MINIMUM ABSOLUTE DEVIATION METHOD OF ESTIMATION OF COBB-DOUGLAS FRONTIER PRODUCTION FUNCTION

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ABSTRACT

This paper contains a number of to measure technical efficiency of Decision Making Units (DMU's). This approach engages the linear programming technique (L.P.P) with parametric and non-parametric production frontiers in easy way. The parametric estimates cannot be subjected to significance tests due to the non-obtainability of standard errors (S.E's).In this Paper we proposed MAD (Minimum Absolute Deviation) method of estimation of Cobb-Douglas frontier production function as a linear programming problem (L.P.P).This method can be stretched in easy way to any parametric frontier production or cost function which is linear in parameters.

I: INTRODUTION :-

Efficiency is critical for organizations that seek to be both environmentally conscious and profitable. Efficiency has implications for a "win-win" situation to arise. Studying and managing organizations from this perspective requires an evaluation of efficiency. To aid researchers and managers develop measures for efficiency we review the use of data envelopment analysis (DEA) for this purpose. DEA theory and application has increased greatly. Its use as a tool for environmental performance evaluation has been limited. In this paper we provide MAD (Minimum Absolute Deviation) method of estimation of Cobb-Douglas frontier production function as a linear programming problem (L.P.P).

II INPUT LEVEL SETS:

$L(u) = \{x : x \text{ produces } u\}$,Where x, u , are input and output vectors respectively.

The input level set $L(u)$ satisfies the following properties.

1. $L(0) = R_{+}^{n}, \quad 0 \notin L(u)$ for $u > 0$
2. $x \in L(u), \quad x' \geq x \Rightarrow x' \in L(u)$
3. $u_2 \geq u_1 \geq 0 \Rightarrow L(u_2) \subseteq L(u_1)$

III FRONTIER PRODUCTION FUNCTION:

Let $\phi(x), x \in R_{+}^{n}$ be a frontier production function. As an optimization problem $\phi(x)$ may be expressed as, $\phi(x) = \text{Max}\{u : x \in L(u)\}, \qquad 0 \leq u < \infty$

$\phi(x)$ succeed properties from $L(u)$

(i) $\phi(0) = 0$, Maximum output produced by a null input vector is zero.

$$\phi(0) = Max\{u : o \in L(u)\} \Rightarrow 0 \in L(u) \Rightarrow u = 0 \Rightarrow \varphi(0) = 0$$

(ii) $x' > x \Rightarrow \phi(x') \geq \phi(x)$, Maximum output produced by a larger input vector is larger.

(iii) $\phi(x)$ is concave function of x

IV INPUT LEVEL SETS INDUCED BY A PRODUCTION FRONTIER $\phi(x)$:

$$L_{\varphi}(u)=\{x:D(u,x)\geq 1\}, \; Where\ D(u,x)=\left[Min\{\lambda:\lambda x\in L(u)\}\right]^{-1}=\frac{\varphi(x)}{u}$$

It can be written $L_{\phi}(u)=L(u)=\left\{x:\frac{\varphi(x)}{u}\geq 1\right\}=\{x:\varphi(x)\geq u\}$

V THE COBB-DOUGLAS PRODUCTION FRONTIER:

The Cobb-Douglas production frontier is given by,

$\hat{y}_i = A\prod_{j=1}^{n} x_{ij}^{\alpha_j}$,It is the i[th] decision making unit

Taking logarithms on both sides,

$$\Rightarrow \ln \hat{y}_i = \ln A + \sum_{j=1}^{n}\alpha_j X_{ij} \quad \Rightarrow \hat{Y}_i = a + \sum_{j=1}^{n} X_{ij}\alpha_j$$

If $\hat{Y}_i \geq Y_i$ then $a+\sum_{j=1}^{n} X_{ij}\alpha_i \geq Y_i$ ….(1.1)

If there are k decision making units(DMUs), then i=1,2,3,……….,k, Introducing slack variables s_i , Then inequation is converted equation.

$$\Rightarrow a+\sum_{j=1}^{n} X_{ij}\alpha_j - s_i = Y_i \Rightarrow \left[a+\sum_{j=1}^{n} X_{ij}\alpha_j - Y_i\right] = s_i$$

Taking summation on both sides

$$\Rightarrow \sum_{i=1}^{k} s_i = ka + \sum_{i=1}^{k}\sum_{j=1}^{n} X_{ij}\alpha_j - \sum_{i=1}^{k} Y_i \qquad \text{….(1.2)}$$

By dividing this equation by k

$$\Rightarrow \bar{s} = a + \sum_{j=1}^{n} \bar{X}_{.j}\alpha_j - \bar{Y}$$

….(1.3)

Minimization of (1.2) is same as minimization of (1.3), $\overline{Y}$ being a constant, minimization of (1.2) is same as minimization of ,

$$\Rightarrow a+\sum_{j=1}^{n}\overline{X}_{.j}\alpha_{j} \qquad \text{....(1.4)}$$

