**IMPLICIT HYBRID LINEAR MULTISTEP METHOD FOR SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS**

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**Abstract**

This paper focuses on the development of implicit hybrid method for the general solution of second order ordinary differential equations with initial value problem (IVPs). The method of collocation and interpolation of power series was used to derive the method, while Taylor series is used to develop and analyze the predictorand. The method is found to be consistent and zero-stable. The method shows to be more efficient and accurate when compared with existing work by other authors.

***Key words:*** *Power Series, Taylor’s series, Implicit Method, Interpolation and Collocation.*

1. **Introduction**

Differential equations which are applicable to our daily lives are often encountered in physical sciences, social sciences and engineering. In this article, general second order initial value problems of ordinary differential equation of the form:

 (1)

is considered.

Customarily, second order ordinary differential equations are solved by reducing it to a system of first order ordinary differential equations and then appropriate numerical method for first order is used to solve the resulting system of equations.

This reduction approach has been extensively discussed by several authors such as [refs 1-4]. The approach was very successful but not without some setbacks as discussed in different literature like [refs 4-8]. These set-backs include computer programs associated with the methods which are mostly complicated when incorporating subroutines to supply the starting values for the methods which invariably resulted into more time for the computer and more computational burden. Many authors have developed methods for the direct solution of (1) without reducing it to systems of first order ordinary differential equations. Linear multistep methods (LMM) for the direct solution of (1) have been considered by refs. [9, 1, 4, 10, 11 and 8]. They independently proposed linear multistep methods with continuous coefficients to solve (1) in the predictor-corrector and block mode based on collocation and interpolation method and used Taylor’s series expansion to supply starting values. Hybrid block method for the solution of third order ordinary differential equations was carried out by ref. [12]. Ref. [13] proposed Taylor series approximation method to improve on the setback usually faced with Predictor-Corrector and Block methods. Ref [14] developed continuous hybrid linear multistep method (CHLMM) of one-step for the generalized solution of second order ordinary differential equations. The work extended the results generated to solve second order ordinary differential equation by multistep collocation and interpolation technique using Taylor series for implementation. This method helps to investigate the impact of the interpolation point which on substitution and evaluation obtained the direct integration of (1) without reduction to systems of first order ordinary differential equations.

In this article, we developed a two-point continuous hybrid method of better accuracy to approximate (1) directly with the use of Taylor series for implementation and evaluation.

This article is divided into sections as follows: section 1, is the introduction and background of the study. Section 2 contains the discussion about the methodology involved in the derivation of a two-point continuous hybrid method and its implementation by Taylor series. Section 3, considers the Taylor’s series algorithm for the implementation of the method.

Section 4 focuses on the analyses of the method in terms of order, error constant, stability, consistence and convergence. Section 5 focuses on the application of the new method on some test problems and Section 6 is on the discussion of the result. We tested our method on some application questions of second order IVP ordinary differential equation and compared our result with existing methods.

1. **Derivation of the Method**

In this section, we apply the interpolation and collocation procedures and we choose our interpolation at the first two points of the method and our collocation points at both grid and off-grid points.

We consider power series as an approximate solution to the general second order ordinary differential equation IVP (1) to be of the form:

 (2)

Where  and  are the number of the interpolation and collocation points respectively.

The first and second derivatives of (2) are

 (3)

and

 (4)

Combining equations (1) and (4) generates the differential system

             (5)

Collocating (5) at  and interpolating (2) at  gives a system of non-linear equation of the form

 (6)

where



and



Using Gaussian elimination method to solve for  in (6) gives a continuous multistep method in the form:



Where  . 

Then, using the transformation and  where  and  give a continuous scheme and the coefficients are put as follows:

 (7)

Differentiating (7) we have:

 (8)

Evaluating (7) and (8) at  which implies that  gives

 (9)

with order  Error constant,  or 

The first derivative is given as:

 (10)

1. **Taylor’s Series Algorithm for the implementation of the method**

To generate y values for the approximate solution, the scheme and its first derivative are expanded term by term, up to the order of the scheme, by Taylor series as follows:

 

and



where



and

by partial derivatives are:











where

 , 

 , 



, 















, 







1. **Analysis of the Basic Properties of the Method**

In verifying the accuracy and applicability of our method, we examine the basic properties which include order, error constant, consistency, convergence and zero stability.

