Packing dimension for β-shifts

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Abstract. This article is devoted to the study of the packing entropy for maps with g-almost product property, a weak form of specification property. In particular, our result can be applied to the packing dimension for β-shifts.

Keywords and phrases: Packing entropy, Packing dimension, β-shifts.

1 Introduction

(X, d, T) (or (X, T) for short) is a topological dynamical system which means that (X, d) is a compact metric space together with a continuous self-map $T : X \to X$. Denote by $M(X), M(X, T)$ and $E(X, T)$ the sets of all Borel probability measures, T-invariant Borel probability measures, and ergodic measures on X, respectively. It is well known that $M(X)$ and $M(X, T)$ equipped with weak* topology are both convex, compact spaces.

For an T-invariant subset $Z \subset X$, let $M(Z, T)$ denote the subset of $M(X, T)$ for which the measures μ satisfy $\mu(Z) = 1$ and $E(Z, f)$ denote those which are ergodic. For a positive integer n, define the n-th empirical measure $\mathcal{E}_n : X \to M(X)$ by

$$\mathcal{E}_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{T^k x},$$

where δ_x denotes the Dirac measure at x. Let $A(x_n)$ be the set of all limit points of sequence $\{x_n\}$.

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This investigation uses the framework introduced and developed by Olsen [3, 4, 5, 6] and Olsen and Winter [7]. Consider the continuous and affine deformations of $\mathcal{E}_n$ i.e. pairs (Y, Ξ) where Y is a vector space with linear compatible metric and $\Xi : M(X) \to Y$ is a continuous and affine map. Let

$$\Delta_{equ}(C) = \{x \in X | A(\Xi\mathcal{E}_n(x)) = C\}$$

and

$$\Delta_{sup}(C) = \{x \in X | A(\Xi\mathcal{E}_n(x)) \subset C\}.$$

where C is a convex and closed subset of $\Xi(M(X, T))$.

There are some interesting results about the description of the structure (Hausdorff dimension or topological entropy or topological pressure) of $\Delta_{equ}(C)$ and $\Delta_{sup}(C)$. Recently, Zhou, Chen and Cheng [10] studied the packing entropy of $\Delta_{equ}(C)$ and $\Delta_{sup}(C)$ for maps with specification property. Pfister and Sullivan [8] obtained the Bowen entropy in a dynamical system with the g-almost product property which is weaker than specification. Zhou and Chen [9] gave topological pressure for maps with g-almost product property.

Motivated by the work of Zhou, Chen and Cheng (see [9, 10]), we study the packing entropy in a dynamical system with the g-almost product property. In particular, our result can be applied to the packing dimension for β-shifts.

2 Definitions and Main result

Let (X, T) be a topological dynamical system and $C(X)$ the space of continuous functions from X to $\mathbb{R}$. For $\mu, \nu \in M(X)$, define a compatible metric d on $M(X)$ as follows:

$$d(\mu, \nu) := \sum_{i \geq 1} 2^{-i} \left| \int f_i d\mu - \int f_i d\nu \right|$$

where $\{f_i\}_{i=1}^{\infty}$ is the subset of $C(X)$ with $0 \leq f_i(x) \leq 1, i = 1, 2, \cdots$. It is convenient to use an equivalent metric on X, still denoted by $d, d(x, y) := d(\delta_x, \delta_y)$.

For every $\epsilon > 0$, denote by $B_n(x, \epsilon), \overline{B}_n(x, \epsilon)$ the open and closed balls of radius $\epsilon > 0$ in the metric d_n around x respectively, i.e.,

$$B_n(x, \epsilon) = \{y \in X : d_n(x, y) < \epsilon\}, \overline{B}_n(x, \epsilon) = \{y \in X : d_n(x, y) \leq \epsilon\}.$$

Where $n \in \mathbb{N}$, the n-th Bowen metric d_n on X is defined by

$$d_n(x, y) = \max\left\{d(T^k x, T^k y) : k = 0, 1, \cdots, n-1\right\}.$$

2.1 Continuous affine deformation Ξ.

Definition 2.1. [1] *If Y is a vector space and d' is a metric in Y, then d' is linearly compatible if*

(1) For all $x_1, x_2, y_1, y_2 \in Y, d'(x_1+x_2, y_1+y_2) \leq d'(x_1, y_1) + d'(x_2, y_2)$;

(2) For all $x, y \in Y$ and all $\lambda \in \mathbb{R}, d'(\lambda x, \lambda y) \leq |\lambda| d'(x, y)$.

