A PRIORI ESTIMATE AND CONTINUOUS DEPENDENCE OF SOLUTIONS TO MIXED BOUNDARY VALUE PROBLEMS FOR PSEUDO-PARABOLIC EQUATION

N'gniamessan GBENOUGA, Aboudou-Fataou MOROU, Mawoussi TODJRO, Kokou TCHARIE

Abstract

One establishes the a priori energy inequality which guarantees the uniqueness of the solution and shows the continuous dependence of the solutions of the form of the boundary conditions.

Mathematics Subject Classification :35K70, 35A35, 34A12.

keywords :pseudoparabolic equations, a priori energetic inequality, continuous dependence

1 Statement of the problem

In a rectangle, $\Omega = (0,l) \times (0,T)$, we study the set of mixed problems with integral conditions

$$\frac{\partial u_\alpha}{\partial t} - \frac{\partial}{\partial x}\left(b(x,t)\frac{\partial^2 u_\alpha}{\partial x \partial t}\right) - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u_\alpha}{\partial x}\right) = f(x,t) \tag{1}$$

$$u_\alpha(x,0) = \varphi_\alpha(x), \quad u_\alpha(0,t) = 0, \quad \frac{1}{l-\alpha}\int_\alpha^l u_\alpha(x,t)dx = h(t) \tag{2}$$

Where $0 \leqslant \alpha < l$ and the mixed problem with local conditions

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(b(x,t)\frac{\partial^2 u}{\partial x \partial t}\right) - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) = f(x,t) \tag{3}$$

$$u(x,0) = \varphi(x), \quad u(0,t) = 0, \quad u(l,t) = h(t) \tag{4}$$

It is assumed that the following conditions are satisfied

Condition 1.
For all $(x,t) \in \Omega$, we suppose that

$a_0 \le a(x,t) \le a_1, \quad \frac{\partial a(x,t)}{\partial t} \le a_2, \quad \frac{\partial a(x,t)}{\partial x} \le a_3, \quad \frac{\partial^2 a(x,t)}{\partial x \partial t} \le a_4$;

$1 \le b(x,t) \le b_1, \quad \frac{\partial b(x,t)}{\partial t} \le b_2, \quad \frac{\partial b(x,t)}{\partial x} \le b_3, \quad \frac{\partial^2 b(x,t)}{\partial t^2} \le b_4$;

$\frac{\partial^2 b(x,t)}{\partial x \partial t} \le b_5, \quad \frac{\partial^2 b(x,t)}{\partial x^2} \le b_6, \quad \frac{\partial^3 b(x,t)}{\partial x \partial t^2} \le b_7$;

$(a_i)_{0\le i\le 4}, \quad (b_k)_{1\le k\le 7}$ *are positive constants.*

Condition 2.
$f \in L_2(\Omega)$, $h \in W_2^1(0,T)$, φ_α, $\varphi \in W_2^1(0,T)$, $\varphi_\alpha(0) = \varphi(0) = 0$, $\varphi(1) = h(0)$,
$\frac{1}{l-\alpha}\int_\alpha^l \varphi_\alpha(x)dx = h(0)$.

2 Preliminary

In a rectangle $\Omega = (0,l) \times (0,T)$, consider equation :

$$\mathcal{L}_\lambda u = \frac{\partial u}{\partial t} - \lambda\frac{\partial}{\partial x}\left(b(x,t)\frac{\partial^2 u}{\partial x \partial t}\right) - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) = f(x,t) \tag{5}$$

with the initial condition

$$u(x,t)|_{t=0} = \varphi(x),\ x \in (0,l), \tag{6}$$

local boundary conditions

$$\begin{cases} u(x,t)|_{x=0} = 0, \text{ (conditions homogeneous to the limits of Dirichlet)} \\ \dfrac{\partial u}{\partial x}(x,t)|_{x=0} = 0, \text{ (conditions homogeneous to the limits of Neumann)} \end{cases} \tag{7}$$

and the homogeneous nonlocal condition

$$\int_{\alpha}^{l} u(x,t)dx = 0,\ 0 \leqslant \alpha < l, t \in (0,T) \tag{8}$$

where the functions,$a(x,t); b(x,t); f(x,t); \varphi(x)$, require the following conditions :

Condition 1

$a_0 \leq a(x,t) \leq a_1;\quad \dfrac{\partial a(x,t)}{\partial t} \leq a_2;\quad \dfrac{\partial a(x,t)}{\partial x} \leq a_3;\quad \dfrac{\partial^2 a(x,t)}{\partial x \partial t} \leq a_4$

$1 \leq b(x,t) \leq b_1;\quad \dfrac{\partial b(x,t)}{\partial t} \leq b_2;\quad \dfrac{\partial b(x,t)}{\partial x} \leq b_3;\quad \dfrac{\partial^2 b(x,t)}{\partial x^2} \leq b_4$

$\dfrac{\partial^2 b(x,t)}{\partial x \partial t} \leq b_5;\ \dfrac{\partial^2 b(x,t)}{\partial t^2} \leq b_6;\quad \dfrac{\partial^3 b(x,t)}{\partial x^2 \partial t} \leq b_7$

with the $(a_i)_{o\leq i\leq 4}$; $(b_k)_{1\leq k\leq 7}$ and λ positive real constants.

Condition 2

$\varphi(0) = 0$ et $\displaystyle\int_{\alpha}^{l} \varphi(x)dx = 0$

When the homogeneous condition at the limits of Dirichet is used .

$\varphi'(0) = 0$ et $\displaystyle\int_{\alpha}^{l} \varphi(x)dx = 0$

When Neumann's homogeneous condition at the limits is used.

