THEOREMS ON BEST PROXIMITY POINTS FOR GENERALIZED RATIONAL PROXIMAL CONTRACTIONS

A.ANTONY RAJ[1] J.MARIA JOSEPH[2] M. MARUDAI[3]

Abstract. If the fixed point equation $Tx = x$ does not posses a solution, then the natural interest is to find an element $x \in X$ such that x is in proximity to Tx in some sense. In other words, we would like to get a desirable estimate for the quantity $d(x, Tx)$. In this paper, we prove best proximity point theorems for generalized rational proximal contraction of the first and second kinds. We also prove a best proximity point theorem for nonself mapping for generalized rational proximal contraction of the first and second kinds without assuming the continuity. Our results unify, generalize various known comparable results from the current literature [6,7] .

1. Introduction and preliminaries:

Fixed point theory is an active area of research with wide range of applications in various directions. It is concerned with the results which state that under certain conditions a self map f on a set X admit one or more fixed points. Fixed point theory started almost immediately after the classical analysis began its rapid development. The further growth was motivated mainly by the need to prove existence theorems for differential and integral equations. Thus the fixed point theory started as purely analytical theory. Fixed point theory can be divided into three major areas: Metric fixed point theory, Topological fixed point theory and Discrete fixed point theory. Classical and major results in these areas are: Banach's fixed point theorem, Brouwer's fixed point theorem and Tarski's fixed point theorem. In 1922, the Polish mathematician Stefan Banach formulated and proved a theorem which concerns under appropriate conditions the existence and uniqueness of a fixed point in a complete metric space. His result is known as Banach's fixed point theorem or the Banach's contraction principle. Due to its simplicity and generality, the contraction principle has drawn attention of a very large number of mathematicians. After the period of enormous development of linear functional analysis the time was ripe to focus on nonlinear problems.

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[1]Department of Mathematics, Jeppiar Engineering College,Chennai, India. Email:rajaloyola16@gmail.com .

[2]Department of Mathematics, St.Joseph's college,Tiruchirappalli, Tamil Nadu, India, Email : joseph80_john@yahoo.co.in .

[3]Department of Mathematics, Bharathidasan University,Tiruchirappalli, Tamil Nadu, India, Email: mmarudaim@yahoo.co.in.

Then the role of the analytical fixed point theory became even more important. The study of fixed points for set valued contractions and nonexpansive maps using the Hausdorff metric was initiated by Markin. Later, an interesting and rich fixed point theory for such maps has been developed. Following the Banach's contraction principle Nadler introduced the concept of set valued contractions and established that a set valued contraction possesses a fixed point in a complete metric space. Subsequently many authors generalized Nadler's fixed point theorem in different ways[1,2,5]. A fundamental result in fixed point theory is the Banach's contraction principle. Several extensions of this result have appeared in the literature; see e.g., Kirk [3]. Srinivasan et al. [4] extended the Banach's contraction theorem for a class of mappings satisfying cyclical contractive conditions. However ,the fixed point theorems do not address the issue of non-existence of a solution to the equation $Tx = x$ when the mapping T is not a self-mapping. On the other hand, the best approximation theorems and best proximity point theorems probe into the existence of an approximate solution to the equation $Tx = x$ when T is non-self mapping, in which case a solution does not necessarily exist. The best proximity point theorems ascertain an optimal approximate solution to the fixed point equation $Tx = x$. In fact, given a non-self mapping $T : A \to B$, a best proximity point theorem examines the conditions that guarantee the existence of an element x which is in some sense closest to Tx. In other words, in the setting of metric spaces, a best proximity point theorem identifies an element x for which $d(x, Tx)$ is minimum. In this case, a point such that $d(z, Tz) = dist(A, B)$ called a best proximity point, has been considered. This notion is more general in the sense that if the sets intersect, then every best proximity point is a fixed point

