Inverse Eigenvalue Problems for a class of Singular Hermitian Matrices

Dickson Y. B. Annor [1], Kwasi B. Gyamfi [2], Richard K. Boadi [3] & Francis T. Oduro [4]

[1, 2, 3, 4] Department of Mathematics, Kwame Nkrumah University of Science and Technology, Kumasi, GHANA

email: [1] **dbannor1111@yahoo.com**, [2] **kwasibaahgyamfi@yahoo.co.uk** [3] **rkboadi.cos@knust.edu.gh** & [4] **francistoduro@yahoo.co.uk**

Abstract

In this paper, we discuss inverse eigenvalue problems for singular Hermitian matrices. In particular, we investigate how to construct $n \times n$ singular Hermitian matrices of rank 2 and 3 from a given prescribed spectral data. It is found that given the spectrum and the multipliers k_i where $i = 1, 2, \cdots n - r$, the inverse eigenvalue problem for $n \times n$ singular Hermitian matrices of rank r is solvable. Numerical examples are presented in each case.

1 Introduction

An *Inverse Eigenvalue Problem* (IEP) is to reconstruct a matrix which possesses both a prescribed eigenvalue and desired structure. Inverse eigenvalue problems arise in broad application areas such as control design, system identification, principle component analysis, structure analysis etc. There are many different types of inverse eigenvalue problems and despite of a great deal of research effort being put into this topic many of them are still open and are hard to be solved.

In [1] Gyamfi studied the solution to the Inverse Eigenvalue Problems (IEP) for a class of singular symmetric and singular Hermitian matrices. On the case of singular Hermitian matrices he presented results up to rank 1. In this paper, we extend earlier results found by Gyamfi [1] on the solution to the IEP for a class of singular Hermitian matrices of rank 1 to ranks 2 and 3. The paper is organised as follows: In section 2 we review basics on how to reconstruct singular Hermitian matrices of rank 1 from prescribed spectrum. Our main work on the solution to inverse eigenvalue problem for singular Hermitian matrices of ranks 2 and 3 is presented in section 3. We give conclusion and recommendations in the fourth section. Readers are referred to Gyamfi [1] for the details on inverse eigenvalue problem for singular symmetric matrices.

2 Background

In this section we review previous results obtained by Gyamfi [1] in respect of the inverse eigenvalue problems for singular Hermitian matrices of rank 1. We begin with 2×2 singular symmetric matrix of rank 1 and extend it to $n \times n$ singular Hermitian matrices of rank 1.

Lemma 1: Let A be a non-traceless, symmetric matrix of rank r with non-vanishing elements. Then there exits an isomorphism between the elements of A and its distinct non-zero eigenvalues if and only if $r = 1$.

Corollary 2: The inverse eigenvalue problem has a unique solution for singular symmetric matrices of rank 1 with prescribed linear dependence relation.

Specific case 1:

Given $n = 2, r = 1$, we begin by consider $A_{(2,1)}$. By definition, $A_{(2,1)}$ has the form:

$$A_{(2,1)} = \begin{pmatrix} a_{11} & ka_{11} \\ ka_{11} & k^2 a_{11} \end{pmatrix} = a_{11} \begin{pmatrix} 1 & k \\ k & k^2 \end{pmatrix}$$

Let $\Lambda_2 = \{\lambda_1, \lambda_2\}$. Since $A_{(2,1)}$ is singular of rank 1, it means that $\lambda_2 = 0$. We have: $\mathrm{tr}(A_{(2,1)}) = \lambda = a_{11} + k^2 a_{11} = a_{11}(1 + k^2)$. Therefore $a_{11} = \dfrac{\lambda}{1 + k^2}$. Hence

$$A_{(2,1)} = \frac{\lambda}{1 + k^2} \begin{pmatrix} 1 & k \\ k & k^2 \end{pmatrix}.$$

Thus $A_{(2,1)}$ has been reconstructed for a given λ and prescribed scalar k.