2.1 Basic lemma

The solution of the problem (5) to (8) can be considered as the solution of the operational equation :

$$\mathcal{L}_\lambda u = \mathcal{F} \equiv (f, \varphi) \tag{9}$$

Where the operator $\mathcal{L}_\lambda$ defined by :

$$\mathcal{L}_\lambda : E_\lambda \subseteq L^2(\Omega) \longrightarrow F$$
$$u \longmapsto \mathcal{L}_\lambda u = (\mathcal{L}_\lambda u, \varphi)$$

• has as its domain :

$D(\mathcal{L}_\lambda) = \{u \in L^2(\Omega) / \dfrac{\partial u}{\partial t};\quad \dfrac{\partial^2 u}{\partial x^2};\quad \dfrac{\partial^3 u}{\partial x^2 \partial t} \in L^2(\Omega);\quad u(x,t)|_{x=0} = \dfrac{\partial u(x,t)}{\partial x}|_{x=0} = \displaystyle\int_{\alpha}^{l} u(x,t)dx = 0\}.$

•E_λ is a space of Banach which is the completeness of $D(\mathcal{L}_\lambda)$ in relation to the norm :

$$\|u\|_{E_\lambda^2} = \int_\Omega (l-x)\left[\frac{1}{2}|\frac{\partial u}{\partial t}|^2 + |\frac{\partial^2 u}{\partial t \partial x}|^2\right] dxdt + \sup_{0\leq t\leq T} \int_0^2 (l-x)|\frac{\partial u}{\partial x}|^2 dx. \tag{10}$$

•F is the Hilbert space with the norm.

$$\|\mathcal{L}_\lambda u\|_F^2 = \int_\Omega |f(x,t)|^2 dxdt + \int_0^l \left[|\varphi(x)|^2 + |\varphi'(x)|^2\right] dx. \tag{11}$$

Definition1

We call a strong generalized solution of the problem 1 to 4 with the conditions 1 and 2, a solution of equation :

$$\overline{\mathcal{L}}u = F \tag{12}$$

and 2 to 4 with the conditions 1 and 2 where $\overline{\mathcal{L}}$ is the closure of $\mathcal{L}$.

lemma 1

Consider the operator M defined by : $Mv = \psi_\alpha v - Jv$ where

$$\psi_\alpha(x) = \begin{cases} 1, \text{ si } \quad 0 \leq x \leq \alpha \\ \dfrac{l-x}{l-\alpha}, \text{ si } \quad \alpha \leq x \leq l \end{cases} \qquad J_v = \begin{cases} 0, \text{ si } \quad 0 \leq x \leq \alpha \\ \dfrac{1}{l-\alpha} \int_x^l v(y)dy, \text{ si} \alpha \leq x \leq l \end{cases}$$

and v a positive function.
We have :

$$1)\forall x \in (0,l), 0 < \frac{l-x}{l-\alpha} \leq \psi_\alpha(x) \leq 1. \quad (13)$$

$$2)\forall v \in D(\mathcal{L}), Mv \leq \psi_\alpha v \quad et \quad Mv \leq v. \quad (14)$$

3)$\forall u$ and v elements of $D(\mathcal{L})$ and $A(x,t)$ defined on $(0,l)\times(0,T)$, we have a :

$$-\int_0^l \frac{\partial}{\partial x}(A(x,t)\frac{\partial u}{\partial x})Mvdx = \int_0^l \psi_\alpha(x)A(x,t)\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}dx. \quad (15)$$

Proof

1) (13) comes from the definition of ψ_α.
2) $J \quad et \quad \psi_\alpha$ are positive by definition, hence $Mv \leq \psi_\alpha v, \forall v \in D(\mathcal{L})$.
And because $0 < \psi_\alpha(x) \leq 1$, it comes :$Mv \leq v$.
3) $\forall u$ and v elements de $D(\mathcal{L})$ we have :

$$-\int_0^l \frac{\partial}{\partial x}(A(x,t)\frac{\partial u}{\partial x})Mvdx = -\int_0^\alpha \frac{\partial}{\partial x}(A(x,t)\frac{\partial u}{\partial x})Mvdx - \int_\alpha^l \frac{\partial}{\partial x}(A(x,t)\frac{\partial u}{\partial x})Mvdx$$

Let's first calculate $\int_0^\alpha \frac{\partial}{\partial x}\left(A(x,t)\frac{\partial u}{\partial x}\right)Mvdx$.

$$\int_0^\alpha \frac{\partial}{\partial x}\left(A(x,t)\frac{\partial u}{\partial x}\right)Mvdx = \int_0^\alpha \frac{\partial}{\partial x}\left(A(x,t)\frac{\partial u}{\partial x}\right)(\Psi_\alpha(x)v - Jv(x))\,dx$$

$$\int_0^\alpha \frac{\partial}{\partial x}\left(A(x,t)\frac{\partial u}{\partial x}\right)Mvdx \quad = \quad \int_0^\alpha \frac{\partial}{\partial x}\left(A(x,t)\frac{\partial u}{\partial x}\right)vdx.$$

By integrating, in parts, we have :

$$\int_0^\alpha \frac{\partial}{\partial x}\left(A(x,t)\frac{\partial u}{\partial x}\right)Mvdx = -\int_0^\alpha A(x,t)\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}dx + \left[\left(A(x,t)\frac{\partial u}{\partial x}\right)v\right]_0^\alpha.$$

Let us now calculate $\int_\alpha^l \frac{\partial}{\partial x}\left(A(x,t)\frac{\partial u}{\partial x}\right)Mvdx$.

$$\int_\alpha^l \frac{\partial}{\partial x}\left(A(x,t)\frac{\partial u}{\partial x}\right)Mvdx = \int_\alpha^l \frac{\partial}{\partial x}\left(A(x,t)\frac{\partial u}{\partial x}\right)(\Psi_\alpha(x)v - Jv(x))\,dx$$

$$\int_\alpha^l \frac{\partial}{\partial x}\left(A(x,t)\frac{\partial u}{\partial x}\right)Mvdx = \frac{1}{l-\alpha}\left\{\int_\alpha^l \frac{\partial}{\partial x}\left(A(x,t)\frac{\partial u}{\partial x}\right)(l-x)vdx - \int_\alpha^l \left\{\frac{\partial}{\partial x}\left(A(x,t)\frac{\partial u}{\partial x}\right)\int_x^l v(y)dy\right\}dx\right\}.$$

By integrating, in parts, the two terms on the right, it comes

$$\int_\alpha^l \frac{\partial}{\partial x}\left(A(x,t)\frac{\partial u}{\partial x}\right)Mvdx = \frac{1}{l-\alpha}\left[\left[A(x,t)\frac{\partial u}{\partial x}(l-x)v\right]_0^l - \int_\alpha^l A(x,t)\frac{\partial u}{\partial x}\left(-v + (l-x)\frac{\partial v}{\partial x}\right)dx\right] -$$

$$-\frac{1}{l-\alpha}\left[\left[A(x,t)\frac{\partial u}{\partial x}\int_x^l v(y)dy\right]_\alpha^l + \int_\alpha^l A(x,t)\frac{\partial u}{\partial x}v(x)dx\right]$$

$$-\int_0^l \frac{\partial}{\partial x}(A(x,t)\frac{\partial u}{\partial x})Mvdx = \int_0^l \psi_\alpha(x)A(x,t)\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}dx.$$