In this section we give some basic definitions and concepts which are useful and related to the context of our results. Define

$$\begin{aligned} P_A(x) &= \{y \in X : d(x, y) = d(x, A)\} \\ d(A, B) &= \inf\{d(x, y) : x \in A, y \in B\} \\ A_0 &= \{x \in A : d(x, y') = d(A, B) \text{ for some } \ y' \in B\} \\ B_0 &= \{y \in B : d(x', y) = d(A, B) \text{ for some } \ \ x' \in A\} \end{aligned}$$

There are some sufficient conditions which guarantee the non emptiness of A_0 and B_0.One such simple condition is that A is compact and B is approximately compact with respect to A(if every sequence $\{x_n\}$ of B such that $d(y, x_n) \to d(y, B)$ for some y in A should have a convergent subsequence)

Definition 1.1. *Let (X, d) be a metric space. Let A and B be two nonempty subsets of X. A mapping $T : A \to B$ is said to be generalized rational proximal contraction of the first kind if there exist $a_i \geq 0, i = 1, 2, ..., 5$ with $a_1 + a_2 + a_3 + 2a_4 + 2a_5 < 1$ such that the conditions $d(u_1, Tx_1) = d(A, B)$ and $d(u_2, Tx_2) = d(A, B)$ imply that*

$$\begin{aligned} d(u_1, u_2) \leq a_1\, d(x_1, x_2) &+ a_2 \frac{\big(1 + d(x_1, u_1)\big)}{1 + d(x_1, x_2)}\, d(x_2, u_2) \\ &+ a_3 \frac{\big(1 + d(x_1, u_1)\big)}{1 + d(x_1, x_2)}\, d(x_1, x_2) + a_4 \big[d(x_1, u_1) + d(x_2, u_2)\big] \end{aligned}$$

$$+ a_5 \big[d(x_1, u_2) + d(x_2, u_1)\big],$$

for all $u_1, u_2, x_1, x_2 \in A$.

Note that, if $a_3 = 0$, we get a rational proximal contraction of the first kind by taking $a_1 = \alpha, a_2 = \beta, a_4 = \gamma, a_5 = \delta, see$[6].

Definition 1.2. *Let* (X, d) *be a metric space. Let* A *and* B *be two nonempty subsets of* X. *A mapping* $T : A \to B$ *is said to be generalized rational proximal contraction of the second kind if there exist* $a_i \geq 0, i = 1, 2, ..., 5$ *with* $a_1+a_2+a_3+2a_4+2a_5 < 1$ *such that the conditions* $d(u_1, Tx_1) = d(A, B)$ *and* $d(u_2, Tx_2) = d(A, B)$ *imply that*

$$d(Tu_1, Tu_2) \leq a_1\, d(Tx_1, Tx_2) + a_2 \frac{\big(1 + d(Tx_1, Tu_1)\big)}{1 + d(Tx_1, Tx_2)}\, d(Tx_2, Tu_2)$$
$$+ a_3 \frac{\big(1 + d(Tx_1, Tu_1)\big)}{1 + d(Tx_1, Tx_2)}\, d(Tx_1, Tx_2) + a_4 \big[d(Tx_1, Tu_1)$$
$$+ \quad d(Tx_2, Tu_2)\big] + a_5 \big[d(Tx_1, Tu_2) + d(Tx_2, Tu_1)\big],$$

for all $u_1, u_2, x_1, x_2 \in A$.

Note that, if $a_3 = 0$, we get a rational proximal contraction of the second kind by taking $a_1 = \alpha, a_2 = \beta, a_4 = \gamma, a_5 = \delta, see$[6].

2. Main Results

Theorem 2.1. *Let* (X, d) *be a complete metric space and* A *and* B *be two non-empty, closed subsets of* X *such that* B *is approximately compact with respect to* A. *Suppose that* A_0 *and* B_0 *are non-empty and* $T : A \to B$ *is a non self-mapping satisfying the following conditions:*
(a) T *is a generalized rational proximal contraction of the first kind,*
(b) $T(A_0) \subseteq B_0$.
Then there exists a unique element x *in* A *such that* $d(x, Tx) = d(A, B)$. *Further, for any* $x_0 \in A_0$, *the sequence* $\{x_n\}$, *defined by* $d(x_{n+1}, Tx_n) = d(A, B)$, *converges to the best proximity point* x.