Combining (1.1) and (1.4) we obtain a linear programming problem (L.P.P) for which decision variables are a and α_j .

$$\begin{aligned} &Min \quad a+\sum_{j=1}^{n}\overline{X}_{.j}\alpha_{j} \\ &subject\ to \\ &\quad a+\sum_{j=1}^{n}X_{ij}\alpha_{j}\geq Y_{i} \\ &\quad \alpha_{j}\geq 0, is\ conditional\ for\ sign. \end{aligned} \qquad \text{... (1.5)}$$

Let $a=a^{+}-a^{-}$, the L.P.P can be expressed as follows:

$$\text{Minimize } Z=a^{+}-a^{-}+\sum_{j=1}^{n}\overline{X}.j\alpha_{j}$$

$$\text{subject to} \quad a^{+}-a^{-}+\sum_{j=1}^{n}X_{ij}\alpha_{j}\geq Y_{i} \qquad \text{...(1.6)}$$

$$a^{+},a^{-},\alpha_{j}\geq 0$$

$$i=1,2,3,..........k$$

VI MAD METHOD OF ESTIMATION OF COBB-DOUGLAS PRODUCTION FRONTIER:

With two errors u and v, one sided and the two sided disturbance terms, then the model is given by

$$a^{+} - a^{-} + \sum_{j=1}^{n} X_{ij}\alpha_j = Y_i + u_i + v_i, \ where \quad 0 \le u_i < \infty, \quad -\infty < v_i < \infty$$

For i[th] Decision making unit(DMU)

$$\Rightarrow a^{+} - a^{-} + \sum_{j=1}^{n} X_{ij}\alpha_j - Y_i - u_i = v_i$$

Taking Modules on both sides

$$\Rightarrow \left| a^{+} - a^{-} + \sum_{j=1}^{n} X_{ij}\alpha_j - Y_i - u_i \right| = |v_i| \quad \Rightarrow a^{+} - a^{-} + \sum_{j=1}^{n} X_{ij}\alpha_j - Y_i - u_i = v_i^{+} + v_i^{-}$$

where $\quad v_i = v_i^{+} + v_i^{-}, \ \mathrm{v_i^{+}} = \mathrm{Max}\{0, \mathrm{v_i}\}, \ v_i^{-} = -Min\{0, v_i\}$

The optimization problem is equal to MAD estimation model and it is given by

Min $\sum_{i=1}^{k} \left(v_i^{+} + v_i^{-}\right)$

subject to

$$a^{+} - a^{-} + \sum_{j=1}^{n} X_{ij}\alpha_j - u_i - v_i^{+} - v_i^{-} = Y_i \qquad \ldots(1.7)$$

$$a^{+}, a^{-}, \alpha_j, u_i, v_i^{+}, and \quad v_i^{-} \ge 0$$

i=1,2,3,……….m, j=1,2,3,……….n

The decision variables of the above Linear Programming are $A, \alpha_j, u_i, v_i^{+} and \quad v_i^{-}$ The optimal solution of L.P.P (1.5.1) tells DMU specific technical efficiency.

VII EFFICIENCY ESTIMATION IN COBB-DOUGLAS PRODUCTION FUNCTION USING MODIFIED LEAST SQUARES:

Consider the Cobb-Douglas production function,

$$y = A\prod_{i=1}^{m} x_i^{\alpha_i} u \quad \text{where} \quad 0 \le u \le 1 \qquad \ldots(1.8)$$

Define $u = e^{-z}; \quad 0 < z < \infty$

Let the random variable Z follow Gamma distribution, so that,

$$f(z,\lambda) = \frac{1}{\Gamma(\lambda)} z^{\lambda-1} \exp(-z), \text{ where } \Gamma(\lambda) = \int_0^\infty z^{\lambda-1} e^{-z} dz$$

$$\ln u = -z \quad \Rightarrow -\ln u = z \quad \Rightarrow dz = -\frac{du}{u} \quad \Rightarrow z = \ln\left(\frac{1}{u}\right)$$

$$z = 0 \Rightarrow u = 1, \; z = \infty \Rightarrow u = 0$$

$$= \frac{1}{\Gamma(\lambda)} \left(\ln \frac{1}{u}\right)^{\lambda-1} du$$

The probability density function of u is given by,

$$g(u,\lambda) = \frac{1}{\Gamma(\lambda)} \left(\ln \frac{1}{u}\right)^{\lambda-1} \qquad \ldots (1.9)$$

Here

1. λ is shape parameter of the distribution, $g(u.\lambda)$
2. $\lambda < 1$ implies that a greater proportion of DMUs are efficient
3. $\lambda = 1$ implies uniform efficiency
4. $\lambda > 1$ implies that a greater proportion of DMUs are inefficient