lemma 2

If $g \in D(\mathcal{L})$ and verifies 2.4 then :

$$\int_0^l Jg.gdx = 0. \quad (16)$$

Proof

$$\begin{aligned}
\int_0^l Jg.gdx &= \int_\alpha^l Jg.gdx \\
&= \frac{1}{l-\alpha}\int_\alpha^l \left[g(x)\int_x^l g(y)dy\right]dx \\
&= -\frac{1}{l-\alpha}\int_\alpha^l \left[\frac{d}{dx}\left(\int_x^l g(y)dy\right)\int_x^l g(y)dy\right]dx \\
&= -\frac{1}{l-\alpha}\int_\alpha^l \frac{1}{2}\left[\frac{d}{dx}\left(\int_x^l g(y)dy\right)\right]^2 dx \\
&= -\frac{1}{2(l-\alpha)}\left[\int_x^l g(y)dy\right]^2 \text{ d'où} \\
\int_0^l Jg.gdx &= 0.
\end{aligned}$$

2.2 Energetic inequality

Theorem 2.1. *If the conditions* 1 *and* 2 *are satisfied then there exists a positive constant* C *such that, for any function* $u \in D(\mathcal{L})$ *solution of the problem* 1 *to* 4*, we have :*

$$\|u\|_{E_\lambda} \leq C\|\mathcal{L}_\lambda u\|_F. \tag{17}$$

Proof

Let $u \in D(\mathcal{L})$ function and check (1) to (4) and the conditions 1 and 2.

By multiplying (1) by $M\dfrac{\partial u}{\partial t}$ and integrating with respect to x on $(0,l)$ and then with respect to t of 0 to τ, it comes :

$$\int_0^\tau\int_0^l \mathcal{L}_\lambda u M\frac{\partial u}{\partial t}dxdt = \int_0^\tau\int_0^l \frac{\partial u}{\partial t}M\frac{\partial u}{\partial t}dxdt - \lambda\int_0^\tau\int_0^l \frac{\partial}{\partial x}\left(b(x,t)\frac{\partial^2 u}{\partial x\partial t}\right)M\frac{\partial u}{\partial t}dxdt-$$
$$-\int_0^\tau\int_0^l \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial^2 u}{\partial x\partial t}\right)M\frac{\partial u}{\partial t}dxdt. \quad (E_1)$$

Let us express each term of the right-hand side of the equality (E_1)

• $\displaystyle\int_0^\tau\int_0^l \frac{\partial u}{\partial t}M\frac{\partial u}{\partial t}dxdt = \int_0^\tau\int_0^l \frac{\partial u}{\partial t}\psi_\alpha(x)\frac{\partial u}{\partial t}dxdt - \int_0^\tau\int_0^l \frac{\partial u}{\partial t}J\frac{\partial u}{\partial t}dxdt.$

As $u \in D(\mathcal{L})$ then $\dfrac{\partial u}{\partial t} \in L^2(\Omega)$ and after (16), we have : $\displaystyle\int_0^l \frac{\partial u}{\partial t}J\frac{\partial u}{\partial t}dx = 0$

Therefore $\displaystyle\int_0^\tau\int_0^l \frac{\partial u}{\partial t}M\frac{\partial u}{\partial t}dxdt = \int_0^\tau\int_0^l \psi_\alpha(x)|\frac{\partial u}{\partial t}|^2|dxdt \quad (E_2)$

• $\displaystyle -\int_0^l \frac{\partial}{\partial x}\left(b(x,t)\frac{\partial^2 u}{\partial x\partial t}\right)M\frac{\partial u}{\partial t}dx = \int_0^l \psi_\alpha(x)b(x,t)\left(\frac{\partial^2 u}{\partial x\partial t}\right)^2 dx.$

By replacing (15) $A(x), u$ and v respectivily by $b(x,t)$, $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial u}{\partial t}$ and by integrating with respect to t from 0 to τ, we obtain :

$$-\int_0^\tau\int_0^l \frac{\partial}{\partial x}\left(b(x,t)\frac{\partial^2 u}{\partial x\partial t}\right)M\frac{\partial u}{\partial t}dxdt = \int_0^\tau\int_0^l \psi_\alpha(x)b(x,t)|\frac{\partial^2 u}{\partial x\partial t}|^2 dxdt. \qquad (E_3)$$

• Similarly, replacing in (15) $A(x)$ and v respectivily by $a(x,t)$ and $\dfrac{\partial u}{\partial t}$, and by integrating with respect to t from 0 to τ, it comes :

$$-\int_0^\tau\int_0^l \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)M\frac{\partial u}{\partial t}dxdt = \int_0^\tau\int_0^l \psi_\alpha(x)a(x,t)\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}dxdt.$$

But $\dfrac{\partial u}{\partial x}\dfrac{\partial^2 u}{\partial x\partial t} = \dfrac{1}{2}\dfrac{\partial}{\partial t}\left(\dfrac{\partial u}{\partial x}\right)^2$ so :

$$\begin{aligned}
-\int_0^\tau\int_0^l \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)M\frac{\partial u}{\partial t}dxdt &= \frac{1}{2}\int_0^\tau\int_0^l \psi_\alpha(x)a(x,t)\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dxdt \\
&= \frac{1}{2}\int_0^l \psi_\alpha(x)\left\{\int_0^\tau a(x,t)\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dt\right\}dx \\
&= \frac{1}{2}\int_0^l \psi_\alpha(x)\left\{\left[a(x,t)\left(\frac{\partial u}{\partial x}\right)^2\right]_0^\tau\right. \\
&\quad \left.-\int_0^\tau \frac{\partial a(x,t)}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dt\right\}dx \\
&= \frac{1}{2}\int_0^l \psi_\alpha(x)\left\{a(x,t)|\frac{\partial u}{\partial x}(x,\tau)|^2\right. \\
&\quad \left.-a(x,0)|\frac{\partial u}{\partial x}(x,0)|^2\right\}dx - \\
&\quad \frac{1}{2}\int_0^\tau\int_0^l \psi_\alpha(x)\frac{\partial a(x,t)}{\partial t}|\frac{\partial u}{\partial x}|^2 dxdt \\
&= \frac{1}{2}\int_0^l \psi_\alpha(x)a(x,\tau)|\frac{\partial u}{\partial x}(x,\tau)|^2 dx \\
&\quad -\frac{1}{2}\int_0^l \psi_\alpha(x)a(x,0)|\frac{\partial u}{\partial x}(x,0)|^2 dx \\
&\quad \frac{1}{2}\int_0^\tau\int_0^l \psi_\alpha(x)\frac{\partial a(x,t)}{\partial t}|\frac{\partial u}{\partial x}|^2 dxdt. \qquad (E_4)
\end{aligned}$$