Proof. Let $x_0 \in A_0$. Since $T(A_0) \subseteq B_0$, then by the definition of B_0, there exists $x_1 \in A_0$ such that

$$d(x_1, Tx_0) = d(A, B)$$

Again, $Tx_1 \in B_0$, it follows that there is $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A, B)$$

Continuing this process, we construct a sequence $\{x_n\}$ in A_0, such that $d(x_{n+1}, Tx_n) = d(A, B)$, for every non-integer n because $T(A_0) \subseteq B_0$.
Also, T is a generalized rational proximal contraction of the first kind, we have

$$d(x_n, x_{n+1}) \leq a_1\, d(x_1, x_2) + a_2 \frac{\big(1 + d(x_1, u_1)\big)}{1 + d(x_1, x_2)}\, d(x_2, u_2)$$
$$+ a_3 \frac{\big(1 + d(x_1, u_1)\big)}{1 + d(x_1, x_2)}\, d(x_1, x_2) + a_4 \big[d(x_1, u_1) + d(x_2, u_2)\big]$$

$$+ a_5 \big[d(x_1,u_2) + d(x_2,u_1)\big]$$

$$\begin{aligned} d(x_n,x_{n+1}) &\le a_1\, d(x_{n-1},x_n) + a_2\, \frac{\big(1+d(x_{n-1},x_n)\big)}{1+d(x_{n-1},x_n)}\, d(x_n,x_{n+1}) \\ &\quad + a_3\, \frac{\big(1+d(x_{n-1},x_n)\big)}{1+d(x_{n-1},x_n)}\, d(x_{n-1},x_n) + a_4 \big[d(x_{n-1},x_n) \\ &\quad + d(x_n,x_{n+1})\big] + a_5 \big[d(x_{n-1},x_{n+1}) + d(x_n,x_n)\big] \\ &\le a_1\, d(x_{n-1},x_n) + a_2\, d(x_n,x_{n+1}) + a_3\, d(x_{n-1},x_n) \\ &\quad + a_4\, d(x_{n-1},x_n) + a_4\, d(x_n,x_{n+1}) + a_5\, d(x_{n-1},x_n) \\ &\quad + a_5 d(x_n,x_{n+1}) \\ &\le \frac{(a_1+a_3+a_4+a_5)}{1-(a_2+a_4+a_5)}\, d(x_{n-1},x_n) \end{aligned}$$

It follows that, $d(x_n,x_{n+1}) \le k\, d(x_{n-1},x_n)$, where, $k = (\frac{a_1+a_3+a_4+a_5}{1-(a_2+a_5+a_5)})$. Similarly, we will show that $d(x_n,x_{n+1}) \le k^2\, d(x_{n-2},x_{n-1})$. By induction, we obtain, $d(x_n,x_{n+1}) \le k^n\, d(x_0,x_1)$. Note that for $m,n \in \mathbb{N}$ such that $m > n$, we have,

$$\begin{aligned} d(x_m,x_n) &\le d(x_m,x_{m-1}) + d(x_{m-1},x_{m-2}) + \cdots + d(x_{n+1},x_n) \\ &\le (k^{m-1} + k^{m-2} + \cdots + k^n)\, d(x_0,x_1) \\ &= k^n(1+k+k^2+k^3+\cdots+k^{m-n-1})\, d(x_0,x_1) \\ &\le k^n \sum_{r=0}^{\infty} k^r\, d(x_0,x_1) \\ &\le \frac{k^n}{1-k}\, d(x_0,x_1) \end{aligned}$$

Since $0 \le k < 1$, then as $n \to \infty$, $k^n(1-k)^{-1} \to 0$ and $d(x_m,x_n) \to 0$ as $m,n \to \infty$

