S D\overline{X}\,\overline{V} \neq \overline{V}^S D\overline{V} = P \; and$$

$$U_1 = V^S X^S F\,\overline{X}\,\overline{V} \neq V^S W\overline{V} = U.$$

## 3. Products of s-Normal Matrices

If $A$, $B$ and $AB$ are s-normal matrices then $BA$ is s-normal; a necessary and sufficient condition that the product $AB$, of two s-normal matrices $A$ and $B$ be s-normal is that each commute with the s-hermitian polar matrix of the other. First a generalization of this theorem is obtained here and then an analog for the con-s-normal case is developed.

**Theorem 2**

Let $A$ be an s-normal matrix. Then $AB$ and $BA$ are s-normal iff $\left(\overline{A}^S A\right)B = B\left(A\overline{A}^S\right)$ and $\left(\overline{B}^S B\right)A = A\left(B\overline{B}^S\right)$. (In a sense, the latter condition might be described as stating that each matrix is s-normal relative to the other).

**Proof**

If $AB$ and $BA$ are s-normal, Let $U$ be a unitary matrix such that $UA\overline{U}^S = D$ is secondary diagonal. $d_i\overline{d}_i \geq d_j\overline{d}_j \geq 0$ for $i < j$, and let $UB\overline{U}^S = B_1 = (b_{ij})$. From $AB\overline{B}^S\overline{A}^S = \overline{B}^S\overline{A}^S AB$ it follows that $DB_1\overline{B}_1^S\overline{D} = \overline{B}^S\overline{D}DB_1$; by equating secondary diagonal elements it follows that $\sum_{j=1}^{n} d_i\overline{d}_i b_{ij}\overline{b}_{ij} = \sum_{j=1}^{n} d_j\overline{d}_j b_{ji}\overline{b}_{ji}$ for $i=1,2\ldots n$. Similarly from $BA\overline{A}^S\overline{B}^S = \overline{A}^S\overline{B}^S BA$ follows $B_1D\overline{D}\overline{B}_1^S = \overline{D}\overline{B}_1^S B_1D$ and $\sum_{j=1}^{n} d_j\overline{d}_j b_{ij}\overline{b}_{ij} = \sum_{j=1}^{n} \overline{d}_i d_i\overline{b}_{ji}b_{ji}$. Let $i=1$ in each of these equations So that $\sum_{j=1}^{n} d_1\overline{d}_1 b_{1j}\overline{b}_{1j} = \sum_{j=1}^{n} d_j\overline{d}_j b_{j1}\overline{b}_{j1}$ and $\sum_{j=1}^{n} d_j\overline{d}_j b_{1j}\overline{b}_{1j} = \sum_{j=1}^{n} \overline{d}_1 d_1\overline{b}_{j1}b_{j1}$ from which follows

$$\sum_{j=1}^{n}\left(d_1\overline{d}_1 - d_j\overline{d}_j\right)b_{1j}\overline{b}_{1j} = \sum_{j=1}^{n}\left(d_j\overline{d}_j - d_1\overline{d}_1\right)d_{j1}\overline{b_{j1}}$$

so that $$\sum_{j=1}^{n}\left(d_1\overline{d}_1 - d_j\overline{d}_j\right)\left(b_{1j}\overline{b_{1j}} + b_{j1}\overline{b_{j1}}\right) = 0.$$

Let $d_1\overline{d_1} = d_2\overline{d_2} = \ldots = d_l\overline{d_l} > d_{l+1}d_{l+1}$, then $b_{1j}\overline{b_{1j}} + b_{j1}\overline{b_{j1}} = 0$ for $j = l+1, l+2, \ldots n$ since $d_1\overline{d_1} - d_j\overline{d_j}$ is zero or positive and is latter for $j > l$. So $b_{1j} = 0$ and $b_{j1} = 0$ for $j = l+1, l+2, \ldots n$. For $i=2,\ldots..l$ in turn it follows that $b_{ij}=0$ and $b_{ji}=0$. For $i=1,2,\ldots.l$ and for $j=l+1,l+2\ldots.n$. Let $UA\overline{U}^S = D = r_1D_1 \oplus r_2D_2 \oplus \ldots \oplus r_SD_S$ where the $r_i$ are real $r_i > r_j$ for $i < j$ and the $D_i$ are s-unitary Then by repeating the above process it follows that $UB\overline{U}^S = B_1 = C_1 \oplus C_2 \oplus \ldots \oplus C_S$ is conformable to $D$.