2.2 Packing entropy.

Given $Z \subset X, \epsilon > 0$ and $N \in \mathbb{N}$, let $\mathcal{P}^*(Z, N, \epsilon)$ be the collection of countable or finite sets $\{(x_i, n_i)\} \subset Z \times \{N, N+1, \cdots\}$ such that $\overline{B}_{n_i}(x_i, \epsilon) \bigcap \overline{B}_{n_j}(x_j, \epsilon) = \emptyset, \forall i \neq j$. For each $s \in \mathbb{R}$, consider the set functions

$$\begin{aligned} m^*(Z, s, N, \epsilon) &= \sup_{\mathcal{P}^*(Z,N,\epsilon)} \sum_{(x_i, n_i)} \exp(-n_i s); \\ m^*(Z, s, \epsilon) &= \lim_{N \to \infty} m^*(Z, s, N, \epsilon); \\ m^{**}(Z, s, \epsilon) &= \inf \left\{ \sum_{i=1}^{\infty} m^*(Z_i, s, \epsilon) : \bigcup_{i=1}^{\infty} Z_i \supset Z \right\}. \end{aligned}$$

Both of these functions are non-increasing in s, and the latter takes values ∞ and 0 at all but at most one value of s. Denoting the critical value of s by

$$\begin{aligned} h^P(Z, \epsilon) &= \inf\{s \in \mathbb{R} : m^{**}(Z, s, \epsilon) = 0\} \\ &= \sup\{s \in \mathbb{R} : m^{**}(Z, s, \epsilon) = \infty\}, \end{aligned}$$

leads to $m^{**}(Z, s, \epsilon) = \infty$ when $s < h^P(Z, \epsilon)$, and 0 when $s > h^P(Z, \epsilon)$.

The *packing entropy* of Z is $h^P(Z) := \lim_{\epsilon \to 0} h^P(Z, \epsilon)$. The limit exists because $h^P(Z, \epsilon)$ increases when ϵ decreases.

2.3 g-almost property and uniform separation property.

In this section, we first present some notations to be used in the paper. Then a weak specification property and a weak expansive property are introduced. A remark about the notation is presented here for convenience.

Remark 2.1. Let (X, T) be a topological dynamical system.

(1) If $F \subset M(X)$ is an open set, set $X_{n,F} := \{x \in X : \mathcal{E}_n x \in F\}$.

(2) Given $\delta > 0$ and $\epsilon > 0$, two points x and y are (δ, n, ϵ)-separated if $\#\{i : d(T^i x, T^i y) > \epsilon, 0 \leq i \leq n-1\} \geqslant \delta n$. A subset E is (δ, n, ϵ)-separated if any pair of different points of E are (δ, n, ϵ)-separated.

(3) Let $F \subset M(X)$ be a neighborhood of ν, and $\epsilon > 0$, and set

$N(F; n, \epsilon)$:=maximal cardinality of an (n, ϵ)-separated subset of $X_{n,F}$;

$N(F; \delta, n, \epsilon)$:=maximal cardinality of an (δ, n, ϵ)-separated subset of $X_{n,F}$.

(4) Let $g : \mathbb{N} \to \mathbb{N}$ be a given nondecreasing unbounded map with the properties $g(n) < n$ and $\lim\limits_{n\to\infty} \frac{g(n)}{n} = 0$. The function g is called a blow-up function. Given $x \in X$ and $\epsilon > 0$; let

$$B_n(g; x, \epsilon) := \{y \in X : \exists \Lambda \subset \Lambda_n, \#(\Lambda_n \setminus \Lambda) \leqslant g(n) \text{ and } \max\{d(T^i x, T^i y) : i \in \Lambda\} \leq \epsilon\},$$

where $\Lambda_n = \{0, 1, \cdots, n-1\}$.

Definition 2.2. ([8]) *The dynamical system (X, d, T) has the g-almost product property with blow-up function g if there exists a non-increasing function $m : \mathbb{R}^+ \to \mathbb{N}$ such that for any $k \in \mathbb{N}$, any $x_1 \in X, \cdots, x_k \in X$, any positive $\varepsilon_1, \epsilon_2, \cdots, \varepsilon_k$, and any integers $n_1 \geq m(\varepsilon_1), \cdots, n_k \geq m(\varepsilon_k)$,*

$$\bigcap_{j=1}^{k} T^{-M_{j-1}} B_{n_j}(g; x_j, \epsilon_j) \neq \emptyset,$$

where $M_0 = 0, M_i = n_1 + n_2 + \cdots + n_i, i = 1, 2 \cdots, k-1$.

