Some common fixed point theorems of sequentially compact and complete cone -2- metric spaces of a contractive mapping

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Abstract: In this paper, we prove some common fixed point theorems of sequentially compact and complete cone 2-metric space of a contractive mappings. We have proved four lemmas for the main theorems. Our results improve and generalize some of the results of B Singh etal(15) etc.

Key-words : Fixed points, common fixed points, compact and complete cone 2-metric spaces, ordered Banach space, contractive mapping, Normal cone with normal constant, sequentially compact cone 2-metric space.

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1. Introduction

The notion of 2-metric space was introduced by S. Gahler in series of paper [1], [2],[3] etc. published in the 60's whose abstract properties were suggested by the area function for a triangle determined by a triplet in the Euclidean Space.

L.G. Huang and X. Zhang [10] introduced the concept of cone metric space as a generalization of a metric space on replacing real numbers by an ordered Banach Space and obtained some fixed points of contractive mappings. Using both the two concepts B.Singh etal [15] introduced cone 2-metric space and proved some common fixed point theorems. We now extend and generalize the results of [15] for a pair of self mappings using a contractive mapping.

We need the following definitions for our main theorems.

Definition 1.1 : A non-empty set X with at least three points together with a mapping $d : X \times X \times X \to R^{+}$, is called a 2-metric space if d satisfies.

i) To each pair of points $x \, and \, y, x \neq y$ in X there exists a point $z \in X$ such that $d(x, y, z) \geq 0$.

ii) $d(x, y, z) = 0$ if at least two of $x, y \, and \, z$ are equal.

iii) $d(x, y, z) = d(y, z, x) = d(z, x, y) \forall x, y, z \in X$

iv) $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z \, w \in X$

It is denoted by the ordered pair (X, d). The mapping d is called a 2-metric defined on X

Definition 1.2 [15]: A cone 2-metric space is defined as follows. Let E be a real Banach Space and P a subset of E. P is called a cone of E if

i) P is closed, non-empty and $P \neq \{0\}$

ii) $a, b \in IR^{+}, a, b \geq 0 \, and \, x, y \in P \Rightarrow ax + by \in P$

iii) $x \in P \, and - x \in P \Rightarrow x = 0$

Given a cone $P \subset E$, we define a partial ordering $\leq$ in E with respect to P by $x \leq y$ if and only if $y - x \in P$. A cone P is normal is if for every $x, y \in P$ there exist a number $K > 0$ such that $x \leq y \Rightarrow \|x\| \leq K \|y\|$.

The least of K is the normal constant of the cone P. The symbol $x << y$ is used to denote that $y - x \in Int(P)$, interior of P in E.

A cone P is regular if for every non-decreasing sequence which is bounded above is convergent i.e. if $\{x_n\}$ be a sequence such that $x_1 \le x_2 \le ....\le x_n \le .....\le y \le ....$ for some $y \in P$ then there exists some $x \in P$ such that $x_n \to x \, or \, \|x_n - x\| \to 0 \, as \, n \to \infty$

Equivalently $x_n \to x \, as \, n \to \infty$ if $\{x_n\}$ is a non-increasing sequence which is bounded from below..

Definition 1.3 : Let X be a non-empty set with at least three points. Let $d : XxXxX \to P$ be a mapping such that

i) $d(x,y,z) \ge 0, \forall x,y,z \in X$ and $d(x,y,z) = 0$ if at least two of $x, y \, or \, z$ are equal.

ii) $d(x,y,z) = d(y,z,x) = d(z,x,y), \forall x,y,z \in X$.

iii) $d(x,y,z) \le d(x,y,w) + d(x,w,z) + d(w,x,z)$ for all $x,y,z,w \in X$.

Such a mapping '*d*' is called a cone 2-metric and the pair *(X,d)* is called a cone 2-metric space.

Definition 1.4 : Let (X,d) be a cone 2-metric space with respect to a cone P in a real Banach Space E, Let $\{x_n\}$ be a sequence in X and let $x \in X$. If for every $c \in Int(P)$ with 0<<C there exist a number N(C) such that $d(x_n, x, a) << c \forall a \in X, n \ge N(c)$.

Then $\{x_n\}$ is called a convergent sequence converging to $x. x$ is the limit of the sequence $\{x_n\}$. We denote it by $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Definition 1.5 : Let $\{x_n\}$ be a sequence in a cone 2-metric space (X,d). For each $c \in Int(P)$ there is a number N(c) such $d(x_n, x_m, a) << c$ for all a $a \in X \, and \, n,m \ge N(c)$. Such a sequence $\{x_n\}$ is called a Caucly Sequence.