The average level of efficiency of the industry comprised of several DMU is,

$$\bar{u} = E(u) = \int_0^\infty \exp(-z) \frac{1}{\Gamma(\lambda)} z^{\lambda-1} \exp(-z) dz$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty \exp(-2z) z^{\lambda-1} dz \qquad [put\, 2z = v,\; 2dz = dv \;\; dz = \frac{1}{2} dv]$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty \exp(-v) \left(\frac{v}{2}\right)^{\lambda-1} 2^{-1} dv$$

$$= \frac{2^{-\lambda}}{\Gamma(\lambda)} \int_0^\infty \exp(-v)(v)^{\lambda-1} dv$$

$$= \frac{2^{-\lambda}}{\Gamma(\lambda)} \Gamma(\lambda)$$

$$\Rightarrow \overline{u} = 2^{-\lambda}$$

VIII THE METHOD OF MODIFIED LEAST SQUARES:

Consider the Cobb-Douglas production function specification

$$y_i = A \prod_{j=1}^{m} x_{ij}^{\beta j} u_i, \qquad i=1,2,3,.......k$$

$$\ln y_i = \ln A + \sum_{j=1}^{m} \beta_j \ln x_{ij} + \ln u_i$$

$$\Rightarrow Y_i = a + \sum_{j=1}^{m} \beta_j X_{ij} - z_i$$

….(1.10)

We have,

$$E(z_i) = \frac{1}{\Gamma(\lambda)} \int_0^\infty z_i z_i^{\lambda-1} \exp(-z_i) dz_i$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty z_i^{\lambda+1-1} \exp(-z_i) dz_i$$

$$= \frac{1}{\Gamma(\lambda)} \Gamma(\lambda+1)$$

$$= \frac{1}{\Gamma(\lambda)} \Gamma(\lambda+1)$$

$$\mathrm{E}(\mathrm{z_i}) = \lambda, \qquad \forall \lambda$$

$$E\left(z_i^2\right)=\frac{1}{\Gamma(\lambda)}\int_0^\infty z_i^{\lambda+2-1}\exp\left(-z_i\right)dz_i$$

$$=\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)}=\frac{(\lambda+1)\lambda\Gamma(\lambda)}{\Gamma(\lambda)}$$

$$=\lambda(\lambda+1)$$

$$V\left(z_i\right)=E\left(z_i^2\right)-\left[E\left(z_i\right)\right]^2=\lambda(\lambda+1)-\lambda^2$$

$$\mathrm{V}\left(\mathrm{z_i}\right)=\lambda,\ \forall \mathrm{i}$$

We shall assume that Cov $\left(z_j, z_l\right)=0 \qquad j \neq l$

Define $\alpha_0 = a-\lambda,\ \ v_i=\lambda - z_i$ …. (1.11)

Combine (1.7.1) and (1.7.2) to obtain,

$$\hat{y}_i=\alpha_0+\sum_{j=1}^{m}\alpha_j X_{ij}+v_i$$

where $Y_i=\ln y_i$, $X_{ij}=\ln x_{ij}$

Let v_t be an disturbance term that satisfies the following properties:

1. $\mathrm{E}\left(\mathrm{v_i}\right)=0, \qquad \forall \mathrm{i}$
2. $V\left(v_i\right)=E\left(v_i^2\right)=\lambda$
3. $\mathrm{Cov}\left(\mathrm{v_i},\mathrm{v_j}\right)=0, \qquad \mathrm{i}\neq \mathrm{j}$
4. $E\left(v/X\right)=0$

Under above conditions the OLS estimators are BLUE of $\alpha_0,\alpha_1,\ldots\ldots..\alpha_{n.}$. Since λ n is variance of v_i , the OLS estimator of λ is,

$$\hat{n} = \frac{\sum_{i=1}^{k}\left(Y_i - \hat{\alpha}_0 - \sum_{j=1}^{m}\hat{\alpha}_j X_{ij}\right)}{k - m - 1} \quad \text{.... (1.12)}$$

$$E(\hat{\alpha}_0) = \alpha_0 = a - \lambda \quad \text{.... (1.13)}$$

$$E(\hat{\alpha}_i) = \alpha_i, i = 1,2,3......$$

$$E(\hat{\lambda}) = \lambda$$

$$E(\hat{\alpha}_0) = a - E(\hat{\lambda})$$

$$E(\hat{\alpha}_0 + \hat{\lambda}) = a \quad \text{....(1.14)}$$

$\hat{\alpha}_0 + \hat{\lambda}$ is an unbiased estimator of a , $2^{-\hat{\lambda}}$ is a consistent but upward biased estimator of average technical efficiency.

$$\hat{\bar{u}} = 2^{-\hat{\lambda}} \quad \text{....(1.15)}$$

In Gamma distribution, we can estimate the proportion of DMUs with efficiency level at least equal to α

$$P[u \geq d] = P[e^{-z} \geq d] \quad \text{....(1.16)}$$

$$= P[z \leq -\ln d]$$

$$= \int_{0}^{-\ln d} \frac{1}{\Gamma(\lambda)} z^{\lambda-1} \exp(-z) dz \quad \text{....(1.17)}$$

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