Definition 2.3. ([8]) *The dynamical system (X, d, T) has the uniform separation property if for any η, there exist $\delta^* > 0$ and $\epsilon^* > 0$ such that for μ ergodic and any neighbourhood $F \subset M(X)$ of μ, there exists $n^*_{F,\mu,\eta}$ such that for $n \geq n^*_{F,\mu,\eta}$,*

$$N(F; \delta^*, n, \epsilon^*) \geq exp(n(h(T, \mu) - \eta)),$$

where $h(T, \mu)$ is the metric entropy of μ.

Proposition 2.1. [8] *Assume that (X, d, T) has the g-almost product property and the uniform separation property. For any η, there exists δ^* and $\epsilon^* > 0$ such that for $\mu \in M(X, T)$ and any neighborhood $F \subset M(X)$ of μ, there exists $n^*_{F,\mu,\eta}$, such that*

$$N(F; \delta^*, n, \epsilon^*) \geq \exp(n(h(T, \mu) - \eta)), \forall n \geqslant n^*_{F,\mu,\eta}.$$

2.4 Statement of main result.

Define

$$\Lambda(y) = \begin{cases} \sup\limits_{\mu \in M(X,T), \Xi\mu = y} h(T, \mu), & y \in \Xi(M(X, T)); \\ -\infty, & \text{otherwise.} \end{cases}$$

The following theorem is the main result of this paper.

Theorem 2.1. *$(X, T, \Xi, \mathcal{E}_n, Y)$ satisfies the g-almost product property and the uniform separation property. If $C \subset Y$ is a convex and closed subset of $\Xi(M(X,T))$, then $\Delta_{equ}(C) \neq \emptyset$ and*

$$h^P(\Delta_{equ}(C)) = h^P(\Delta_{sup}(C)) = \sup_{y \in C} \Lambda(y).$$

3 Proof of Theorem 2.1.

In this section, we are going to prove Theorem 2.1. The upper bound of $h^P(\Delta_{sup}(C))$ holds without extra assumption. From the second part proof of Theorem 1.1 in [10], we have

$$h^P(\Delta_{sup}(C)) \leq \sup_{y \in C} \Lambda(y).$$

Now we prove the lower bound of $h^P(\Delta_{equ}(C))$. We need the following lemma.

Lemma 3.1. ([2]) *Let (X, T) be a topological dynamical systems. If $K \subset X$ is non-empty and compact, then*

$$h^P(T, K) = \sup\{\ \overline{h}_\mu(T) : \mu \in M(X), \mu(K) = 1\}.$$

where

$$\overline{h}_\mu(T) = \int \overline{h}_\mu(T, x) d\mu(x), \overline{h}_\mu(T, x) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu(B_n(x, \varepsilon)).$$

For any $\eta > 0$, there exists sufficiently small $\epsilon > 0$ (see below) and $p \in C$ such that

$$\sup_{q \in C} \Lambda(q) - \eta \leq \Lambda(p).$$

Let $n \in \mathbb{N} \backslash \{0\}$. Since C is compact and connected, it is possible to choose $q_{n,1}, \cdots, q_{n,M_n} \in C$ such that

$$C \subset \bigcup_{i=1}^{M_n} B\left(q_{n,i}, \frac{1}{n}\right),$$

$$|d'(q_{n,i} - q_{n,i+1})| \leq \frac{1}{n} \ \ \forall i, |d'(q_{n,M_n} - q_{n+1,1})| \leq \frac{1}{n},$$

$$q_{n,M_n} = p \ \ \forall n.$$

Let $\{\alpha''_1, \alpha''_2, \alpha''_3, \cdots\} = \{q_{1,1}, q_{1,2}, \cdots, q_{1,M_1}, q_{2,1}, q_{2,2}, \cdots\}$; then for any $n \in \mathbb{N} \setminus \{0\}$,

$$\overline{\{\alpha''_j : j \in \mathbb{N} \setminus \{0\}, j \geq n\}} = C$$

and $\lim\limits_{j \to \infty} d'(\alpha''_j, \alpha''_{j+1}) = 0$.

We will construct a subset $F \subset \Delta_{equ}(C)$ such that for each $x \in F$, $\{\Xi \mathcal{E}_n(x)\}$ has the same limit-point set as the sequence $\{\alpha''_k\}$ and $h^P(F) \geq \sup\limits_{x \in C} \Lambda(x)$.