By using (E_2),(E_3) and (E_4) the equality (E_1) becomes :
$$\int_0^\tau \mathcal{L}_\lambda u M\frac{\partial u}{\partial t}dxdt = \int_0^\tau\int_0^l \psi_\alpha(x)|\frac{\partial u}{\partial t}|^2 dxdt + \lambda\int_0^\tau\int_0^l \psi_\alpha(x)b(x,t)|\frac{\partial^2 u}{\partial x\partial t}|^2 dxdt+$$
$$\frac{1}{2}\int_0^l \psi_\alpha(x)a(x,\tau)|\frac{\partial u}{\partial x}(x,\tau)|^2dx - \frac{1}{2}\int_0^l \psi_\alpha(x)a(x,0)|\frac{\partial u}{\partial x}(x,0)|^2dx - \frac{1}{2}\int_0^\tau\int_0^l \psi_\alpha(x)\frac{\partial a(x,t)}{\partial t}|\frac{\partial u}{\partial x}|^2dxdt. \quad (E_5)$$
According to the inequality of Cauchy Bougnyakosky,
$$2\int_0^\tau \mathcal{L}_\lambda u M\frac{\partial u}{\partial t}dxdt \le 2\sqrt{\int_0^\tau\int_0^l (\mathcal{L}_\lambda u)^2\,dxdt}\sqrt{\int_0^\tau\int_0^l \left(M\frac{\partial u}{\partial t}\right)^2 dxdt}$$
the application of the inequality $2ab \le a^2 + b^2$ to the right-hand side of the above-mentioned one makes it possible to obtain :

$$2\sqrt{\int_0^\tau\int_0^l (\mathcal{L}_\lambda u)^2\,dxdt}\sqrt{\int_0^\tau\int_0^l \left(M\frac{\partial u}{\partial t}\right)^2 dxdt} \le \int_0^\tau\int_0^l (\mathcal{L}_\lambda u)^2\,dxdt + \int_0^\tau\int_0^l \left(M\frac{\partial u}{\partial t}\right)^2 dxdt.$$

We can deduce : $\int_0^\tau\int_0^l \mathcal{L}_\lambda u M\frac{\partial u}{\partial t}dxdt \le \frac{1}{2}\int_0^\tau\int_0^l (\mathcal{L}_\lambda u)^2\,dxdt + \frac{1}{2}\int_0^\tau\int_0^l \left(M\frac{\partial u}{\partial t}\right)^2 dxdt. \quad (E_6)$

But $\int_0^\tau\int_0^l \left(M\frac{\partial u}{\partial t}\right)^2 dxdt \le \int_0^\tau\int_0^l \frac{\partial u}{\partial t}M\frac{\partial u}{\partial t}dxdt \le \int_0^\tau\int_0^l \psi_\alpha(x)\left(\frac{\partial u}{\partial t}\right)^2 dxdt. \quad (E_7)$

Applying (10) to $\frac{\partial u}{\partial t}$ we have :$M\frac{\partial u}{\partial t} \le \frac{\partial u}{\partial t}$ et $M\frac{\partial u}{\partial t} \le \psi_\alpha\frac{\partial u}{\partial t}$.Thus the inequality (E_6) becomes :
$$\int_0^\tau\int_0^l \mathcal{L}_\lambda u M\frac{\partial u}{\partial t}dxdt \le \frac{1}{2}\int_0^\tau\int_0^l (\mathcal{L}_\lambda u)^2\,dxdt + \frac{1}{2}\int_0^\tau\int_0^l \psi_\alpha(x)\left(\frac{\partial u}{\partial t}\right)^2 dxdt. \quad (E_8)$$
By replacing in the relation (E_8) the left term by its expression of the equality (E_5), one obtains :

$$\int_0^\tau\int_0^l \psi_\alpha(x)|\frac{\partial u}{\partial t}|^2dxdt + \lambda\int_0^\tau\int_0^l \psi_\alpha(x)b(x,t)|\frac{\partial^2 u}{\partial x\partial t}|^2dxdt + \frac{1}{2}\int_0^l \psi_\alpha(x)a(x,\tau)|\frac{\partial u}{\partial x}(x,\tau)|^2dx-$$
$$\frac{1}{2}\int_0^l \psi_\alpha(x)a(x,0)|\frac{\partial u}{\partial x}(x,0)|^2dx - \frac{1}{2}\int_0^\tau\int_0^l \psi_\alpha(x)\frac{\partial a(x,t)}{\partial t}|\frac{\partial u}{\partial x}|^2dxd \le \frac{1}{2}\int_0^\tau\int_0^l (\mathcal{L}_\lambda u)^2dxdt+$$
$$\frac{1}{2}\int_0^\tau\int_0^l \psi_\alpha(x)(\frac{\partial u}{\partial t})^2dxdt.$$
It then comes :
$$\int_0^\tau\int_0^l \psi_\alpha(x)\left[\frac{1}{2}|\frac{\partial u}{\partial t}|^2 + \lambda b(x,t)|\frac{\partial^2 u}{\partial x\partial t}|^2\right]dxdt + \frac{1}{2}\int_0^l \psi_\alpha(x)a(x,\tau)|\frac{\partial u)}{\partial x}(x,\tau)|^2dx \le \frac{1}{2}\int_0^\tau\int_0^l |\mathcal{L}_\lambda u|^2dxdt +$$
$$\frac{1}{2}\int_0^\tau\int_0^l \psi_\alpha(x)\frac{\partial a(x,t)}{\partial t}|\frac{\partial u}{\partial x}|^2dxdt. \quad (E_9)$$
However, according to (2), $u(x,0) = \phi(x)$ therefore $\frac{\partial u}{\partial x}(x,0) = \phi'(x)$. In addition, using **condition**1, especially $a_0 \le a(x,t) \le a_1$; $\frac{\partial a}{\partial t}(x,t) \le a_2; 1 \le b(x,t) \le b_1$, (E_9) becomes :
$$\int_0^\tau\int_0^l \Psi_\alpha(x)\left[\frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 + \lambda\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2\right]dxdt + \frac{a_0}{2}\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x\partial t}(x,\tau)\right|^2dx \leqslant \frac{1}{2}\int_0^\tau\int_0^l |\mathcal{L}_\lambda u|^2\,dxdt+$$
$$\frac{a_1}{2}\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2dx + \frac{a_2}{2}\int_0^\tau\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x}\right|^2 dxdt. \quad (E_{10})$$
As
$\Psi_\alpha(x)$ et $\left[\frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 + \lambda\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2\right]$ are positive, by removing the first term from the left member of $E(10)$ we get :
$$\frac{a_0}{2}\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x\partial t}(x,\tau)\right|^2dx \leqslant \frac{1}{2}\int_0^\tau\int_0^l |\mathcal{L}_\lambda u|^2\,dxdt + \frac{a_1}{2}\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2dx + \frac{a_2}{2}\int_0^\tau\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x}\right|^2 dxdt,$$
which can also be written in the form :
$$\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x\partial t}(x,\tau)\right|^2dx \leqslant \frac{1}{a_0}\int_0^\tau\int_0^l |\mathcal{L}_\lambda u|^2\,dxdt + \frac{a_1}{a_0}\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2dx+$$
$$\frac{a_2}{a_0}\int_0^\tau\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x}\right|^2 dxdt. \quad (E_{11})$$
By setting
$h(t) = \frac{1}{a_0}\int_0^\tau\int_0^l |\mathcal{L}_\lambda u|^2\,dxdt + \frac{a_1}{a_0}\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2dx$, $c = \frac{a_2}{a_0}$ and $g(\tau) = \int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x}(x,\tau)\right|^2dx$, the inequality (E_{11}) can be written in the form :