It follows from the given conditions that $r_iD_iC_i\overline{C_i}^S\overline{D_i r_i} = \overline{C_i}^S\left(r_i\overline{D_i}\right)\left(D_ir_i\right)C_i$ and $C_ir_iD_i\overline{D_i}r_i\overline{C_i}^S = r_i\overline{D_i}\overline{C_i}^S C_iD_ir_i$ or that $D_iC_i\overline{C_i}^S = \overline{C_i}^S C_iD_i$ and $D_iC_i\overline{C_i}^S = \overline{C_i}^S C_iD_i$ if $r_i > 0$. If $r_s=0$, $D_s$ is arbitrary insofar as $D$ is concerned and so may be chosen so that $D_SC_S\overline{C_S}^S = \overline{C_S}^S C_SD_S$ in which case $D_s$ may not be secondary diagonal. But whether or not this is done, it follows that $DB_1\overline{B_1}^S = \overline{B_1}^S B_1D$ and that $B_1D\overline{D}^S = \overline{D}^S DB_1$ so that $A\left(B\overline{B}^S\right) = \left(\overline{B}^S B\right)A$ and $B\left(A\overline{A}^S\right) = \left(\overline{A}^S A\right)B$. The converse is immediate. It may be noted that if the roots of $A$ are all distinct in absolute value, $B$ must be s-normal. The following further clarifies the situation.

**Theorem 3**

Let $A = LW = WL$ be the polar form of the s-normal matrix $A$. Then $AB$ and $BA$ are s-normal iff $B = N\overline{W}^S$ where $N$ is s-normal and $LN = NL$.

**Proof**

In the proof of the above theorem, let $C_i = H_iU_i = U_iK_i$ be polar forms of the $C_i$. Then $\overline{U_i}^S\ H_iU_i = K_i$ so that $\overline{U_i}^S\ C_i\overline{C_i}^S U_i = \overline{C_i}^S C_i$ or $\overline{U_i}^S\ C_i\overline{C_i}^S = \overline{C_i}^S C_i\overline{U_i}^S$. Also, from the above $D_iC_i\overline{C_i}^S = \overline{C_i}^S C_iD_i$.

Let $R_i = \overline{D_i}\,\overline{U_i}^S$ then $R_iC_i\overline{C_i}^S = \overline{D_i}\,\overline{U_i}^S C_i\overline{C_i}^S = \overline{D_i}\,\overline{C_i}^S C_i\overline{U_i}^S = C_i\overline{C_i}^S\overline{D_i}\,\overline{U_i}^S = C_i\overline{C_i}^S R_i$ where $R_i$ is s-unitary (if $r_s = 0$, $D_S$ may be chosen $= \overline{U_S}^S$ as described above). So $R_iH_i^2 = H_i^2R_i$ and since $H_i$ has positive or zero roots, $R_iH_i = H_iR_i$ and so $H_i\overline{R_i}^S = \overline{R_i}^S H_i$. Then $A = \overline{U}^S DU = \overline{U}^S D_rU\overline{U}^S D_UU = LW = WL$ and

$$
\begin{aligned}
B &= \overline{U}^S B_1U = \overline{U}^S\left(C_1 \oplus C_2 \oplus \ldots \oplus C_S\right)U \\
&= \overline{U}^S\left(H_1U_1 \oplus H_2U_2 \oplus \ldots \oplus H_SC_S\right)U \\
&= \overline{U}^S\left(H_1\overline{R_1}^S\overline{D_1} \oplus H_2\overline{R_2}^S\overline{D_2} \oplus \ldots \oplus H_S\overline{R_S}^S\overline{D_S}\right)U \\
&= NWC^{-S}
\end{aligned}
$$

where $N=\overline{U}^S\left(H_1\overline{R}_1^S\oplus H_2\overline{R}_2^S\oplus\ldots\oplus H_S\overline{R}_S^S\right)U$ (which is s-normal since the s-hermitian $H_i$ and s-unitary $\overline{R}_i^S$ commute) and $\overline{W}^S=\overline{U}^S\left(\overline{D}_1\oplus\overline{D}_2\oplus\ldots\oplus\overline{D}_S\right)U$. It is evident that $LN = NL$.

Conversely, if $A = LW = WL$ and $B=N\overline{W}^S$ as described, then $AB=WLN\overline{W}^S$ which is obviously s-normal as is $BA=N\overline{W}^S WL=NL$.

It is easy seen that $B=N\overline{W}^S$ is s-normal iff $N\overline{W}^S=\overline{W}^S N$. if $B=N\overline{W}^S=(HR)\overline{W}^S$ is con-s-normal; then $B=H\left(R\overline{W}^S\right)=\left(R\overline{W}^S\right)H^S=RH\overline{W}^S$ (form property **(a)**) so $\overline{W}^S H^S=H\overline{W}^S$ or $WH=H^SW$ and $W\left(B\overline{B}^S\right)=\left(\overline{B}^S B\right)W$.