For $\frac{\eta}{2}$ and $\alpha_k'' \in C$, there exists $\alpha_k \in \Xi^{-1}C \cap M(X,T)$ such that $\Lambda(\alpha_k'') - \frac{\eta}{2} < h(T,\alpha_k)$. By Proposition 2.1, it is easy to see that for $\frac{\eta}{2} > 0$, there exist $\delta^* > 0$ and $\epsilon^* > 0$, such that for any neighborhood $F'' \subset \Xi(M(X))$ of α_k'' (choose $F'' = B(\alpha_k'', \xi_k'')$), there exist $B(\alpha_k, \xi_k) \subseteq \Xi^{-1}F''$ and $n^*_{B(\alpha_k,\xi_k),\alpha_k,\frac{\eta}{2}}$ satisfying

$$N(B(\alpha_k, \xi_k); \delta^*, n, \epsilon^*) \geq \exp\left(n\left(h(T,\alpha_k) - \frac{\eta}{2}\right)\right), \tag{3.1}$$

where $n \geq n^*_{B(\alpha_k,\xi_k),\alpha_k),\frac{\eta}{2}}$ and ξ_k, ξ_k'' will be determined later.

We choose strictly decreasing sequences $\{\xi_k\}_k, \{\xi_k''\}_k$ and $\{\epsilon_k\}_k$ such that $\lim\limits_k \xi_k = 0, \lim\limits_k \xi_k'' = 0$ with $\epsilon_1 < \epsilon^*$. From (3.1), we deduce the existence of n_k and a $(\delta^*, n_k, \epsilon^*)$-separated subset $\Gamma_k \subseteq X_{n_k, B(\alpha_k,\xi_k)} \subseteq X_{n_k, \Xi^{-1}B(\alpha_k'',\xi_k'')}$ with

$$\sharp\Gamma_k \geq \exp\left(n_k\left(h(T,\alpha_k) - \frac{\eta}{2}\right)\right) \geq \exp\left(n_k(\Lambda(\alpha_k'') - \eta)\right).$$

We may assume that n_k satisfies

$$\delta^* n_k > 2g(n_k) + 1, \frac{g(n_k)}{n_k} \leq \epsilon_k.$$

We choose a strictly increasing sequence $\{N_k\}_{k=0}^{\infty}$ with $N_0 = 0$ and $N_k \in \mathbb{N} \setminus \{0\}$ such that

$$n_{k+1} \leqslant \xi_k \sum_{j=1}^{k} n_j N_j$$

and

$$\sum_{j=1}^{k-1} n_j N_j \leq \xi_k \sum_{j=1}^{k} n_j N_j. \tag{3.2}$$

We enumerate the points in the set Γ_k and consider the set $\Gamma_i^{N_i}, i = 1, 2, \cdots,$

Let $\underline{x}_i = (x_1^i, \cdots, x_{N_i}^i) \in \Gamma_i^{N_i}$, for any $(\underline{x}_1, \cdots, \underline{x}_k) \in \Gamma_1^{N_1} \times \cdots \times \Gamma_k^{N_k}$, by g-almost product property, we have

$$B(\underline{x}_1, \cdots, \underline{x}_k) = \bigcap_{i=1}^{k} \bigcap_{j=1}^{N_i} T^{-\sum\limits_{l=0}^{i-1} N_l n_l - (j-1)n_i} B_{n_i}(g; x_j^i, \varepsilon_j)$$

is a non-empty closed set. We define F_k by

$$F_k = \bigcup \left\{B(\underline{x}_1, \cdots, \underline{x}_k) : (\underline{x}_1, \cdots, \underline{x}_k) \in \Gamma_1^{N_1} \times \cdots \times \Gamma_k^{N_k}\right\}.$$

Note that F_k is compact and $F_{k+1} \subseteq F_k$. Define $F = \bigcap_{i=1}^{\infty} F_k$. Let $t_k = \sum_{i=1}^{k} n_i N_i$.

The proof of the following lemma is same as the proof of Lemma 3.2 in [9].

Lemma 3.2. *Let ϵ be such that $4\epsilon = \epsilon^*$, then*

(1) Let $x_i, y_i \in \Gamma_i$ with $x_i \neq y_i$. If $x \in B_{n_i}(g; x_i, \epsilon_i)$ and $y \in B_{n_i}(g; y_i, \epsilon_i)$, then

$$d_{n_i}(x, y) = \max\{d(T^j x, T^j y) : j = 0, 1, \cdots, n_i - 1\} > 2\epsilon.$$

(2) $F \subset \triangle_{equ}(C)$.