We see from this formula that for any given λ and parameter k, we can generate any 2×2 singular matrix of rank 1. For example if $k = 3, \lambda = 10$ we have

$$A_{(2,1)} = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}.$$

2.1 Extension to Hermitian matrices

We now extend the above to Hermitian matrices of dimension 2×2. A is Hermitian implies that $a_{21} = \overline{a}_{12}$. Linear dependence of rows is given by $a_{21} = ka_{11}$ and $a_{22} = ka_{12}$, so that $a_{21} = \overline{a}_{12} = \overline{k}a_{11}$. Then $a_{22} = k(\overline{k}a_{11}) = |k|^2 a_{11}$. We now write the matrix as

$$A_{(2,1)} = \begin{pmatrix} a_{11} & \overline{k}a_{11} \\ ka_{11} & |k|^2 a_{11} \end{pmatrix} = a_{11} \begin{pmatrix} 1 & \overline{k} \\ k & |k|^2 \end{pmatrix}.$$

Hence $\mathrm{tr}(A_{(2,1)}) = \lambda = a_{11} + |k|^2 a_{11} = a_{11}(1 + |k|^2) \Rightarrow a_{11} = \dfrac{\lambda}{1 + |k|^2}$. From this, we see that any 2×2 Hermitian matrix which has a parameter with the same value as the modulus k satisfies the above formula.

Example: Let $k = 1 + 2i, \lambda = 3$ and $\overline{k} = 1 - 2i$. We have $a_{11} = \dfrac{1}{2}$ and

$$A_{(2,1)} = \frac{1}{2} \begin{pmatrix} 1 & 1 - 2i \\ 1 + 2i & 5 \end{pmatrix}.$$

We use numerical example to illustrate small singular Hermitian matrices of size $3 \leq n \leq 4$ of rank 1.

Example: For $n = 3, r = 1, A_{(3,1)}$ is of the form:

$$A_{(3,1)} = a_{11} \begin{pmatrix} 1 & \overline{k}_1 & \overline{k}_1\overline{k}_2 \\ k_1 & |k_1|^2 & |k_1|^2\overline{k}_2 \\ k_1k_2 & |k_1|k_2 & |k_1|^2|k_2|^2 \end{pmatrix}$$

where $a_{11} = \dfrac{\lambda}{1 + |k_1|^2 + |k_1|^2|k_2|^2}$. Any parameter which has the same value as the modulus of k_1 and k_2 generates the 3×3 Hermitian matrix. Suppose $\lambda = 3, k_1 = 2i, \overline{k}_1 = -2i, k_2 = 1 + i, \overline{k}_2 = 1 - i$, we have $a_{11} = \dfrac{3}{13}$ and

$$A_{(3,1)} = \frac{3}{13} \begin{pmatrix} 1 & -2i & -2 - 2i \\ 2i & 4 & 4 - 4i \\ -2 + 2i & 4 + 4i & 8 \end{pmatrix}.$$

Example: For $n = 4, r = 1$. Given k_1, k_2 and k_3 we obtain the following singular Hermitian matrix:

$$A_{(4,1)} = a_{11} \begin{pmatrix} 1 & \overline{k}_1 & \overline{k}_1\overline{k}_2 & \overline{k}_1\overline{k}_2\overline{k}_3 \\ k_1 & |k_1|^2 & |k_1|^2\overline{k}_2 & |k_1|^2\overline{k}_2\overline{k}_3 \\ k_1k_2 & |k_1|k_2 & |k_1|^2|k_2|^2 & |k_1|^2|k_2|^2\overline{k}_3 \\ k_1k_2k_3 & |k_1|^2k_2k_3 & |k_1|^2|k_2|^2k_3 & |k_1|^2|k_2|^2|k_3|^2 \end{pmatrix}.$$

In this case $a_{11} = \frac{\lambda}{1+|k_1|^2+|k_1|^2|k_2|^2+|k_1|^2|k_2|^2|k_3|^2}$. When $\lambda = 2, k_1 = 2i, k_2 = 2+i, k_3 = i$, we have $a_{11} = \frac{2}{45}$ and hence

$$A_{(4,1)} = \frac{2}{45}\begin{pmatrix} 1 & -2i & -2-4i & -4+2i \\ 2i & 4 & 8-4i & -4-8i \\ -2+4i & 8+4i & 40 & -20i \\ -4-2i & -4+8i & 20i & 20 \end{pmatrix}.$$