If $A$ is s-normal and $B$ is con-s-normal then $AB$ is s-normal, it does not necessarily follow that $BA$ is s-normal though it can occur. For example, if $B=HU=UH^S$ is con-s-normal and if $A=\overline{U}^S$ then $AB=\overline{U}^S UH^S$ and $BA=HU\overline{U}^S=H$ are both s-normal. But the following is an example in which $AB$ is s-normal but not $BA$. Let $B = HU = UH^S$ be con-s-normal but not s-normal (i.e, $H$ is not real by property **(b)**) and let $H$ be non-singular. Let $A=H^{-1}$ is s-hermitian (So s-normal) and not con-s-normal (since $H^{-1}$ is not real). Then $AB=H^{-1}HU=U$ is s-normal if $BA$ were also s-normal, then by the above theorem $\left(\overline{A}^S A\right)B=B\left(A\overline{A}^S\right)$ and $\left(\overline{B}^S B\right)A=A\left(B\overline{B}^S\right)$. But $\left(\overline{B}^S B\right)A=\left(H^s\right)^2H^{-1}$ and $A\left(B\overline{B}^S\right)=\left(\overline{H}^{-1}\right)\left(H^2\right)$ and if these were equal, $\left(H^s\right)^2=H^2$ would follow which means that $H^2=\left(H^s\right)^2=\left(\overline{H}^S\right)^2$ so that $H^2$ real. But this is not possible for if $H=VD\overline{V}^S$ where $D$ is secondary diagonal with positive real elements (since $H$ is non singular), then $H^2=VD^2\overline{V}^S=\overline{V}DV^S$ if $H^2$ is real so that $V^SVD^2=D^2V^SV$ so $V^SVD=DV^SV$ so $VD\overline{V}^S=\overline{V}DV^S=H$ is real which contradicts the above assumption.

**Theorem 4**

If $A$ and $B$ are con-s-normal and if $AB$ is s-normal then $BA$ is s-normal.

**Proof**

Let $U$ be a s-unitary matrix such that $UAU^S=F$ is the s-normal from described in **Theorem 1** and where $F\overline{F}^S=FF^S=r_1^2I_1\oplus r_2^2I_2\oplus\ldots\oplus r_k^2I_k$ which is real s-diagonal with $r_1^2>r_2^2>\ldots>r_k^2\geqq 0$ There $r_i^2$ may be either the squares of secondary diagonal elements of F or they may arise when matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ are squared. Assume that any of the latter whose $r_i^2$ are equal are arranged first in a given block followed by any secondary diagonal elements whose square is the same $r_i^2$.

Let $\overline{U}B\overline{U}^S = B_1$ which is con-s-normal and then $UAU^S\overline{U}B\overline{U}^S = F\,B_1$ is s-normal Let $V$ be the s-unitary matrix.

$$V = \begin{bmatrix} \sqrt{1/2} & i\sqrt{1/2} \\ i\sqrt{1/2} & \sqrt{1/2} \end{bmatrix}$$

Then the following matrix relation holds, independent of $a$ and $b$:

$$V\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\overline{V}^S = \begin{bmatrix} a-bi & 0 \\ 0 & a+bi \end{bmatrix}$$

Let $F = F_1 \oplus F_2 \oplus ... \oplus F_k$ where the direct sum is conformable to that of $F\overline{F}^S$ given above $\left(i.e,\ F_i\overline{F}_i^S = r_i^2 I_i\right)$ and consider $F_1 = G_1 \oplus G_2 \oplus ... \oplus G_i \oplus r_i I$ where each $G_i$ is $2$x$2$ as described above and $I$ is an identity matrix of proper size. Let $W_1 = V \oplus V \oplus ... \oplus V \oplus I$ be conformable to $F_1$; define $W_i$ for each $F_i$ in like manner and let $W = W_1 \oplus W_2 \oplus ... \oplus W_K$. If $r_k = 0, W_k = I$. Then $WF\overline{W}^S = D$ is complex secondary diagonal, where if $d_i$ is the $i^{th}$ secondary diagonal element $d_i\overline{d}_i \geqq d_{i+1}\overline{d}_{i+1}$. Then $W\left(UAU^S\right)\overline{W}^S W\left(\overline{U}B\overline{U}^S\right)\overline{W}^S = \left(WF\overline{W}^S\right)\left(WB_1\overline{W}^S\right) = DB_2$ is s-normal for $B_2 = WB_1\overline{W}^S$ (or $B_1 = \overline{W}^S B_2 W$). Since $B_1$ is con-s-normal, $B_1\overline{B}_1^S = B_1^S\overline{B}_1$ so that $\overline{W}^S B_2 W\overline{W}^S\overline{B}_2^S W = W^S B_2^S \overline{W} W^S \overline{B}_2 W$ or that $B_2\overline{B}_2^S WW^S = WW^S B_2^S\overline{B}_2$. Now $VV^S$ is a matrix of the form $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$. So that $WW^S$ is a direct sum of matrices of this form and one's.