Definition 1.6 : If every sequence $\{x_n\}$ in a cone 2-metric space has a convergent subsequence then X is said to be a sequentially compact cone 2-metric space. A compact space is a sequentially compact space.

2. Main results.

We prove some results in cone 2-metric spaces.

Lamma : 2.1 : Every regular cone is a normal cone with a normal constant.

Proof : Let $\{x_n\}$ be a non-decreasing sequence which is bounded. Then exists a real number $M \in N$ such that each $x_i \le M, \forall i$ if $\{x_n\} \subset X \, i.e. M - x_i \in Int(P)$ i.e $\|M - x_i\| \to 0 \, as \, i \to \infty$.

i.e $\{x_n\}$ is convergent. Also for some $i \, and \, j, i \le j$

We have,

$$\|x_i\| \le K \|x_j\|, \forall i \, and \, j .$$

Such K has least value and hence P is a normal cone with a normal constant.

If $\{x_n\}$ be a non-increasing sequence and $x_{n+1} \le x_n \forall n \in N$ then $\|x_{n+1}\| \le K\|x_n\|, \forall n$ and hence P is normal cone with a normal constant K.

Lamma : 2.2 : Every convergent sequence $\{x_n\}$ in a cone 2-metric space has a convergent subsequence in *X* or Every subsequence of a convergent sequence in a cone 2-metric space converges.

Proof : Let $\{x_n\}$ be convergent to some $x \in X$. Then for each $c \in Int(P), P$ is normal cone with normal constant K. there exists a number N(c) such that

$$d(x_n, x, a) << c, \forall n \geq N(c)$$

Choosing $c, 0 << c$ such that $K\|c\| < \in, for \in > 0$.

We have

$\|d(x_n, x, a)\| \leq K\|c\| < \in \forall n \geq N(c)$

$\Rightarrow \|d(x_{n(\lambda)}, x, a)\| \leq \in \forall n(\lambda) \geq N(c), a \in X$

$\Rightarrow d(x_{n(\lambda)}, x, a) \to 0 \, as \, \lambda \to \infty$

$\Rightarrow \lim_{\lambda \to \infty} x_{n(\lambda)} = x$

$\therefore \{x_{n(\lambda)}\} \subset \{x_n\}$ converges to $x \, in \, X$.

As $\{x_{n(\lambda)}\}$ is arbitrary, any subsequence of a convergent sequence $\{x_n\}$ convereges.

Lemma 2.3 : Every convergent sequence is Cauchy in a cone 2-metric space but the converse is not true.

Proof : Let $\{x_n\}$ be convergent

There for each $c \in Int(P)$ there exists a number $N(c)$ such that $d(x_n, x, a) << c$ for all $a \in X, n \geq N(c)$, Now for some $m, n \geq N(c)$

$d(x_n, x, a)$

$\leq d(x_n, x, a) + d(x, x_m, a) + d(x_n, x_m, x)$

$< d(x_n, x, a) + d(x, x_m, a) + d(x_n, x_m, x)$

$< \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c$

$i.e. d(x_n, x_m, a) \to 0 \, as \, n, m \to \infty$

$\therefore \{x_n\}$ is a Cauchy sequence.

Conversely if $\{x_n\}$ is Cauchy sequence, then for some $c \in Int(P)$. We have,

$\therefore d(x_n, x_m, a) \leq d(x_n, x_m, x) + d(x_n, x, a) + d(x, x_m, a), \forall \text{ n}, \text{m} \geq \text{N(c)}$

$\therefore 0 < d(x_n, x, a), \forall n \geq N(c)$

$Hence, \lim d(x_n, x, a) \neq 0$

$or \lim_{n \to \infty} x_n \neq x$

$\therefore \{x_n\}$ does not converge to x.

This completes the proof.

Lemma 2.4 : If every subsequence of a sequence $\{x_n\}$ converges to the same limit x in a cone 2-metric space (X, d) then x is the limit of the sequence $\{x_n\}$ in X.

Proof :

Let $\{x_{n(\lambda)}\} \subset \{x_n\}$ be a subsequence such that $x_{n(\lambda)} \to x \, as \, \lambda \to \infty$

Therefore, given $c \in Int(P),$ there exists a number N(c) and $x \in X$ such that $d(x_{n(\lambda)}, x, a) << c$ or equivalently.