Therefore $\{x_n\}$ is a Cauchy sequence in X. But X is complete and A is closed, the sequence $\{x_n\}$ converges to some $x \in A$. Further, we have,

$$\begin{aligned} d(x,B) &\le d(x,Tx) \\ &\le d(x,x_{n+1}) + d(x_{n+1},Tx_n) \\ &= d(x,x_{n+1}) + d(A,B) \\ &\le d(x_n x_{n+1}) + d(x,B) \end{aligned}$$

Therefore, $d(x,Tx_n) \to d(x,B)$. But B is approximately compact with respect to A, then the sequence $\{Tx_n\}$ has a subsequence $\{Tx_{n_k}\}$ that converges to some $y \in B$. Then, $d(x,y) = \lim_{k\to\infty} d(x_{n_{k+1}},Tx_{n_k}) = d(A,B)$ and hence x must be in A_0. Since $T(A_0) \subseteq B_0$, then, $d(u,Tx) = d(A,B)$, for some $u \in A$. Using the fact that T is generalized rational proximal contraction of the first kind, we obtain,

$$\begin{aligned} d(u,x_{n+1}) &\le a_1\, d(x,x_n) + a_2\, \tfrac{\big(1+d(x,u)\big)}{1+d(x,x_n)}\, d(x_n,x_{n+1}) + a_3\, \tfrac{\big(1+d(x,u)\big)}{1+d(x,x_n)}\, d(x,x_n) \\ &\quad + a_4\big[d(x,u) + d(x_n,x_{n+1})\big] + a_5\big[d(x,x_{n+1}) + d(x_n,u)\big] \end{aligned}$$

. Taking the limit as $n \to \infty$, we get, $d(u,x) \le (a_4+a_5)\, d(x,u)$, which yields $x = u$, since $a_4 + a_5 < 1$. Therefore, $d(x,Tx) = d(u,Tx) = d(A,B)$.

Hence, x is a best proximity point of T.
Uniqueness: Let y be the other best proximity points of T. So that $d(y,Ty) = d(A,B)$. Since T is a generalized rational proximity contraction of the first kind, we have,

$$\begin{aligned} d(x,y) &\leq a_1\, d(x,y) + a_2\, \frac{\big(1+d(x,x)\big)}{1+d(x,y)}\, d(y,y) + a_3\, \frac{\big(1+d(x,x)\big)}{1+d(x,y)}\, d(x,y) \\ &\quad + a_4 \big[d(x,x) + d(y,y)\big] + a_5 \big[d(x,y) + d(x,y)\big] \end{aligned}$$

which yields $d(x,y) < (a_1 + a_3 + 2a_5)\, d(x,y)$. It follows that $x = y$, since $a_1 + a_3 + 2a_5 < 1$.
Hence T has a unique best proximal points. ∎

Theorem 2.2. *Let (X,d) be a complete metric space. Let A and B be two non-empty, closed subsets of X such that A is approximately compact with respect to B. Suppose that A_0 and B_0 are non-empty and $T : A \to B$ is a non self-mapping satisfies the following conditions:*

(a) T is continuous generalized rational proximity contraction of the second kind.
(b) $T(A_0) \subseteq B_0$.

Then there exists an element x in A such that $d(x,Tx) = d(A,B)$ and the sequence $\{x_n\}$, defined by $d(x_{n+1}, x_n) = d(A,B)$ converges to the best proximity point x, where x_0 is any fixed element in A_0 and $d(x_{n+1}, x_n) = d(A,B)$ for $n \geq 0$.

Moreover, if y is another best proximity point of T, then $Tx = Ty$ and hence T is a constant on the set of all best proximity points of T.