Let $B_2 = (b_{ij})$ and consider $\overline{\left(WW^S\right)}^S B_2\overline{B}_2^S\left(WW^S\right) = B_2^S\overline{B}_2$. Let $B_2\overline{B}_2^S = \left(c_{ij}\right)$, $B_2^S\overline{B}_2 = \left(f_{ij}\right)$. $c_{ij}$ and $f_{ij}$ are identifiable with the $b_{ij}$, both matrices being s-hermitian. Consider two cases:

a) If $d_1\overline{d}_1 = d_j\overline{d}_j$ for all $j$ (where $d_j$ is the $j^{th}$ secondary diagonal element of $D$), then $D$=$KD_u$ where $D_u$ is s-unitary diagonal. Since $WFB_1\overline{W}^S = DB_2 = KD_uB_2 = D_u\left(KB_2\right)$ is s-normal, then $\overline{D}_u\left(D_uB_2K\right)D_u = B_2D = WB_1F\overline{W}^S$ is s-normal, as is $B_1F = \overline{U}B\overline{U}^S UAU^S$ so $BA$ is s-normal.

b) If $d_1\overline{d}_1 \neq d_j\overline{d}_j$ for some $j$, let $d_1\overline{d}_1 = d_2\overline{d}_2 ... = d_l\overline{d}_l \; for 1 \leq l < n$ (so that $d_l\overline{d}_l > d_{l+1}\overline{d}_{l+1}$).

Suppose $F_1 = G_1 \oplus G_2 \oplus r_1 I_1$ where $I_1$ is the $2x2$ matrix (The general case will be seen to follow from this example). From $\left(\overline{WW^S}\right)^S B_2\overline{B}_2^S \left(ww^S\right) = B_2^S\overline{B}_2$ and the fact that $W_1 = V \oplus V \oplus I_1$ it follows that $C_{11}=f_{22}, C_{22}=f_{11}$, $C_{33}=f_{44}$, $C_{44}=f_{33}$, $C_{55}=f_{55}$, $C_{66}=f_{66}$ (and $\overline{C}_{12} = f_{12}.\overline{C}_{34} = f_{34}$ etc) there equalities supply the following relation (where the summation is over $i=1$ to $n$).

$$C_{11} = \sum b_{1i}\overline{b}_{1i} = \sum b_{i2}\overline{b}_{i2} = f_{22};$$
$$C_{22} = \sum b_{2i}\overline{b}_{i2} = \sum b_{i1}\overline{b}_{i1} = f_{11};$$
$$C_{33} = \sum b_{3i}\overline{b}_{3i} = \sum b_{i4}\overline{b}_{i4} = f_{44};$$
$$C_{44} = \sum b_{4i}\overline{b}_{4i} = \sum b_{i3}\overline{b}_{i3} = f_{33};$$
$$C_{55} = \sum b_{5i}\overline{b}_{5i} = \sum b_{i5}\overline{b}_{i5} = f_{55};$$
$$C_{66} = \sum b_{6i}\overline{b}_{6i} = \sum b_{i6}\overline{b}_{i6} = f_{66};$$

$DB_2$ is s-normal so that the following relations also hold:

$$d_1\overline{d}_1, \sum b_{1i}\overline{b}_{1i} = \sum d_i\overline{d_i}\, b_{i1}\overline{b}_{i1};$$
$$d_1\overline{d}_2, \sum b_{2i}\overline{b}_{2i} = \sum d_i\overline{d_i}\, b_{i2}\overline{b}_{i2};$$
$$d_3\overline{d}_3, \sum b_{3i}\overline{b}_{3i} = \sum d_i\overline{d_i}\, b_{i3}\overline{b}_{i3};$$
$$d_4\overline{d}_4, \sum b_{4i}\overline{b}_{4i} = \sum d_i\overline{d_i}\, b_{i4}\overline{b}_{i4};$$
$$d_5\overline{d}_5, \sum b_{5i}\overline{b}_{5i} = \sum d_i\overline{d_i}\, b_{i5}\overline{b}_{i5};$$
$$d_6\overline{d}_6, \sum b_{6i}\overline{b}_{6i} = \sum d_i\overline{d_i}\, b_{i6}\overline{b}_{i6};$$

Since $d_1\overline{d}_1 = d_2\overline{d}_2$ on combining the first 2 relation in each of these sets, $d_1\overline{d}_1\left(\sum b_{1i}\overline{b_{1i}} + \sum b_{2i}\overline{b_{2i}}\right) = d_1\overline{d}_1\left(\sum b_{i1}\overline{b}_{i1} + \sum b_{i2}\overline{b}_{i2}\right) = \sum d_i\overline{d_i}\left(b_{i1}\overline{b_{i1}} + b_{i2}\overline{b_{i2}}\right)$ so that $\sum\left(d_1\overline{d}_1 - d_i\overline{d}_i\right)\left(b_{i1}\overline{b_{i1}} + b_{i2}\overline{b}_{i2}\right) = 0$ $d_1\overline{d}_1 = d_j\overline{d}_j$ for $j=1,2...6$ but for $j$ beyond 6, $d_1\overline{d}_1 = d_j\overline{d}_j > 0$ or $b_{i1}\overline{b_{i1}} + b_{i2}\overline{b_{i2}} = 0$ or $b_{i1} = 0 \; and \; b_{i2} = 0$ for $i=7,8....n$ similarly, $b_{i3}=0$ and $b_{i4}=0$ for $i>6$ the third relation in each set give $b_{i5}=0$ and $b_{i6}=0$ for $i>6$.