$d(x_{n(\lambda)}, x, a) < \frac{\in}{3} \, if \, K\|c\| < \in and \in < 0$ be chosen.

Now,

$$
\begin{aligned}
& d(x_n, x, a) \\
& \leq d(x_n, x, x_{n(\lambda)}) + d(x_n, x_{n(\lambda)}, a) + d(x_{n(\lambda)}, x, a) \\
& \leq \frac{\in}{3} + \frac{\in}{3} + \frac{\in}{3} = \in \\
& i.e, d(x_n, x, a) \to 0 \, as \, n \to \infty \forall a \in X \; \& \, n \geq N(\mathrm{c})
\end{aligned}
$$

Thus,

$$\lim_{n\to\infty} x_n = x, as\, n \to \infty$$

Hence the sequence $\{x_n\}$ also converges to x.

Remark 2.5 [15];[12] : If E be a real Banach space with cone P and if $a \leq \lambda a$ where $a \in P \, and \, 0 < \lambda < 1, then \, a = 0.$

Theorem 2.6 : Let ((X, d) be a sequentially compact cone 2- metric space and let P be a cone with a normal constant K. Suppose the mappings $S, T : X \to X$ satisfy the following for all $x, y, z \in X$:

(2.6.1) T(X) or S(X) is closed subset of X.

(2.6.2) $d(Sx, Ty, z) \leq Kd(x, y, z) + \lambda d(Sx, x, z)$
$+\mu d(Ty, y, z) + \beta \max\{d(Sx, y, z), d(Ty, x, z)\}$

Where $0 \leq K, \lambda, \mu, \beta < 1$

and $K + \lambda + \mu + 2\beta < 1, K + \beta < 1, \mu + \beta < 1$

Then S and T have coincident points which are the unique common fixed points of S and T.

Proof : For $x_0 \in X$, define a sequence $\{x_n\}$ as follows

$x_n = Tx_{n-1} \; if \; n$ is odd

$x_{n+1} = Sx_n$ if n is even.

Now,

$$
\begin{aligned}
d(x_{n+1}, x_n, z) &= d(Sx_n, Tx_{n-1}, \mathrm{z}) \\
&\leq Kd(x_n, x_{n-1}, z) + \lambda d(Sx_n, x_n, Z) + \mu d(Tx_{n-1}, x_{n-1}, Z) \\
&\quad + \beta \max\left\{d(Sx_n, x_{n-1}, Z), d(Tx_{n-1}, x_n, z)\right\} \\
&\leq Kd(x_n, x_{n-1}, z) + \lambda d(x_{n+1}, x_n, Z) + \mu d(x_n, x_{n-1}, z) \\
&\quad + \beta \max\left\{d(x_{n+1}, x_{n-1}, z), d(x_n, x_n, z)\right\} \\
&\leq Kd(x_n, x_{n+1}, z) + \lambda d(x_{n+1}, x_n, Z) + \mu d(x_n, x_{n-1}, z) \\
&\quad + \beta \, d(x_{n+1}, x_{n-1}, z)
\end{aligned}
$$

$$\Rightarrow (1 - \lambda - \beta) d(x_{n+1}, x_n, z) \leq (K + \mu + \beta) d(x_n, x_{n-1}, z)$$

$$\Rightarrow d(x_{n=1}, x_n, z) \leq \left(\frac{K + \mu + \beta}{1 - \lambda - \beta} \right) d(x_n, x_{n-1}, z)$$

$$\Rightarrow d(x_{n=1}, x_n, z) \leq \rho d(x_n, x_{n-1}, z)$$

$$if \;\; \rho = \frac{K + \mu + \beta}{1 - \lambda - \beta} < 1 \; i.e. if \;\; K + \mu + \lambda + 2\beta < 1$$

Also,

$$
\begin{aligned}
\Rightarrow d(x_{n=1}, x_n, z) &\leq \rho d(x_{n-1}, x_{n-2}, z) \\
&\leq \rho^2 d(x_{n-1}, x_{n-2}, z) \leq \ldots
\end{aligned}
$$

$$\leq \rho^n d(x_n, x_0, z)$$

Now for some $m, n \in N, n > m$.