$$g(\tau) \leqslant c\int_0^\tau g(t)dt + h(t)$$

and using Grönwall's inequality which states :Let g and h be two integrable functions such that $g(t) \ge 0$, $h(t) \ge 0$ and h increasing on $(0,T)$. If $g(\tau) \le c\int_0^\tau g(t)dt + h(\tau)$ then : $g(\tau) \le exp(c\tau)h(\tau)$ where $c \in \Re_+^*$, of the inequality (E_{11}), we deduce :

$$\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x\partial t}(x,\tau)\right|^2 dx \leqslant e^{\frac{a_2}{a_0}\tau}\left\{\frac{1}{a_0}\int_0^\tau\int_0^l |\mathcal{L}_\lambda u|^2\,dxdt + \frac{a_1}{a_0}\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dx\right\}.$$

As $\tau \in [0;T]$, then $e^{\frac{a_2}{a_0}\tau} \leqslant e^{\frac{a_2}{a_0}T}$ and $\int_0^\tau .dt \leqslant \int_0^T$;

then the last inequality can be written as :

$$\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x\partial t}(x,\tau)\right|^2 dx \leqslant e^{\frac{a_2}{a_0}T}\left\{\frac{1}{a_0}\int_0^T\int_0^l |\mathcal{L}_\lambda u|^2\,dxdt + \frac{a_1}{a_0}\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dx\right\}.(E_{12})$$

By replacing in (E_{12}), τ by y then integrating with respect to y from 0 to τ we get :

$$\int_0^\tau\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x\partial t}(x,y)\right|^2 dxdy \leqslant e^{\frac{a_2}{a_0}T}\left\{\frac{1}{a_0}\int_0^\tau\int_0^T\int_0^l |\mathcal{L}_\lambda u|^2\,dxdtdy + \frac{a_1}{a_0}\int_0^\tau\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dxdy\right\}.(E_{13})$$

Replacing y by t in the left-hand member and applying $\int_0^\tau \leqslant \int_0^T$ to the member of (U_{13}) then multiplying on both sides by $\frac{a_2}{2}$ we find :

$$\frac{a_2}{2}\int_0^\tau\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x\partial t}(x,\tau)\right|^2 dxdt \leqslant \frac{a_2}{2a_0}Te^{\frac{a_2}{a_0}T}\left\{\int_0^T\int_0^l |\mathcal{L}_\lambda u|^2\,dxdt + a_1\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dx\right\}.(E_{14})$$

When (E_{14}) is used to bound above the right hand side of (E_{10}), we get : :

$$\int_0^\tau\int_0^l \Psi_\alpha(x)\left[\frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 + \lambda\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2\right] dxdt + \frac{a_0}{2}\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x\partial t}(x,\tau)\right|^2 dx \leqslant$$
$$\frac{1}{2}\int_0^\tau\int_0^l |\mathcal{L}_\lambda u|^2\,dxdt + \frac{a_2}{2a_0}Te^{\frac{a_2}{a_0}T}\left\{\int_0^T\int_0^l |\mathcal{L}_\lambda u|^2\,dxdt + a_1\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dx\right\} + \frac{a_1}{2}\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dx.$$

And, again using inequality $\int_0^\tau .dt \leqslant \int_0^T .dt$, we have :

$$\int_0^\tau\int_0^l \Psi_\alpha(x)\left[\frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 + \lambda\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2\right] dxdt + \frac{a_0}{2}\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x\partial t}(x,\tau)\right|^2 dx \leqslant$$
$$\frac{a_2}{2a_0}Te^{\frac{a_2}{a_0}T}\left\{\int_0^T\int_0^l |\mathcal{L}_\lambda u|^2\,dxdt + a_1\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dx\right\} + \left\{\frac{1}{2}\int_0^\tau\int_0^l |\mathcal{L}_\lambda u|^2\,dxdt + \frac{a_1}{2}\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dx\right\};$$

which leads to

$$\int_0^\tau\int_0^l \Psi_\alpha(x)\left[\frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 + \lambda\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2\right] dxdt + \frac{a_0}{2}\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x\partial t}(x,\tau)\right|^2 dx \leqslant$$
$$\left(\frac{1}{2} + \frac{a_2}{2a_0}Te^{\frac{a_2}{a_0}T}\right)\left\{\int_0^T\int_0^l |\mathcal{L}_\lambda u|^2\,dxdt + a_1\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dx\right\}.(E_{15})$$

Note that the right hand side does not depend on τ. In this way,

$$\sup_{0\leqslant\tau\leqslant T}\left\{\int_0^\tau\int_0^l \Psi_\alpha(x)\left[\frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 + \lambda\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2\right] dxdt + \frac{a_0}{2}\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x\partial t}(x,\tau)\right|^2 dx\right\} \leqslant$$
$$\left(\frac{1}{2} + \frac{a_2}{2a_0}Te^{\frac{a_2}{a_0}T}\right)\left\{\int_0^T\int_0^l |\mathcal{L}_\lambda u|^2\,dxdt + a_1\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dx\right\},$$

which leads to

$$\int_0^\tau\int_0^l \Psi_\alpha(x)\left[\frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 + \lambda\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2\right] dxdt + \frac{a_0}{2}\sup_{0\leqslant\tau\leqslant T}\left\{\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x\partial t}(x,\tau)\right|^2 dx\right\} \leqslant$$
$$\left(\frac{1}{2} + \frac{a_2}{2a_0}Te^{\frac{a_2}{a_0}T}\right)\left\{\int_0^T\int_0^l |\mathcal{L}_\lambda u|^2\,dxdt + a_1\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dx\right\}.(E_{16})$$