For each $(\underline{x}_1, \cdots, \underline{x}_k) \in \Gamma_1^{N_1} \times \cdots \times \Gamma_k^{N_k}$, we choose one point $z = z(\underline{x}_1, \cdots, \underline{x}_k)$ such that $z \in B(\underline{x}_1, \cdots, \underline{x}_k)$. Let L_k be the set of all points constructed in this way. From Lemma 3.2, we have $\sharp L_k = \sharp\Gamma_1^{N_1} \sharp\Gamma_2^{N_2} \cdots \sharp\Gamma_k^{N_k}$. We define for each k, an atomic measure centred on L_k. Precisely, let

$$\nu_k = \sum_{z \in L_k} \delta_z.$$

We normalise ν_k to obtain a sequence of probality measure μ_k, i.e. we let

$$\mu_k = \frac{1}{\sharp L_k} \nu_k.$$

Lemma 3.3. *Suppose μ is a limit point of the sequence of probability measures μ_k, then $\mu(F) = 1$.*

Proof. Suppose $\mu = \lim_{k\to\infty} \mu_{l_k}$ for $l_k \to \infty$. For any fixed l and all $p \geq 0, \mu_{l+p}(F_l) = 1$ since $F_{l+p} \subset F_l$. Thus, $\mu(F_l) \geq \limsup_{k\to\infty} \mu_{l_k}(F_l) = 1$. It follows that $\mu(F) = 1$. □

Lemma 3.4. *Let μ be limit point of the sequence of probability measure μ_k and $\varepsilon = \frac{1}{4}\epsilon^*$. For any $x \in F$ and $\delta > 0$, there exists a increasing sequence $\{l_i\}$ with $\lim\limits_{i\to\infty} l_i = \infty$ such that for sufficiently large i, we have*

$$\mu(B_{l_i}(x, \epsilon)) \leq e^{-l_i}(\overline{s} - \delta),$$

where $\overline{s} = \sup_{x\in C} \Lambda(x) - 2\eta$.

Proof. Choose $l_i = t_{M_1+\cdots+M_i}$. Let $\underline{s} = \inf_{x\in C} \Lambda(x) - \eta$. First we show that

$$\mu_{M_1+\cdots+M_i+p}(B_{l_i}(x, \varepsilon)) \leq \sharp L^{-1}_{M_1+\cdots+M_i}, \forall p \in \mathbb{N} \setminus \{0\}.$$

If $\mu_{M_1+\cdots+M_i+p}(B_{l_i}(x, \varepsilon)) > 0$, then $L_{M_1+\cdots+M_i+p} \cap B_{l_i}(x, \varepsilon) \neq \emptyset$. Let $z = z(\underline{x}, \underline{y}) \in L_{M_1+\cdots+M_i+p} \cap B_{l_i}(x, \varepsilon)$, $z' = z(\underline{x}', \underline{y}') \in L_{M_1+\cdots+M_i+p} \cap B_{l_i}(x, \varepsilon)$, where

$$\underline{x}, \underline{x}' \in \Gamma_1^{N_1} \times \cdots \times \Gamma_{M_1+\cdots+M_i}^{N_{M_1+\cdots+M_i}},$$

$$\underline{y}, \underline{y}' \in \Gamma_{M_1+\cdots+M_i+1}^{N_{M_1+\cdots+M_i+1}}, \times \cdots \times \Gamma_{M_1+\cdots+M_i+p}^{N_{M_1+\cdots+M_i+p}}.$$

Since $d_{l_i}(z,z') \le 2\epsilon$, from Lemma 3.2, we have $\underline{x} = \underline{x}'$. Thus we have

$$\begin{aligned}&\mu_{M_1+\cdots+M_i+p}(B_{l_i}(x,\varepsilon))\\ \le&\frac{1 \times \sharp\Gamma_{M_1+\cdots+M_i+1}^{N_{M_1+\cdots+M_i+1}}, \times \cdots \times \sharp\Gamma_{M_1+\cdots+M_i+p}^{N_{M_1+\cdots+M_i+p}}}{\sharp L_{M_1+\cdots+M_i+p}}\\ =&\frac{1}{\sharp L_{M_1+\cdots+M_i}}.\end{aligned}$$