Proposition 3: If the row dependence relations for a Hermitian or anti-Hermitian matrix of rank 1 are specified as follows $R_i = k_{i-1}R_{i-1}, i = 2, 3, ..., n-1$ where R_i is the ith row and each k_i is a non-zero scalar. The matrix can be generated from its non-zero eigenvalue λ:

$$A = a_{11}\begin{pmatrix} 1 & \overline{k}_1 & \overline{k}_1\overline{k}_2 & \cdots & \overline{k}_1\cdots\overline{k}_{n-1} \\ k_1 & |k_1|^2 & |k_1|^2\overline{k}_2 & \cdots & |k_1|^2\overline{k}_2\cdots\overline{k}_{n-1} \\ k_1k_2 & |k_1|^2k_2 & |k_1|^2|k_2|^2 & \cdots & \cdots \\ \vdots & \vdots & \vdots & & \vdots \\ k_1\cdots k_{n-1} & |k_1|^2k_2\cdots k_{n-1} & |k_1|^2|k_2|^2k_3\cdots k_{n-1} & \cdots & |k_1|^2\cdots|k_{n-1}|^2 \end{pmatrix} \quad (1)$$

where

$$a_{11} = \frac{\lambda}{1+|k_1|^2+|k_1|^2|k_2|^2+\cdots+|k_1|^2\times\cdots\times|k_{n-1}|^2}.$$

3 Main work

We now consider the IEP for $n \times n$ singular Hermitian matrices of rank 2. $A_{(3,2)}$ is of the form:

$$A_{(3,2)} = \begin{pmatrix} a_{11} & \overline{k}a_{11} & \overline{a}_{13} \\ ka_{11} & |k|^2a_{11} & k\overline{a}_{13} \\ a_{13} & \overline{k}a_{13} & a_{33} \end{pmatrix}.$$

Here, $\mathrm{tr}(A_{(3,2)}) = \lambda_1 + \lambda_2 = a_{11} + |k|^2a_{11} + a_{33} = a_{11}(1+|k|^2) + a_{33}$. But $\lambda_1\lambda_2 = a_{11}(1+|k|^2)a_{33} \Rightarrow a_{33} = \dfrac{\lambda_1\lambda_2}{a_{11}(1+|k|^2)}$. Thus

$$\lambda_1 + \lambda_2 = a_{11}(1+|k|^2) + \frac{\lambda_1\lambda_2}{a_{11}(1+|k|^2)},$$

which implies

$$a_{11}^2(1+|k|^2)^2 - a_{11}(1+|k|^2)(\lambda_1+\lambda_2) + \lambda_1\lambda_2 = 0$$

which yields $a_{11} = \dfrac{\lambda_1}{1+|k|^2}$ and $\lambda_2 = a_{33}$. Therefore a_{13} becomes a free variable. When $\lambda_1 = 1, \lambda_2 = 2, k = 3i$ and $a_{13} = 2+i$, for example, we obtain the following singular Hermitian matrix:

$$A_{(3,2)} = \begin{pmatrix} \frac{1}{10} & \frac{-3i}{10} & 2-i \\ \frac{3i}{10} & \frac{9}{10} & 3+6i \\ 2+i & 3-6i & 2 \end{pmatrix}.$$

We illustrate the results for 4×4 singular Hermitian matrices of rank 2. $A_{(4,2)}$ is of the form:

$$A_{(4,2)} = \begin{pmatrix} a_{11} & \overline{k}_1 a_{11} & \overline{k}_1\overline{k}_2 a_{11} & \overline{a}_{14} \\ k_1 a_{11} & |k_1|^2 a_{11} & |k_1|^2 \overline{k}_2 a_{11} & k_1 \overline{a}_{14} \\ k_1 k_2 a_{11} & |k_1|^2 k_2 a_{11} & |k_1|^2 |k_2|^2 a_{11} & k_1 k_2 \overline{a}_{14} \\ a_{14} & \overline{k}_1 a_{14} & \overline{k}_1 \overline{k}_2 a_{14} & a_{44} \end{pmatrix}.$$

Then $\operatorname{tr}(A_{(4,2)}) = \lambda_1 + \lambda_2 = a_{11}(1+|k_1|^2+|k_1|^2|k_2|^2) + a_{44}$. This implies $a_{11} = \dfrac{\lambda_1}{1+|k_1|^2+|k_1|^2|k_2|^2}, \lambda_2 = a_{44}$ and a_{14} becomes a free variable.