From (1.9) the inequality gives :

$$\frac{1}{l-\alpha}\int_0^\tau\int_0^l (l-x)\left[\frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 + \lambda\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2\right] dxdt + \frac{a_0}{2(l-\alpha)}\sup_{0\leqslant\tau\leqslant T}\left\{\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x}(x,\tau)\right|^2 dx\right\} \leqslant$$
$$\left(\frac{1}{2} + \frac{a_2}{2a_0}Te^{\frac{a_2}{a_0}T}\right)\left\{\int_0^T\int_0^l |\mathcal{L}_\lambda u|^2\,dxdt + a_1\int_0^l |\varphi'(x)|^2 dx\right\}.(E_{17})$$

From this we deduce that

$$\min\left(\frac{1}{l-\alpha};\frac{a_0}{2(l-\alpha)}\right)\left\{\int_0^\tau\int_0^l (l-x)\left[\frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2+\lambda\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2\right]dxdt+\sup_{0\leqslant\tau\leqslant T}\left\{\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x}(x,\tau)\right|^2 dx\right\}\right\}\leqslant$$

$$\max(1;a_1)\left[\frac{1}{2}+\frac{a_2}{2a_0}Te^{\frac{a_2}{a_0}T}\right]\left\{\int_0^T\int_0^l |\mathcal{L}_\lambda u|^2\,dxdt+a_1\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2dx\right\},$$

hence

$$||u||_{E_\lambda}\leqslant C||\mathcal{L}_\lambda||_F,\ \text{avec}\ C=\frac{\max(1;a_1)}{\min\left(\frac{1}{l-\alpha};\frac{a_0}{2(l-\alpha)}\right)}\left[\frac{1}{2}+\frac{a_2}{2a_0}Te^{\frac{a_2}{a_0}T}\right].$$

We know that for every function $g\in C[0,l]$ the following equality is true

$$g(l)=\lim_{\alpha\longrightarrow l}\frac{1}{l-\alpha}\int_\alpha^l g(x)dx.$$

Therefore, the problem (3), (4) is the limit when $\alpha\to l$ of all problems (1), (2).
In the present work we establish the a priori estimate for the difference $u_\alpha-u$ and using this estimate, we will prove that if $\alpha\longrightarrow l$ and $\varphi_\alpha\longrightarrow\varphi$, then $u_\alpha\longrightarrow u$.

3 The a priori estimate

Theorem 3.1.
The conditions 1 and 2 are satisfied. Then there exists the constant C independent of u_α, u and α, such that there exists the a priori estimate

$$\int_0^l\int_0^T (l-x)\left[\left|\frac{\partial u_\alpha}{\partial t}-\frac{\partial u}{\partial t}\right|^2+\left|\frac{\partial^2 u_\alpha}{\partial t\partial x}-\frac{\partial^2 u}{\partial t\partial x}\right|^2\right]dxdt+\sup_{0\leqslant t\leqslant T}\int_0^l (l-x)\left|\frac{\partial u_\alpha}{\partial t}-\frac{\partial u}{\partial t}\right|^2dx\leqslant$$

$$\leqslant C\left[\int_0^l \left|\varphi'_\alpha(x)-\varphi'(x)\right|^2dx+\left|\frac{1}{l-\alpha}\int_\alpha^l u(\xi,0)d\xi-h(0)\right|^2+\int_0^T\left|\frac{1}{l-\alpha}\int_\alpha^l u(\xi,t)d\xi-h(t)\right|^2dt+\right.$$

$$\left.+\sup_{0\leqslant t\leqslant T}\left|\frac{1}{l-\alpha}\int_\alpha^l u(\xi,t)d\xi-h(t)\right|^2+\int_0^T\left|\frac{1}{l-\alpha}\int_\alpha^l\frac{\partial u(\xi,t)}{\partial t}d\xi-h'(t)\right|^2dt\right].\tag{18}$$

Proof.
Consider the problem (1), (2) and make the change of the unknown function :

$$u_\alpha(x,t)=v_\alpha(x,t)+\frac{2x}{l-\alpha}h(t)\tag{19}$$

Where $v_\alpha(x,t)$ are solutions to problems

$$\frac{\partial v_\alpha}{\partial t}-\frac{\partial}{\partial x}\left(b(x,t)\frac{\partial^2 v_\alpha}{\partial x\partial t}\right)-\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial v_\alpha}{\partial x}\right)=f(x,t)+\frac{2}{l+\alpha}\frac{\partial b}{\partial x}h'(t)+\frac{2}{l+\alpha}\frac{\partial a}{\partial x}h(t)-\frac{2x}{l+\alpha}h'(t),$$

$$v_\alpha(x,0)=\varphi_\alpha(x)-\frac{2x}{l+\alpha}h(0),\ \frac{1}{l-\alpha}\int_\alpha^l v_\alpha(x,t)dx=0,\ v_\alpha(0,t)=0.$$

In the problem (3) , (4) make the change of the unknown function

$$u=v+\frac{2x}{l^2-\alpha^2}\int_\alpha^l u(\xi,t)d\xi\tag{20}$$

where $v(x,t)$ is the solution of the problem

$$\frac{\partial v}{\partial t}-\frac{\partial}{\partial x}\left(b(x,t)\frac{\partial^2 v}{\partial x\partial t}\right)-\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial v}{\partial x}\right)=f(x,t)+\frac{2}{l^2+\alpha^2}\frac{\partial b}{\partial x}\int_\alpha^l\frac{\partial u(\xi,t)}{\partial t}d\xi+$$

$$+\frac{2}{l^2-\alpha^2}\frac{\partial a}{\partial x}\int_\alpha^l u(\xi,t)d\xi-\frac{2x}{l^2-\alpha^2}\int_\alpha^l\frac{\partial u(\xi,t)}{\partial t}d\xi,$$