* 1. **Order and Error Constant:**

**Definition 1:** According to ref. [1], Linear Multistep Method (9) is said to be of order if  is the largest positive integer for which  but 

Expanding (9) by Taylor series and comparing coefficients of the expansion equating it to zero, we get the values for the method:



Hence, the method is of orderwith principal truncation error 

**4.2 Consistency**

For (9) to be consistent, the following criteria must be met.

Condition 1: 

Condition 2:  where 

Condition 3: when 

Condition 4: when 

where  and  are the first and second characteristic polynomials of (9), applying these conditions to (9), the method was found to be consistent.

**4.3 Zero Stability**

**Definition 2** [see ref. 1]**:** A linear multistep method is said to be zero-stable if no root  has modulus greater than one (that is, if all roots oflie in or on the unit circle). A numerical solution to class of system (1) is stable if the difference between the numerical and the theoretical solutions can be made as small as possible. Hence, (9) is found to be zero-stable since none of the roots has modulus greater than one.

**4.4 Convergence**

**Definition 3**: a linear multistep method of the form (9) is convergent if it is consistent and zero stable. Hence the necessary and sufficient condition for the method (9) to be convergent is that it must both be consistent and zero-stable. Since, these conditions are satisfied, then the method (9) is said to be convergent. [See refs. 1 and 2]

1. **Numerical Experiments**

The accuracy of the method (9) for the direct solution of (1) is tested on some linear and non-linear problems.

**Problem 1:**

****

**Analytical Solution**

****

**Problem 2**

****

**Analytical Solution**



**Problem 3**

The temperature  degrees of a body,  minutes after being placed in a certain room, satisfies the differential equation  By using the substitution  or otherwise, find  in terms of  given that  when  and  when  Find after how many minutes the rate of cooling of the body will have fallen below one degree per minute, giving your answer correct to the nearest minute. How cool does the body get?

Formulating the Problem we have;

**, **

**Analytical Solution**



1. **Results**

**Table 1a: Result from the new method for Test Problem 1**

|  |  |  |  |
| --- | --- | --- | --- |
| *x* | Exact Solution | Computed Solution | Error |
| 0.1  0.2  0.3  0.4  0.5  0.6  0.7  0.8  0.9  1.0 | 0.255432048815786460  0.258162456471933590  0.260902310876373440  0.263651505003892010  0.266409931464443030  0.269177482507343860  0.271954050025483560  0.274739525559546730  0.277533800302249440  0.280336765102589810 | 0.255432048815786570  0.258162456471933200  0.260902310876372720  0.263651505003890950  0.266409931464441200  0.269177482507341140  0.271954050025480120  0.274739525559542170  0.277533800302243670  0.280336765102582490 | 1.110223e-016  3.885781e-016  7.216450e-016  1.054712e-015  1.831868e-015  2.720046e-015  3.441691e-015  4.551914e-015  5.773160e-015  7.327472e-015 |

**Table 1b: Comparison of Errors in our new method with ref [15]**

|  |  |  |
| --- | --- | --- |
| *x* | Error in New Method | Error in [ref 15] |
| 0.1 | 1.110223E-16 | 0.64811445E-07 |
| 0.2 | 3.885781E-16 | 0.80343529E-07 |
| 0.3 | 7.216450E-16 | 0.93317005E-07 |
| 0.4 | 1.054712E-15 | 0.10334724E-06 |
| 0.5 | 1.831868E-15 | 0.11012633E-06 |
| 0.6 | 2.720044E-15 | 0.11342972E-06 |
| 0.7 | 3.441691E-15 | 0.11312237E-06 |
| 0.8 | 4.551914E-15 | 0.10916432E-06 |
| 0.9 | 5.773160E-15 | 0.10161543E-06 |
| 1.0 | 7.327472E-15 | 0.90639024E-07 |