Numerical example, for $\lambda_1 = 3, \lambda_2 = 5, k_1 = 2i, k_2 = 1+2i$ and $a_{14} = 1+i$, we obtain a singular Hermitian matrix below:

$$A_{(4,2)} = \begin{pmatrix} \frac{3}{25} & \frac{-6i}{25} & \frac{-12-6i}{25} & 1-i \\ \frac{6i}{25} & \frac{12}{25} & \frac{12-24i}{25} & 2i+2 \\ \frac{-12+6i}{25} & \frac{12+24i}{25} & \frac{60}{25} & -2+6i \\ 1+i & 2-2i & -2-6i & 5 \end{pmatrix}$$

In general, the solution of the IEP for $A_{(n,r)}$ leads to the solution of an rth degree polynomial equation in a_{11} of the form:

$$
\begin{aligned}
0 =& a_{11}^r(1+|k_1|^2+\cdots+|k_{n-r}|^2)^r - (\sum_{i=1}^{r}\lambda_i)(1+|k_1|^2+\cdots+|k_{n-r}|^2)a_{11}^{r-1} \\
&+ \sum_{k=1}^{r}(\prod_{i=k}^{k+1}\lambda_i)(1+|k_1|^2+\cdots+|k_{n-r}|^2)a_{11}^{r-2} \\
&- \sum_{k=1}^{r}(\prod_{i=k}^{k+2}\lambda_i)(1+|k_1|^2+\cdots+|k_{n-r}|^2)a_{11}^{r-3} + \cdots - (\prod_{i=1}^{r}\lambda_i) - - - - - (1).
\end{aligned}
$$

We generalise the method above in the following two theorems, first an $n \times n$ singular Hermitian matrix of rank 2 and then of rank r, where $2 \leq r < n$.

Theorem 4: Given the spectrum and the row multipliers $k_i, i = 1, \cdots, n-2$, the inverse eigenvalue problem for an $n \times n$ singular Hermitian matrix of rank 2 is solvable.

Proof:

Given the spectrum $\Lambda_n = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, since the rank of $\Lambda_2 = 2$, it follows from our notation above that $\lambda_1 \neq 0 \neq \lambda_2$ and $\lambda_i = 0$, for $i = 3, 4, \cdots, n$. Let $k_i, i = 1, 2, \cdots, k_{n-2}$ be row multiples. Letting

$$
A_{(n,2)} = \begin{pmatrix}
a_{11} & \overline{k}_1 a_{11} & \overline{k}_1\overline{k}_2 a_{11} & \cdots & \overline{k}_1\overline{k}_2\cdots\overline{k}_{n-2}a_{11} & \overline{a}_{1n} \\
k_1 a_{11} & |k_1|^2 a_{11} & |k_1|^2\overline{k}_2 a_{11} & \cdots & |k_1|^2\overline{k}_2\cdots\overline{k}_{n-2}a_{11} & k_1\overline{a}_{1n} \\
k_1k_2a_{11} & |k_1|^2k_2a_{11} & |k_1|^2|k_2|^2a_{11} & \cdots & |k_1|^2|k_2|^2k_3\cdots\overline{k}_{n-2}a_{11} & k_1k_2\overline{a}_{1n} \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
a_{1n} & \overline{k}_1 a_{1n} & \overline{k}_1\overline{k}_2 a_{1n} & \cdots & \overline{k}_1\overline{k}_2\cdots\overline{k}_{n-2}a_{1n} & a_{nn}
\end{pmatrix}. \qquad (2)
$$

Then

$$
\operatorname{tr}(A_{(n,2)}) = \lambda_1+\lambda_2 = a_{11}(1+|k_1|^2+|k_1|^2|k_2|^2+\cdots+|k_1|^2\times\cdots\times|k_{n-2}|^2)+a_{nn}.
$$

Hence

$$
a_{11} = \frac{\lambda_1}{1+|k_1|^2+|k_1|^2|k_2|^2+\cdots+|k_1|^2\times\cdots\times|k_{n-2}|^2},
$$

$\lambda_2 = a_{nn}$ and a_{1n} becomes a free variable. The result follows by induction on n.