On adding all 6 relation in the first set,

$$\sum_{i,j=1}^{6} b_{ij}\overline{b}_{ij} + \sum_{i=1}^{6}\sum_{j=7}^{n} b_{ij}\overline{b}_{ij} = \sum_{i,j=1}^{6} b_{ij}\overline{b}_{ij} + \sum_{i=7}^{n}\sum_{j=1}^{6} b_{ij}\overline{b}_{ij}$$

and on canceling the first summations on each side,

$$\sum_{i=1}^{6}\sum_{j=7}^{n} b_{ij}\overline{b}_{ij} = \sum_{i=7}^{n}\sum_{j=1}^{6} b_{ij}\overline{b}_{ij}.$$

But the right side is zero from the above, so the left side is 0 and so $b_{ij}=0$ for $i=1,2\ldots6$ and $j>6$.

From this it is evident that this procedure may be repeated and that if $D=r_1D_1\oplus r_2D_2\oplus\ldots\oplus r_kD_k$. Where the $D_i$ are s-unitary and the $r_i$ non–negative real, as above, then $B_2=C_1\oplus C_2\oplus\ldots\oplus C_k$ Conformable to $D$ then $r_iD_iC_i$ is s–normal so $\overline{D}_i^S\left(D_iC_ir_i\right)D_i=C_ir_iD_i$ is s-normal so $B_2D$ is s-normal. So $B_1F$ and so $\overline{U}B\overline{U}^S UAU^S$ and $BA$.

**Theorem 5**

If $A$ and $B$ are con-s-normal then $AB$ is s-normal iff $\overline{A}^S AB=BA\overline{A}^S$ and $AB\overline{B}^S=\overline{B}^S BA$ (ie, iff each is s-normal relative to the other).

**Proof**

If $AB$ is s-normal, from the above $\overline{D}^S DB_2=B_2D\overline{D}^S$ so that $\overline{F}^S FB_1=B_1F\overline{F}^S$ or $\overline{A}^S AB=BA\overline{A}^S$.

Similarly $DB_2$ is s-normal, $DB_2\overline{B_2}^S\overline{D}=\overline{B}^S_2\overline{D}DB_2$ *so* $DB_2\overline{B}^S_2=\overline{B}^S_2B_2D$ or $FB_1\overline{B}^S_1=\overline{B}^S_1B_1F$ or $AB\overline{B}^S=\overline{B}^S BA.$ the converse is directly verifiable.

**Theorem 6**

Let $A$ and $B$ be con-s-normal, if $AB$ is s-normal, then $A=LW=WL^S$ (with $L$ s-hermitian and $W$ s-unitary) and $B=N\overline{W}^S$. Where $N$ is s-normal and $L^SN=NL^S$; and conversely.

**Proof**

As above, let $UAU^S=F=\overline{W}^S DW=\overline{w}^S D_rw\overline{w}^S D_uw$ where $D_r$ and $D_u$ are the s-hermitian and s-unitary polar matrices of $D$) and $\overline{U}B\overline{U}^S=B_1=\overline{W}^S B_2W=\overline{W}^S\left(C_1\oplus\ldots\oplus C_K\right)W.$ As in the proof of **Theorem 3** if follows that for all $i$, $D_iC_i\overline{C}^S_i=\overline{C}^S_iC_iD_i$ and $\overline{U}^S_iC_i\overline{C}^S_i=\overline{C}^S_iC_i\overline{U}^S_i$ with $U_i$ as defined there, so that when $R_i=\overline{D}_i\overline{U}^S_i$ (where $D$, here, $=r_1D_1\oplus r_2D_2\oplus\ldots\oplus r_kD_k$ as earlier) then $C_i=H_iU_i=H_i\overline{R}^S_i\overline{D}_i$ *with* $H_iR_i=R_iH_i$.

Then since, $WD_r=D_rW$, $UAU^S=\overline{W}^S D_rw\overline{W}^S D_uw=D_r\left(\overline{W}^S D_uw\right)$ and

$$A=\left(\overline{U}^S D_rU\right)\left(\overline{U}^S\overline{w}^S D_uw\overline{U}\right)=LX$$

$$=\left(\overline{U}^S\overline{w}^S D_uw\overline{U}\right)\left(U^S D_r\overline{U}\right)=XL^S$$

with $L=\overline{U}^S D_rU$ s-hermitian and $X=\overline{U}^S\overline{w}^S D_uw\overline{U}$ s-unitary.