$$d(x_n, x_m, z) \leq d(x_n, x_m, x_{n-1}) + d(x_n, x_{n-1}, z) + d(x_{n-1}, x_m, z)$$
$$\leq \rho^{n-1} d(x_1, x_0, z) + d(x_n, x_{n-1}, z) + d(x_{n-1}, x_m, z)$$
$$\leq (\rho^{n-1} + \rho^{n-2}) d(x_1, x_0, z) + d(x_{n-2}, x_m, z)$$
$$\leq \left(\rho^{n-1}\rho^{n-2} + ... + \rho^m\right) d\left(x_1, x_0, z\right)$$
$$\leq \rho^m \left(1 + \rho + \rho^2 ... + \rho^{n-m-1}\right) d\left(x_1, x_0, z\right)$$
$$\leq \frac{\rho^m}{1-\rho} d\left(x_1, x_0, z\right)$$
$$\Rightarrow \quad \| d\left(x_n, x_m, z\right) \| \leq \frac{\rho^m}{1-\rho} \| d\left(x_1, x_0, z\right) \|$$
$$\Rightarrow \quad \| d\left(x_n, x_m, z\right) \| \to o \text{ as } n, m \to \infty$$ by remark (2.5)
$$\Rightarrow \quad \lim_{n,m\to\infty} d\left(x_n, x_m, z\right) = o$$

Thus, $\{x_n\}$ is a Cauchy sequence in $T(X)$ *or* $S(X)$ which has a convergent subsequence. Let $\{x_{n(k+1)}\} \subset \{x_n\}$ be the convergent subsequence such that $x_{n(k+1)} \to x \text{ as } k \to \infty$

$\therefore \lim_{k\to\infty} x_{n(k+1)} = \lim_{k\to\infty} S\, x_{n(k)} \Rightarrow \quad Sx = x$

To prove $x = Tx$. *If not*, $x \neq Tx$.

Now, $d\left(x, Tx, z\right)$

$$= d\left(Sx, \lim Tx_n, z\right)$$
$$= d\left(Sx, \lim_{n\to\infty} Tx_n, z\right)$$
$$= \lim_{n\to\infty} d\left(Sx, Tx_n, z\right)$$
$$= \lim_{n\to\infty} d\left\{Kd\left(x, x_n, z\right) + \lambda d\left(Sx, x, z\right) + md\left(Tx_n, x_n, z\right) + \beta \max\left(d\left(Sx, x_n, z\right), d\left(Tx_n, x, z\right)\right)\right\}$$
$$\leq \left(Kd\left(x, x, z\right) + \lambda d\left(sx, x, z\right) + md\left(Tx, x, Z\right) + \beta \max\left(d\left(Sx, x, z\right), d\left(Tx, x, z\right)\right)\right)$$

$\therefore \ d\left(x, Tx, z\right) \leq \left(m + \beta\right) d\left(Tx, x, z\right)$

$\Rightarrow T\left(x, Tx, z\right) = 0$ *by remark* $[2.5]$

$\Rightarrow Tx = x$

Thus, $Sx = Tx = x \Rightarrow$ The coincident points of S and T is the common fixed point of the mappings S and T.

We prove that the fixed point is unique .

If possible, let z' be another fixed point of S and T, then $z = z'$.

Now,

$$d\left(z, z', a\right)$$
$$= d\left(Sz, Tz', a\right)$$
$$\leq Kd\left(z, z', a\right) + \lambda d\left(sz, z, a\right) + ud\left(Tz', z', a\right) + \beta \max\left\{d\left(Sz, z', a\right), d\left(Tz, z', a\right)\right\}$$
$$\leq Kd\left(z, z', a\right) + \lambda d\left(z, z, a\right) + ud\left(z', z', a\right) + \beta \max\left\{d\left(z, z', a\right), d\left(z', z', a\right)\right\}$$
$$\leq \left(K + \beta\right) d\left(z, z', a\right)$$
$$i.e, d\left(z, z', a\right) \leq \left(k + \beta\right) d\left(z, z'\, a\right)$$
$$i.e, d\left(z, z', a\right) = o \quad \text{by remark}(2.5)$$
$$\therefore z = z'$$

Thus, the fixed point s of S and T is unique.

Theorem 2.7 : Let (X, d) be a complete cone 2-metric space and let P be a normal cone with normal constant K suppose the mappings $S,T : X \to X$ satisfy the contractive conditions :

$(2.6.1)$ $\quad d(SxTy,z) \le Kd(x,y,z) + \lambda d(Sx,x,z) + \mu d(Ty,y,z) + \beta max\{d(Sx,y,z),d(Ty,x,z)\}$

where K,λ,μ and β are constants and $o \le K,\lambda,u,\beta < 1$ and

$(2.6.2)$ $\quad K + \lambda + u + 2\beta < 1,$ with $K + \beta < 1, \mu + \beta < 1$

Thus S and T have coincident points which are the unique common fixed point of the mappings S and T

Proof : For $x_0 \in X$, We define a sequence $\{x_n\}$ as

$x_n = Tx_{n-1}$ if n is odd.