$$v(x,0)=\varphi(x)-\frac{2x}{l^2-\alpha^2}\int_\alpha^l u(\xi,0)d\xi$$

$$v(0,t)=0,\ \frac{1}{l-\alpha}\int_\alpha^l v(x,t)dx=0$$

Then the function $w_\alpha=v-v_\alpha$ is solution of the problem

$$\frac{\partial w_\alpha}{\partial t}-\frac{\partial}{\partial x}\left(b(x,t)\frac{\partial^2 w_\alpha}{\partial x\partial t}\right)-\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial w_\alpha}{\partial x}\right)=\mathfrak{F}(x,t)\tag{21}$$

$$w_\alpha(x,0)=\phi_\alpha(x),\quad w_\alpha(0,t)=0,\quad \frac{1}{l-\alpha}\int_\alpha^l w_\alpha(x,t)dx=0\tag{22}$$

where

$$F_\alpha(x,t)=\frac{\partial b}{\partial x}\frac{2}{l-\alpha}\left[\frac{1}{l-\alpha}\int_\alpha^l\frac{\partial u(\xi,t)}{\partial t}d\xi-h'(t)\right]+\frac{\partial a}{\partial x}\frac{2}{l+\alpha}\left[\frac{1}{l-\alpha}\int_\alpha^l u(\xi,t)d\xi-h(t)\right]-$$

$$-\frac{2x}{l+\alpha}\left[\frac{1}{l-\alpha}\int_\alpha^l\frac{\partial u(\xi,t)}{\partial t}d\xi-h'(t)\right], \tag{23}$$

$$\phi_\alpha(x)=\varphi(x)-\varphi_\alpha-\frac{2x}{l+\alpha}\left(\frac{1}{l-\alpha}\int_\alpha^l u(\xi,0)d\xi-h(0)\right). \tag{24}$$

In preliminary for the problem (21), (22), it has been demonstrated that the following a priori estimate

$$\int_\Omega(l-x)\left[\left|\frac{\partial w_\alpha}{\partial t}\right|^2+\left|\frac{\partial^2 w_\alpha}{\partial x\partial t}\right|^2\right]dxdt+\sup_{0\leqslant t\leqslant T}\int_0^l(l-x)\left|\frac{\partial w_\alpha}{\partial x}\right|^2dx\leqslant C_1\left\{\int_0^l\left|\frac{d\phi_\alpha}{dx}\right|^2dx+\int_\Omega|F_\alpha(x,t)|^2\,dxdt\right\}, \tag{25}$$

where the constant C_1 does not depend on $w_\alpha, \quad \phi_\alpha, \quad F_\alpha$.
From Equality
$u-u_\alpha=w_\alpha+\frac{2x}{l+\alpha}\left[\frac{1}{l-\alpha}\int_\alpha^l u(\xi,t)d\xi-h(t)\right]$
it results in inequalities

$$\int_\Omega(l-x)|\frac{\partial u}{\partial t}-\frac{\partial u_\alpha}{\partial t}|^2dxdt\leqslant 2\int_\Omega(l-x)|\frac{\partial w_\alpha}{\partial t}|^2dxdt+\frac{7l^2}{3}\int_0^T|h'(t)-\frac{1}{l-\alpha}\int_\alpha^l\frac{\partial u(\xi,t)}{\partial t}d\xi|^2dt \tag{26}$$

$$\int_\Omega(l-x)\left|\frac{\partial^2 u}{\partial x\partial t}-\frac{\partial^2 u_\alpha}{\partial x\partial t}\right|^2dxdt\leqslant 2\int_\Omega(l-x)\left|\frac{\partial^2 w_\alpha}{\partial x\partial t}\right|^2dxdt+2\int_0^T\left|h'(t)-\frac{1}{l-\alpha}\int_\alpha^l\frac{\partial u(\xi,t}{\partial t}d\xi\right|^2dt \tag{27}$$

$$\sup_{0\leqslant t\leqslant T}\int_0^l(l-x)\left|\frac{\partial u}{\partial x}-\frac{\partial u_\alpha}{\partial x}\right|^2dx\leqslant 2\sup_{0\leqslant t\leqslant T}\int_0^l(l-x)\left|\frac{\partial w_\alpha}{\partial x}\right|^2dx+4\sup_{0\leqslant t\leqslant T}\left|h(t)-\frac{1}{l-\alpha}\int_\alpha^l u(\xi,t)d\xi\right|^2. \tag{28}$$

of both ties (23),and (24),it results in inequalities

$$\int_\Omega|F_\alpha(x,t)|^2\,dx\leqslant\frac{12}{l}a_2\int_0^T\left|\frac{1}{l-\alpha}\int_\alpha^l u(\xi,t)d\xi-h(t)\right|dt+12\left(\frac{b_2}{l}+\frac{l}{3}\right)\int_0^T\left|\frac{1}{l-\alpha}\int_\alpha^l\frac{\partial u(\xi,t)}{\partial\xi}d\xi-h'(t)\right|dt, \tag{29}$$

$$\int_0^l\left|\frac{d\phi_\alpha}{dx}\right|^2dx\leqslant 2\int_0^l\left|\varphi'(x)-\varphi'_\alpha(x)\right|^2dx+\frac{8}{l}\left|\frac{1}{l-\alpha}\int_\alpha^l u(\xi,0)d\xi-h(0)\right|^2. \tag{30}$$

On the basis of the inequalities (26) - (30), the inequality (25) implies the inequality (18) in which
$C=\max\left(\frac{7}{3}l^2+2+6C_1\left(\frac{4b_2}{l}+\frac{4l}{3}\right),8\frac{C_1a_2}{l},4,\frac{16C_1}{l}\right)$.
The theorem 2.1 is thus proved.

4 Continuous dependence of solutions of mixed problems of the form of boundary conditions

Using the estimation (18) for the difference $u_\alpha-u$ of the solutions u_α of the problems (1), (2) with integral conditions and the solution u of the mixed problem (3), (4) with the local condition, we obtain the following result.