We consider the Inverse Eigenvalue Problem (IEP) for $n \times n$ singular Hermitian matrices of rank 3. $A_{(4,3)}$ is of the form

$$A_{(4,3)} = \begin{pmatrix} a_{11} & \overline{k}a_{11} & \overline{a}_{13} & \overline{a}_{14} \\ ka_{11} & |k|^2 a_{11} & k\overline{a}_{13} & k\overline{a}_{14} \\ a_{13} & \overline{k}a_{13} & a_{33} & \overline{a}_{34} \\ a_{14} & \overline{k}a_{14} & a_{34} & a_{44} \end{pmatrix} \tag{3}$$

Here, $\text{tr}(A_{(4,3)}) = \lambda_1 + \lambda_2 + \lambda_3 = a_{11}(1+|k|^2) + a_{33} + a_{44}$. Using equation(1), $A_{(4,3)}$ leads to the following cubic equation in a_{11} where λ_1, λ_2 and λ_3 are nonzero members of the spectrum.

$$a_{11}^3(1+|k|^2)^3 - a_{11}^2(1+|k|^2)^2(\lambda_1+\lambda_2+\lambda_3) + a_{11}(1+|k|^2)(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3) - \lambda_1\lambda_2\lambda_3 = 0.$$

Solving the above cubic equation we obtain the following roots, $\lambda_1 = a_{11}(1+|k|^2) \Rightarrow a_{11} = \dfrac{\lambda_1}{1+|k|^2}$, $\lambda_2 = a_{33}$ and $\lambda_3 = a_{44}$, where a_{13}, a_{14} and a_{34} are free variables. For instance, when $\lambda_1 = 2, \lambda_2 = -1, \lambda_3 = 5, k = -i, a_{13} = 2+i, a_{14} = 2i$ and $a_{34} = 1-3i$ we have

$$A_{(4,3)} = \begin{pmatrix} 1 & i & 2-i & -2i \\ -i & 1 & -1-2i & -2 \\ 2+i & -1+2i & -1 & 1+3i \\ 2i & -2 & 1-3i & 5 \end{pmatrix}.$$

Finally, we present 5×5 singular Hermitian matrix of rank 3. Using the same method, we obtain the following equation in a_{11} where λ_1, λ_2 and λ_3 are nonzero members of the spectrum.

$$\begin{aligned} 0 =& a_{11}^3(1+|k_1|^2+|k_1|^2|k_2|^2)^3 - a_{11}^2(1+|k_1|^2+|k_1|^2|k_2|^2)^2(\lambda_1+\lambda_2+\lambda_3) \\ &+ a_{11}(1+|k_1|^2+|k_1|^2|k_2|^2)(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3) - \lambda_1\lambda_2\lambda_3. \end{aligned}$$

Factoring the above equation gives the following results:

$a_{11} = \frac{\lambda_1}{1+|k_1|^2+|k_1|^2|k_2|^2}$, $\lambda_2 = a_{44}$ and $\lambda_3 = a_{55}$. The free variables are a_{14}, a_{15} and a_{45}.

Example, let $\lambda_1 = 13, \lambda_2 = -3, \lambda_3 = 5, k_1 = 2i, k_2 = 1+i, a_{14} = 4, a_{15} = i$ and $a_{54} = 3-i$ we get the following 5×5 singular Hermitian matrix of rank 3:

$$A_{(5,3)} = \begin{pmatrix} 1 & -2i & -2(1+i) & 4 & -i \\ 2i & 4 & 4(1-i) & 8i & 2 \\ -2(1-i) & 4(1+i) & 8 & -8(1-i) & 2(1+i) \\ 4 & -8i & -8(1+i) & -3 & 3+i \\ i & 2 & 2(1-i) & 3-i & 5 \end{pmatrix}.$$

We state the following theorem for the general case where A_n has rank r:

Theorem 5: The inverse eigenvalue problem for an $n \times n$ singular Hermitian matrix of rank r is solvable provided that $n - r$ arbitrary parameters are described.

4 Conclusion and Recommendation

We found in this study that when the eigenvalues and some parameters are given the inverse eigenvalue problem for $n \times n$ singular Hermitian matrices of rank 2 and 3 are solvable. To illustrate the results, numerical examples were provided.

Finally, we recommend that Singular Hermitian matrices of rank ≥ 4 could be studied for future research work.

References

1. Gyamfi, K. B. (2012). Solution of inverse eigenvalue problem of certain singular Hermitian matrices. PhD thesis, Kwame Nkrumah University of Science and Technology.

2. Oduro, F. T. (2014). Solution of the inverse eigenvalue problem for certain (anti-) Hermitian matrices using Newton's method. Retrieved from www.ccsenet.org/journal/index.php/jmr/article/viewFile/27848/20358