$x_{n+1} = Sx$ if n is even.

Then continuing as in the proof of Theorem 2.6 are can prove that the sequence $\{x_n\}$ is Cauchy

Since X is complete the sequence $\{x_n\}$ converges to a unique limit i.e. if $z \in X$ such that $x_n \to z$ as $n \to \infty$

Then $Sz = Tz = z,$ Such a fixed point is unique.

The details of the proof are omitted.

Theorem 2.8 : Let (X,d) be a complete cone -2 metric space and Let P be a normal cone with a normal constant K Suppose the mapping $T : X \to X$ satisfies for all $x,y,z \in X : d(Tx,Ty,Z) \le Kd(x,y,z) + \lambda d(Tx,x,z) + \mu d(Ty,y,z) + \beta max\{d(Tx,y,z),d(Ty,x,z)\}$

For some constants $K,\lambda,u\mu \& \beta$ such that $o \le k,\lambda,\mu,\beta < 1$ and

$$K + T + \mu + 2\beta < 1, \lambda + \mu + \beta < 1, K + \beta < 1$$

Then T has a unique fixed point in X.

Proof: For some $x_o \in X$ define a sequence $\{x_n\}$ in X by

$x_1 = Tx_o$

$x_2 = Tx_1 = T^2 x_o ...., x_{n+1} = Tx_n = T^{(n+1)} x_o$

Now,

$$d(x_{n+1},x_n,z) = d(Tx_n,Tx_{n-1},z)$$
$$\le Kd(x_n,x_{n-1},Z) + \lambda d(Tx_n,x_n,z) + \mu d(Tx_{n-1},x_{n-1},z) + \beta max\{d(Tx_n,x_{n-1},z),d(Tx_{n-1},x_n,z)\}$$
$$\Rightarrow d(x_{n+1},x_n,x) \le Kd(x_n,x_{n-1},x) + \lambda d(x_{n+1},x_n,z) + \mu d(x_n,x_{n-1},z) + \beta max\{d(x_{n+1},x_{n-1},Z),d(x_n,xn,z)\}$$
$$\Rightarrow (1 - \lambda - \beta) d(x_{n+1},x_n,z) \le (K + \mu + \beta) d(x_n,x_{n-1},z)$$
$$\Rightarrow d(x_{n+1},x_n,z) \le \rho d(x_n,x_{n-1},x), if\ \rho = \left(\frac{k + u + \beta}{1 - \lambda - \beta}\right) < 1$$

$i.e.\ K + u + \lambda + 2\beta < 1$

$Also, d(x_n,x_{n-1},x) \le \rho d(x_{n-1},x_{n-2},x)$

$Thus,\ d(x_{n+1},x_n,z) \le \rho^n d(x_1,x_o,x)$

For some $m,n \in N$ *and* $n > m$

We can prove that

$$d(x_n, x_m, z) \to o \text{ as } n,m \to \infty$$

$\therefore \{x_n\}$ is a Cauchy sequence in X and hence if converges to some $x \in X$.

To prove $x = Tx = \lim_{n\to\infty} Tx_n$. Assume that $x \neq Tx$.

Now,

$$d(Tx,x,z)$$
$$\leq d(Tx,x,Tx_n) + d(Tx,Tx_n,z) + d(Tx_n,Tx_n,z)$$
$$\leq d(Tx,Tx_n,x) + d(Tx,Tx_n,z) + d(Tx_n,x,z)$$
$$\leq \left[Kd(x,x_n,x) + \lambda d(Tx,x,x) + \mu d(Tx_n,x_n,x) + \beta max\{d(Tx,x_n,x), d(Tx_n,x,x)\} \right]$$
$$+\left[Kd(x,x_n,z) + \lambda d(Tx,x,x) + \mu d(Tx_n,x_n,z) \right]$$
$$+\beta\, max\{d(Tx,x_n,z), d(Tx_n,x,z)\} + d(Tx_n,x,z)$$
$$\leq \mu d(Tx_n,x_n,x) + \beta . d(Tx,x_n,x) + Kd(x,x_n,z)$$
$$+\lambda d(Tx,x,z) + \mu d(Tx_n,x_n,z)$$
$$+\beta\, max\{d(Tx,x_n,z), d(Tx_n,x,z)\} + d(Tx_n,x,z)$$