Theorem 4.1.
Let the conditions 1 *and* 2.
If

$$\lim_{\alpha\longrightarrow l}\int_0^l\left|\varphi'_\alpha(x)-\varphi'(x)\right|^2dx=0 \tag{31}$$

then

$$\lim_{\alpha\longrightarrow 0}\left\{\int_0^l\int_0^T(l-x)\left[\left|\frac{\partial u_\alpha}{\partial t}-\frac{\partial u}{\partial t}\right|^2+\left|\frac{\partial^2 u_\alpha}{\partial x\partial t}-\frac{\partial^2 u}{\partial x\partial t}\right|^2\right]dxdt+\sup_{0\leqslant t\leqslant T}\int_0^l(l-x)\left|\frac{\partial u_\alpha}{\partial x}-\frac{\partial u}{\partial x}\right|^2dx\right\}=0. \tag{32}$$

Proof.
As $h(t)=u(l,t)$, then

$$\lim_{\alpha\longrightarrow l}\sup_{0\leqslant t\leqslant T}\left|\frac{1}{l-\alpha}\int_\alpha^l u(\xi,t)d\xi-h(t)\right|^2=\lim_{\alpha\longrightarrow l}\sup_{0\leqslant t\leqslant T}\left|\frac{1}{l-\alpha}\int_\alpha^l u(\xi,t)d\xi-u(l,t)\right|^2=0 \tag{33}$$

$$\lim_{\alpha\longrightarrow l}\int_0^T\left|\frac{1}{l-\alpha}\int_\alpha^l\frac{\partial u(\xi,t)}{\partial t}d\xi-h'(t)\right|^2dt=\lim_{\alpha\longrightarrow l}\int_0^T\left|\frac{1}{l-\alpha}\int_\alpha^l\frac{\partial u(\xi,t)}{\partial t}-\frac{\partial u(l,t)}{\partial t}\right|^2dt=0. \tag{34}$$

From the equality (18) and the equations (31), (33), (34), comes the equality (32).
This proves the theorem (3.1). To complete our search, let us show that for every $\varphi \in W_2^1(0,T)$ function satisfying the conditions $\varphi(0)=0$ and $\varphi(l)=h(0)$,there exists $\varphi_\alpha \in W_2^1(0,T)$ the functions such that

$$\varphi_\alpha(0)=0, \quad \frac{1}{l-\alpha}\int_\alpha^l \varphi_\alpha(x)dx = h(0) \tag{35}$$

and the equality (31) is checked.
Let
$\varphi_\alpha(x)=\varphi(x)-\frac{2x}{l+\alpha}\left(\frac{1}{l-\alpha}\int_\alpha^l \varphi(x)dx - h(0)\right)$,
then $\varphi_\alpha \in W_2^1(0,T), \varphi_\alpha(0)=0$,

$$\frac{1}{l-\alpha}\int_\alpha^l \varphi_\alpha(x)dx = \frac{1}{l-\alpha}\int_\alpha^l \varphi(x)dx - \frac{2}{l-\alpha}\int_\alpha^l \frac{2xdx}{l+\alpha}\left(\frac{1}{l-\alpha}\int_\alpha^l \varphi(x)dx - h(0)\right) =$$
$$= \frac{1}{l-\alpha}\int_0^l \varphi(x)dx - \left(\frac{1}{l-\alpha}\int_\alpha^l \varphi(x)dx - h(0)\right) = h(0).$$

And

$$\lim_{\alpha\longrightarrow l}\int_0^l \left|\varphi_\alpha' - \varphi'(x)\right| dx = \lim_{\alpha\longrightarrow l}\frac{4l}{(l+\alpha)^2}\left|\int_0^l \frac{1}{l-\alpha}\varphi(x)dx - h(0)\right|^2 = \frac{4}{l}\left|\varphi(l)-h(0)\right|^2 = 0.$$

In other words, we have just demonstrated the continuous dependence of the solutions of the mixed problems for the pseudo-parabolic equations of the form of the boundary conditions.

Références

[1] A. BOUZIANI . On the third order parabolic with a nonlocal boundary condition. *Journal of applied Mathematics and Stochastic Analysis,13*, pages 181–195, 2000.

[2] A. BOUZIANI . On the solvability of nonlocal pluriparabolic problem. *Electronic journal of differential Equations*, 2006(N 115) :1–18, 2006.

[3] A. BOUZIANI. Mixed problem with an integral conditions. *Journal of applied Mathematics and Stochastic Analysis,9*, pages 323–330, 3 1996.

[4] A. BOUZIANI. On the solvability of a class of singular parabolic equations with nonlocal boundary conditions in non classacal function spaces. *IJMMS.,30*, pages 435–447, 2002.

[5] A. BOUZIANI. On three-point boundary value problem with a weighted integral condition for a class of singular parabolic equations. *Abstract and Applied Analysis,7*, pages 517–530, 10 2002.

[6] A. BOUZIANI. Solution to semilinear pseudoparabolic problem with integral conditions. *Electronic journal of differential Equations*, 9 2006.

[7] A.A. SAMARKII. Some problems in differential equations theory,differents. *Uravn.16*, pages 1925–1935, 11 1980.

[8] A.V. KARTYNNIK. Three point boundary value problem with an integral space variable conditions for second order parabolic equations, differents. *Uravn.26*, pages 1568–1575, 1990.

[9] K. TCHARIE et M.Z. DJIBIBE . Problème mixte pour une équation parabolique linéaire du deuxième ordre avec des conditions aux limites non locales en deux points. *J.Rev.Ivoir.Sci. Technol,09*, pages 153–164, 2007.

[10] K. TCHARIE et M.Z. DJIBIBE. Inégalité énergétique a priori d'un problème mixte pour les équations aux dérivées partielles paraboliques du second ordre avec conditions aux limites non locales en deux points. *J.Rech.Sci.Univ.Lomé(Togo)*, 2006.

[11] K. TCHARIE et M.Z. DJIBIBE. Continuous dependance of solutions to mixed boundary value problem for parabolic equation. *Electronic journal of differential Equations*, 2008(N 17) :1–10, 2008.

[12] M.Z. DJIBIBE. Problèmes mixtes pour les équations parboliques du deuxième ordre avec les conditions aux limites intégrales,locales et non locales en deux points. *Thèse, Doctorat unique dirigé par K.TCHARIE, Université de Lomé, Togo*, 12 2008.

[13] N. GBENOUGA et A. MOROU et M. TODJRO et K. TCHARIE. Problème initial et aux limites avec une condition non locale pour une équation pseudoparabolique de transfert d'humidité de troisième ordre. *Annales Mathématiques Africaines*, 6 :55–64, 2017.

[14] N.I. YURCHUK . Mixed problem with an integral condition for certain parabolic equations, differents. *Uravn.22*, 12 1986.

GBENOUGA N'Gniamessan 06 BP :62337 Lomé Togo polycarpesgbenouga@gmail.com
MOROU Aboudou-Fataou BP : 1515 Lomé Togo mofata1062@yahoo.fr
TODJRO Mawoussi BP :1515 Lomé Togo todjrom7@gmail.com
TCHARIE Kokou BP :1515 Lomé Togo tkokou09@yahoo.fr