This leads to

$$\begin{aligned}&\mu(B_{l_i}(x,\varepsilon)) \le \liminf_{k\to\infty} \mu_k(B_{l_i}(x,\varepsilon))\\ =&\frac{1}{\sharp\Gamma_1^{N_1}\sharp\Gamma_2^{N_2}\cdots\sharp\Gamma_{M_1+\cdots+M_i}^{N_{M_1+\cdots+M_i}}}\\ \le&\frac{1}{\exp\{n_1N_1\underline{s} + n_2N_2\underline{s}\cdots + n_{M_1+\cdots+M_i-1}N_{M_1+\cdots+M_i-1}\underline{s} + n_{M_1+\cdots+M_i}N_{M_1+\cdots+M_i}\overline{s}\}}\\ =&\exp\left\{-l_i\left(\frac{n_1N_1+\cdots+n_{M_1+\cdots+M_i-1}N_{M_1+\cdots+M_i-1}}{l_i}\underline{s} + \frac{n_{M_1+\cdots+M_i}N_{M_1+\cdots+M_i}}{l_i}\overline{s}\right)\right\}.\end{aligned}$$

It follow from (3.2),we have

$$\lim_{i\to\infty}\frac{n_1N_1+\cdots+n_{M_1+\cdots+M_i-1}N_{M_1+\cdots+M_i-1}}{l_i} = 0.$$

$$\lim_{i\to\infty}\frac{n_1N_1+\cdots+n_{M_1+\cdots+M_i}N_{M_1+\cdots+M_i}}{l_i} = 1.$$

Thus for sufficiently large i, we have $\mu(B_{l_i}(x,\epsilon) \le e^{-l_i(\overline{s}-\delta)}$. □

Applying Lemma 3.1, we have

$$h^P(F) \geqslant \overline{s} - \delta = \sup_{x\in C}\Lambda(x) - 2\eta - \delta.$$

Since η and δ are arbitrary, we have

$$h^P(\triangle_{equ}\ (C)) \geqslant h^P(C) \ge \sup_{x\in C}\Lambda(x).$$

Thus the proof of Theorem 2.1 is completed.

4 Application

In this section, we apply our result to the packing dimension for β-shift. Let $n = \lceil\beta\rceil$. Let $\beta > 1$ be fixed. For $t \in \mathbb{R}$, we define

$$\lfloor t\rfloor = \max\{i \in \mathbb{Z} : i \le t\}, \lceil t\rceil := \min\{i \in \mathbb{Z} : i \ge t\}.$$

Consider the β-expansion of 1,

$$1 = \sum_{i=1}^{\infty} c_i \beta^{-j},$$

which is given by the algorithm

$$r_0 = 1, c_{i+1} = \lceil \beta r_i \rceil - 1, r_{i+1} = \beta r_i - c_{i+1}, \quad i \in \mathbb{Z}_+.$$

For sequences $\{a_i\}_{i\geq 1}$ and $\{b_i\}_{i\geq 1}$ the lexicographical order is defined by $\{a_i\} < \{b_i\}$ if and only if for the least index i with $a_i \neq b_i$, $a_i < b_i$. The β-shift is the subshift of the full shift on the alphabet with n characters, $A := \{0, 1, \cdots, n-1\}$, which is given by

$$X^\beta = \{\omega = \{\omega_i\}_{i\geq 1} : \omega_i \in A, T^k\{\omega_i\} \leq \{c_i\} \forall k \in \mathbb{Z}_+\},$$

where $T(\omega_1, \omega_2, \omega_3, \cdots) = (\omega_2, \omega_3, \cdots)$. Pfister and Sullivan [8] proved that (X^β, T) satisfies g-almost product property and uniform separation property.

Endow X^β with the metric $d(x, y) = e^{-n}$ for $x = (x_i)_{i=1}^\infty$ and $y = (y_i)_{i=1}^\infty$, where n is the largest integer such that $x_i = y_i, 1 \leq i \leq n$. It is easy to check that for any $Z \subset X^\beta$, $h^P(Z) = \dim_P(Z)$, where $\dim_P(Z)$ denotes the packing dimension of Z. Hence, if C is a closed and convex subset of $\Xi(M(X^\beta, T))$, then

$$\dim_P \left\{x \in X^\beta | A(\Xi L_n x) = C\right\} = \dim_P \left\{x \in X^\beta | A(\Xi L_n x) \subset C\right\} = \sup_{y \in C} \Lambda(y).$$

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