Also, $\overline{U}B\overline{U}^S = \overline{w}^S\left(H_1\overline{R}_1^S\overline{D}_1 \oplus H_2\overline{R}_2^S\overline{D}_2 \oplus \ldots \oplus H_k\overline{R}_k^S\overline{D}_k\right)w = N_1Y$

Where $N_1 = \overline{w}^S\left(H_1\overline{R}_1^S \oplus H_2\overline{R}_2^S \oplus \ldots \oplus H_k\overline{R}_k^S\right)w$ is s-normal and $Y = \overline{w}^S\left(\overline{D}_1 \oplus \overline{D}_2 \oplus \ldots \oplus \overline{D}_k\right)w$ is s-unitary; then $B = U^S N_1 Y U = \left(U^S N_1 \overline{U}\right)\left(U^S Y U\right) = N\overline{X}^S$.

Where $N = U^S N_1 \overline{U}$ is s-normal and $\overline{X}^S = U^S Y U = U^S \overline{W}^S \overline{D}_u w U$. Also $L^S N = NL^S$ since $D_r N_1 = N_1 D_r, \overline{D}_r N_1 = N_1 \overline{D}_r$ so $\left(\overline{U}LU^S\right)\left(\overline{U}NU^S\right) = \left(\overline{U}NU^S\right)\left(\overline{U}LU^S\right)$ so $L^S N = NL^S$.

The converse is immediate.

## 4. Products of Con-s-Normal Matrices

It is possible if $A$ is s-normal and $B$ con-s-normal that $AB$ is con-s-normal. For example, any con-s-normal matrix $C=HU=UH^S$ is such a product with $A=H$ and $B=U$. Or if $C=HU=UH^S$ and $A=H$, then $AC=H^2U=HUH^S=U(H^S)^2$ is con-s-normal. The following theorems clarify this matter.

**Theorem 7**

If $A$ is s-normal and $B$ is con-s-normal then $AB$ is con-s-normal iff $AB\overline{B}^S = B\overline{B}^S A$ *and* $\overline{B}A\overline{A}^S = A^S\overline{A}\overline{B}(or B\overline{A}A^S = \overline{A}^S AB)$.

(If one were to define $N$ is s-normal with respect to $M$' to mean $N\overline{N}^S M = M\overline{N}^S N$ and $Q$ is con-s-normal with respect to $P$ to mean $PQ\overline{Q}^S = Q^S\overline{Q}P$ the above theorem would say that if $A$ is s-normal and $B$ is con-s-normal then $AB$ is con-s-normal iff (con-s-normal) $B$ is s-normal with respect to $A$ and (s-normal) $A$ is con-s-normal with respect to $\overline{B}$).

**Proof**

If the latter condition hold, then; $(AB)\left(\overline{AB}\right)^S = AB\overline{B}^S\overline{A}^S = B\overline{B}^S A\overline{A}^S$ and $(AB)^S\left(\overline{AB}\right) = B^S A^S \overline{A}\,\overline{B} = B^S\overline{B}A\overline{A}^S$ which are equal.

Conversely, let $AB$ be con-s-normal and let $UA\overline{U}^S = D = d_1I_1 \oplus d_2I_2 \oplus \ldots \oplus d_kI_k$ where $d_i\overline{d}_i > d_j\overline{d}_j,\ i > j$.

$Let\ UB^S U^S = B_1 = (bij)$,

$if\ (AB)(\overline{AB})^S = AB\overline{B}^S\overline{A}^S = AB^S\overline{B}\ \overline{A}^S = (AB)^S(\overline{AB})$

$= B^S A^S \overline{A}\,\overline{B} = B^S\overline{A}A^S\overline{B}$,

then $\left(UA\overline{U}^S\right)\left(UB^S U^S\overline{U}\,\overline{B}\,\overline{U}^S\right)\left(U\overline{A}^S\overline{U}^S\right) = \left(UB^S U^S\right)\left(\overline{U}\,\overline{A}U^S\overline{U}A^S U^S\right)\left(\overline{U}\,\overline{B}\overline{U}^S\right)$

So that $DB_1\overline{B}_1^S\overline{D}^S = B_1\overline{D}D\overline{B}_1^S$.

Equating secondary diagonal elements on each side of this relation, we get $\sum_{j=1}^{n} d_i\overline{d}_i b_{ij}\overline{b}_{ij} = \sum_{j=1}^{n} d_j\overline{d}_j b_{ij}\overline{b}_{ij}$ , $i=1,2,\ldots n$ or

$\sum_{j=1}^{n}\left(d_i\overline{d}_i - d_j\overline{d}_j\right)b_{ij}\overline{b}_{ij} = 0$ .

Let $d_1\overline{d}_1 = d_2\overline{d}_2 = \ldots d_l\overline{d}_l > d_{l+1}\overline{d}_{l+1}$ then $b_{ij} = 0$ for $i=1,2\ldots l$ and $j = l+1, l+2\ldots n$ since $B_1$ is con-s-normal, $\sum_{j=1}^{n} b_{ij}\overline{b}_{ij} = \sum_{j=1}^{n} b_{ji}\overline{b}_{ji}$ for $i = 1,2,\ldots n$ on adding the first $l$ of these equation and canceling, $b_{ij} = 0$ for $\mathrm{i} = l+1, l+2\ldots\mathrm{n}$ and $\mathrm{j} = 1,2,\ldots,l$. In this manner if $D = r_1D_1 \oplus r_2D_2 \oplus \ldots \oplus r_tD_t$ with $r_i > r_{i+1}$ and $D_i$ s-unitary, then $B_1 = C_1 \oplus C_2 \oplus \ldots \oplus C_t$ conformable to $D$.