**Table 2a: Result of our New Method for Problem 2**

|  |  |  |  |
| --- | --- | --- | --- |
| *x* | Exact Solution | Computed Solution | Error |
| 0.1  0.2  0.3  0.4  0.5  0.6  0.7  0.8  0.9  1.0 | 1.050041729278491400  1.100335347731075300  1.151140435936466500  1.202732554054081600  1.255412811882994600  1.309519604203111900  1.365443754271397100  1.423648930193603500  1.484700278594054600  1.549306144334058600 | 1.050041729278491400  1.100335347731069100  1.151140435936445000  1.202732554054025900  1.255412811882878500  1.309519604202889800  1.365443754270997600  1.423648930192909200  1.484700278592867400  1.549306144332032000 | 0.000000e+000  6.217249e-015  2.153833e-014  5.573320e-014  1.161293e-013  2.220446e-013  3.994582e-013  6.943335e-013  1.187273e-012  2.026601e-012 |

**Table 2b: Comparison of Errors in our new method with ref [15] and [16]**

|  |  |  |  |
| --- | --- | --- | --- |
| *x* | Error in New Method | Error in [ref 15] | Error in ref [16] |
| 0.1 | 0.000000E+000 | 0.61853700E-08 | 6.4420468E-11 |
| 0.2 | 6.217249E-015 | 0.31695117E-07 | 5.4567017E-10 |
| 0.3 | 2.153833E-014 | 0.75714456E-07 | 1.921674E-09 |
| 0.4 | 5.573320E-014 | 0.14304432E-06 | 4.797029E-09 |
| 0.5 | 1.161293E-013 | 0.24120724E-06 | 9.998000E-09 |
| 0.6 | 2.220446E-013 | 0.38177170E-06 | 1.871478E-08 |
| 0.7 | 3.994582E-013 | 0.58268768E-06 | 3.272868E-08 |
| 0.8 | 6.943335E-013 | 0.87233773E-07 | 5.4792477E-08 |
| 0.9 | 1.187273E-012 | 0.12968951E-07 | 8.929446E-08 |
| 1.0 | 2.026601E-012 | 0.19343897E-06 | 1.4347036E-07 |

**Table 3: Results for Test Problem 3 (Cooling Problem)**

|  |  |  |  |
| --- | --- | --- | --- |
| *x* | Exact Solution | Computed Solution | Error |
| 0.1  0.2  0.3  0.4  0.5  0.6  0.7  0.8  0.9  1.0 | 59.125762679520  58.280186267509  57.462331147625  56.671288507811  55.906179330416  55.166153415412  54.450388435647  53.758089023057  53.088485884845  52.440834948634 | 59.125770155947  58.280215661700  57.462395949280  56.671401298345  55.906351828020  55.166396518144  54.450712262611  53.758502953777  53.088998596888  52.441454453559 | 7.476427E-06  2.939419E-05  6.480165E-05  1.127905E-05  1.724976E-04  2.431027E-04  3.238270E-04  4.139307E-04  5.127120E-04  6.195049E-04 |

1. **Discussion of Results**

Tables 1-3 presented the numerical solutions in terms of the maximum errors obtained for each of the problems considered respectively. The error of the new method is compared with those of predictor-corrector and block method of refs. [15] and [16] respectively.

In table 1, the new method converges faster than that of ref [15] when solving problem 1 with the same order of the method but different approach. This makes the new method to be more efficient than previous method as displayed in global maximum errors obtained for the method in (9). Also, we compared the new method with refs. [15] and [16] in table 2. These authors respectively solved problem 2 with predictor-corrector and Block mode. With our results as displayed in table 2, the error in our new method shows to be more efficient and converges faster. Also the new method was used to solve an engineering (cooling) problem which shows that the body become more cooler and fallen below one degree Celsius () as the step length

() of our method is been reduced which makes the temperature of the body in the room to satisfy our differential equations and the new method developed as it has been demonstrated in the computed result and the error displayed in table 3.

1. **Conclusion**

In this paper, a new method with the use of Taylor’s series for the approximation of  variables has enabled us to compute the derivatives of the method to any possible order which allows direct solution of Initial Value Problems (IVPs) of ordinary differential equations. This new method and approach enable us to compute directly the solution of second order ordinary differentials equations with initial value problems (IVPs) directly without reducing to system of first order. Based on this new approach, it is evident that the new method is considerably more efficient than other numerical methods with the same properties of consistency, zero-stability and convergence.

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