Letting $n \to \infty$ and lim $x_n = x$, we have

$$d(Tx,x,z) \leq \mu d(Tx,x,z) + \beta d(Tx,x,x) + Kd(x,x,z)$$
$$+\lambda d(Tx,x,z) + \mu d(Tx,x,z) + \beta\, max\{d(Tx,x,z), d(Tx,x,z)\}$$
$$\Rightarrow d(Tx,x,z) \leq (\lambda + \mu + \beta) d(Tx,x,z)$$
$$\Rightarrow d(Tx,x,z) = 0 \, as\, \lambda + \mu + \beta < 1$$
$$\Rightarrow Tx = x$$

i.e. x is as fixed point of T.

This fixed point is unique. Let, if possible, $Ty = y$ for some $y \in X$.

Now,

$$d(x,y,z)$$
$$= d(Tx,Ty,z)$$
$$\leq Kd(x,y,z) + \lambda d(Tx,x,z) + \mu d(Ty,y,z)$$
$$+\beta \max\{d(Tx,y,z), d(Ty,x,z)\} \quad \text{using (2.8.1)}$$
$$\leq Kd(x,y,z) + \lambda d(x,x,z) + \mu d(y,y,z)$$
$$+\beta \max\{d(x,y,z), d(y,x,z)\}$$
$$\leq (K+\beta) d(x,y,z)$$

i.e. $d(x,y,z) \leq (K+\beta) d(x,y,z), K+\beta < 1.$

$\therefore d(x,y,z) = 0 \forall z \in X$ using remark (2.5)

i.e. $x = y$

$\therefore$ The fixed point of T is unique in (X,d).

Corollary Theorem 2.9:

Let (X,d) be a complete cone 2-metric space and P be a normal cone with a normal constant K. Suppose $T: X \to X$ satisfies,

$d(Tx,Ty,c) \leq Kd(x,y,c) + \lambda d(Tx,\mathrm{x},c) + \mu d(Ty,y,c)$ for some $k,\lambda,\mu \in [0,1)$ with $k + \lambda + \mu < 1$, then T has a unique fixed point on X and for every $x \in X$, the sequence $\{T^n x\}$ coverages to the fixed point. This is the main Theorem 2.1 of [15] as a particular case of our theorem 2.7.

Proof: Putting $\beta = 0$ in Theorem 2.8 above, we get a sequence $\{T^n x_0\} \subset X$ which is a Cauchy sequence and hence converges to a point $x \in X$ as X is x_0 -orbitally complete. The uniqueness of such fixed point follows easily.

Example: Let $E = R^2$ and $P = \{(x, y) \in R^2 : x, y \geq 0\}$ be a normal cone in *E*. Let X be $X = \{(x,0) : 0 \leq x \leq 1\} \cup \{(0, y) : 0 \leq y \leq 1\}$. Define a mapping d by $d : X \times X \times X \to E$

$d(a_1, a_2, a_3) = d_1(b_1, b_2), a_i \in X, \forall i = 1,2,3 \; \& \; b_1, b_2 \in \{a_1, a_2, a_3\}$ such that $\|b_1 - b_2\| = \min\{\|a_1 - a_2\|, \|a_2 - a_3\|, \|a_3 - a_1\|\}$ and

$$d_1((x,0),(y,0)) = \left(\frac{3}{2}|x-y|, |x-y|\right),$$

$$d_1((0,x),(0,y)) = \left(|x-y|, \frac{1}{2}|x-y|\right),$$

$$d_1((x,0),(0,y)) = d_1((0,y),(x,0)) = \left(\frac{3}{2}x + y, x + \frac{1}{2}y\right)$$

Then (X, d) is a complete cone 2-metric space. Define the mappings $S, T : X \to X$ by

$$S(x,0) = T(x,0) = (0, \frac{1}{2}x)$$

$$S(0,y) = T(0,y) = (\frac{1}{12}y, 0)$$

then the mappings S and T satisfy the contractive condition

$$d(Sx, Ty, z) \leq kd(x, y, z) + \lambda d(Sx, x, z) + \mu d(Ty, y, z) + \beta \max\{d(Sx, y, z), d(Ty, x, z)\}$$

for all $x, y, z \in X$ with the constants assuming the values $k = \frac{1}{4}, \lambda = \mu = \frac{1}{6}$ and $\beta = \frac{1}{12}$.

The mappings S and T have a coincident point (0,0) which the unique common fixed points. One can check that S and T are not contractive in the 2-metric space on X.

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