Since $r_iD_i\overline{D}_i^S r_i\overline{C}_i^S = r_i^2C_i^S = C_i^S r_i^2 = C_i^S r_iD_i\overline{D}_i^S r_i$, for all $i$, $D\overline{D}^S B_1^S = B_1^S D\overline{D}^S$ and so $\overline{U}^S D\overline{D}^S U\overline{U}^S B_1^S\,\overline{U} = \overline{U}^S B_1^S\overline{U}U^S D\overline{D}^S\overline{U}$ or $A\overline{A}^S B = BA^S\overline{A}$ or $\overline{A}^S AB = BA^S\overline{A}$ or $A^S\overline{A}\overline{B} = \overline{B}A\overline{A}^S$.

Also, $D\left(B_1\overline{B}_1^S\overline{D}^S\right) = B_1\overline{D}D\overline{B}_1^S = \overline{D}D\overline{B}_1^S = D\left(\overline{D}B_1\overline{B}_1^S\right)$ so that $C_i\overline{C}_i^S\left(r_i\overline{D}_i\right) = \left(r_i\overline{D}_i\right)C_i\overline{C}_i^S$ for $i = 1,2\ldots t$. (if $r_t = 0$, this is still true and $D_t$ may be chosen to be identity matrix). Therefore $B_1\overline{B}_1^S\ \overline{D}^S = \overline{D}^S B_1\overline{B}_1^S$ and $UB^S U^S\ \overline{U}\ \overline{B}\,\overline{U}^S U\overline{A}^S\overline{U}^S = U\overline{A}^S\overline{U}^S UB^S U^S\overline{U}\,\overline{B}_1\overline{U}^S$ so $B^S\overline{B}\overline{A}^S = \overline{A}^S B^S\overline{B}$ $or$ $AB^S\overline{B} = B^S\overline{B}A$.

**Corollary 1**

Let $A$ be s-normal, $B$ con-s-normal; if $AB$ is con-s-normal, then $B\overline{A}$ is con-s-normal, and conversely.

**Proof**

From the above, $UA\overline{U}^S UBU^S = DB_1^S$ is con-s-normal, and if $D = D_r D_u$, $D_r$ real and $D_u$ s-unitary, then since

$$\overline{D_u} = \overline{D_u}^S, \overline{D_u}\left(DB_1^S\right)\overline{D_u} = D_r B_1^S \overline{D_u} = B_1^S D_r \overline{D_u} = B_1^S \overline{D} \text{ is con-s-normal,}$$

as are $U\,BU^S\,\overline{U}\,\overline{A}\,U^S$ and $B\,\overline{A}$ . Reversing the steps proves the converse.

If $A$ is s-normal and $B$ is con-s-normal, $B\overline{A}$ is con-s-normal iff $AB$ is con-s-normal, iff $\left(B^S\,\overline{B}\right)A = A\left(B\overline{B}^S\right)$ and $\left(A^S\,\overline{A}\right)\overline{B} = \overline{B}\left(A\overline{A}^S\right)$. Therefore if $A$ is s-normal $B$ is con-s-normal $BA$ is con-s-normal iff $\left(B^S\,\overline{B}\right)\overline{A} = \overline{A}\left(B\overline{B}^S\right)$ and $\left(\overline{A}^S A\right)\overline{B} = \overline{B}\left(\overline{A}A^S\right)$ that is replace $A$ by $\overline{A}$ in the proceeding or $\left(\overline{B}^S B\right)A = A\left(\overline{B}B^S\right) = A\left(\overline{B}^S B\right)$ and $\left(\overline{A}^S A\right)\overline{B} = \overline{B}\left(\overline{A}A^S\right)$, thus exhibiting the fact that when $AB$ is con-s-normal, $BA$ is not necessarily so.

**Theorem 8**

If $A = LW = WL$ is s-normal and $B = KV = VK^S$ is con-s-normal (where $L$ and $K$ are s-hermitian and $W$ and $V$ are s-unitary) then $AB$ is con-s-normal iff $LK = KL$, $LV = VL^S$ and $WK = KW$.

**Proof**

If the three relations in the theorem hold, then $AB = LWKV = LKWV$, and $AB = WLKV = WKLV = WKVL^S = WVK^S L^S = WV\left(LK\right)^S$ is con-s-normal since $LK$ is s-hermitian and $WV$ is s-unitary.

Conversely, Let $A = \overline{U}^S DU = \left(\overline{U}^S D_r U\right)\left(\overline{U}^S D_u U\right) = LW$ and

$$B = \left(\overline{U}^S B_1{}^S \overline{U}\right) = \left(\overline{U}^S K_1 U\right)\left(\overline{U}^S V_1 \overline{U}\right) = KV = VK^S$$

where $K_1$ and $V_1$ are s-hermitian and s-unitary and direct sums conformable to $B_1^S$ and $D$. A direct check shows that $LK = KL$ and $LV = VL^S$, also $WK = \overline{U}^S D_u K_1 U = \overline{U}^S K_1 D_u U = KW$ since $D_u B_1 \overline{B_1}^S = B_1 \overline{B_1}^S D_u$ implies $D_u K_1 = K_1 D_u$.

A sufficient condition for the simultaneous reduction of $A$ and $B$ is given by the following:

**Theorem 9**

If $A$ is s-normal, $B$ is con-s-normal and $AB = BA^S$, then $WA\overline{W}^S = D$ and $WB^S W = F$, the s-normal form of **Theorem 1**, where $W$ is an s-unitary matrix; also $AB$ is con-s-normal.

**Proof**

Let $UA\overline{U}^S = D$ secondary diagonal and $UBU^S = B_2$ which is con-s-normal. Then $AB = BA^S$ implies $DB_2 = UA\overline{U}^S UBU^S = UBU^S \overline{U} A^S U^S = B_2 D^S = B_2 D.$

Let $D = C_1 I_1 \oplus C_2 I_2 \oplus .... \oplus C_K I_K$. Where the $C_i$ are complex and $C_i \neq C_j$ for $i \neq j$ and $B_2 = C_1 \oplus C_2 \oplus .... \oplus C_K$ let $V_i$ be s-unitary such that $V_i C_i V_i^S = F_i$ the real s-normal form of **Theorem 1**, and let $V = V_1 \oplus V_2 \oplus ... \oplus V_k$.

Then $VUA\overline{U}^S \overline{V}^S = D$, $VUBU^S V^S = F = a$ direct sum of the $F_i$.

Also, $AB = BA^S$ implies $B^S A^S = AB^S$ and so

$$AB\overline{B}^S \overline{A}^S = AB^S \overline{B}\overline{A}^S = B^S A^S \overline{A}\overline{B} = (AB)^S \left(\overline{AB}\right).$$

It is also possible for the product of two s-normal matrices $A$ and $B$ to be con-s-normal if $Q = HU = UH^S$ is con-s-normal and if $A = U$ and $B = H$ this is so or if $KV = VK^S$ is con-s-normal and if $A=UK=KU$ is s-normal with $K$ s-hermitian and $V$ and $U$ s-unitary, for $B = V, AB = (UK)V = K(UV) = (UV)K^S$ con-s-normal. But if in the first example, $U^2 H$ is not s-normal then $HU$ is not con-s-normal so that $BA$ is not necessarily con-s-normal though $AB$ is. When $A$ alone is s-normal an analog of **Theorem 2** can be obtained which states the following: if $A$ is s-normal, then $AB$ and $AB^S$ are con-s-normal iff $AB\overline{B}^S = B^S \overline{B} A$, $B\overline{B}^S A = AB^S \overline{B}$ and $\overline{B} A \overline{A}^S = A^S \overline{A}\overline{B}$. (The proof is not included here because of its similarity to that above) when $B$ is con-s-normal, two of these conditions merge into one in **Theorem 7**.

It is possible for the product of two con-s-normal matrices to be con-s-normal but no such simple analogous necessary and sufficient conditions as exhibited above are available. This may be seen as follows two non-real complex commutative matrices $P = P^S$ and $Q = Q^S$ can form a con-s-normal (and non-real s-symmetric) matrix $PQ$ which need not be s-normal. Then two s-symmetric matrices $X = \begin{bmatrix} -i & -i \\ i & -i \end{bmatrix}$ $Y = \begin{bmatrix} 2i & 0 \\ 0 & 2i \end{bmatrix}$ are such that $XY=Z$ is real, s-normal and con-s-normal (s-symmetric).

Finally if $U$ and $V$ are two complex s-unitary matrices of the same order, they can be chosen so $UV$ is non-real that is complex, s-normal and con-s-normal. If $A = P \oplus X \oplus U$ and $B = Q \oplus Y \oplus V$ $AB = PQ \oplus XY \oplus UV$ where $A$ and $B$ are con-s-normal as in $AB$ (s-symmetric). A simple inspection of these matrices shows that relations on the order of $\left(B^S \overline{B}\right) A = A\left(B\overline{B}^S\right) = \left(B\overline{B}^S\right) A$ and $\left(A^S \overline{A}\right)\overline{B} = \left(A\overline{A}^S\right)\overline{B} = \overline{B}\left(A\overline{A}^S\right)$ do not necessarily hold; these are sufficient, however, to guarantee that $AB$ is con-s-normal (as direct verification from the definition).

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