Proof. Proceeding as in Theorem 2.1, it is possible to construct a sequence $\{x_n\}$ in A_0 such that $d(x_{n+1}, x_n) = d(A,B)$,for any non-negative integer n. Since T is a generalized rational proximal contraction of the second kind, we obtain

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq a_1\, (Tx_{n-1}, Tx_n) + a_2\, \frac{\big(1 + d(Tx_{n-1}, Tx_n)\big)}{1 + d(Tx_{n-1}, Tx_n)}\, d(Tx_n, Tx_{n+1}) \\ &\quad + a_3\, \frac{\big(1 + d(Tx_{n-1}, Tx_n)\big)}{1 + d(Tx_{n-1}, Tx_n)}\, d(Tx_{n-1}, Tx_n) \\ &\quad + a_4 \big[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})\big] \\ &\quad + a_5 \big[d(Tx_{n-1}, Tx_{n+1}) + d(Tx_n, Tx_n)\big] \\ &\leq a_1\, d(Tx_{n-1}, Tx_n) + a_2\, d(Tx_n, Tx_{n+1}) \\ &\quad + a_3\, d(Tx_{n-1}, Tx_n) + a_4\, d(Tx_{n-1}, Tx_n) \\ &\quad + a_4\, d(Tx_n, Tx_{n+1}) + a_5\, d(Tx_{n-1}, Tx_{n+1}) \\ &\leq a_1\, d(Tx_{n-1}, Tx_n) + a_2\, d(Tx_n, Tx_{n+1}) \\ &\quad + a_3\, d(Tx_{n-1}, Tx_n) + a_4\, d(Tx_{n-1}, Tx_n) \\ &\quad + a_4\, d(Tx_n, Tx_{n+1}) + a_5\, d(Tx_{n-1}, Tx_n) \\ &\quad + a_5\, d(Tx_n, Tx_{n+1}) \end{aligned}$$

It follows that $d(Tx_n, Tx_{n+1}) \leq k\; d(Tx_{n-1}, Tx_n)$, where $k = (\frac{a_1+a_3+a_4+a_5}{1-(a_2+a_4+a_5)})$
Following the same proof of the Theorem 2.1, we can show that $\{Tx_n\}$ is a

Cauchy sequence. Since X is complete, then the sequence $\{Tx_n\}$ converges to some $y \in B$. Moreover, we have,

$$\begin{aligned} d(y,x) &\leq d(y,x_{n+1}) \\ &\leq d(y,Tx_n) + d(Tx_n,x_{n+1}) \\ &= d(y,Tx_n) + d(A,B) \\ &\leq d(y,Tx_n) + d(y,A) \end{aligned}$$

Therefore, $d(y,x_n) \to d(y,A)$. But, A is approximately compact with respect to B, then the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to some $x \in A$. Since T is continuous mapping, $d(x,Tx) = \lim_{k\to\infty} \, d(x_{n_{k+1}},Tx_{n_k}) = d(A,B)$. Hence x is a best proximity point of T.

Uniqueness: Let y be another best proximity point of T so that $d(y,Ty) = d(A,B)$ Using the fact that T is a generalized rational proximal contraction of the second kind, we obtain,

$$\begin{aligned} d(Tx,Ty) \leq a_1\, d(Tx,Ty) &+ a_2\, \frac{\big(1+d(Tx,Tx)\big)}{1+d(Tx,Ty)}\, d(Ty,Ty) \\ &+ a_3\, \frac{\big(1+d(Tx,Tx)}{1+d(Tx,Ty)}\, d(Tx,Ty) + a_4\, \big[d(Tx,Tx) \\ &+ d(Ty,Ty)\big] + a_5\, \big[d(Tx,Ty) + d(Tx,Ty)\big] \end{aligned}$$

which yields $d(Tx,Ty) < (a_1 + a_3 + 2a_5)\, d(Tx,Ty)$,since $a_1 + a_3 + 2a_5 < 1$. Hence, $Tx = Ty$. This completes the proof. ∎

Theorem 2.3. *Let (X,d) be a complete metric space. Let A and B be two non-empty, closed subsets of X. Suppose that A_0 and B_0 are non-empty . Let $T : A \to B$ satisfies the following conditions:*

(a) T is a generalized rational proximal contraction of the first kind and second kind.

(b) $T(A_0) \subseteq B_0$.

Then, there exists a unique element x in A such that $d(x,Tx) = d(A,B)$. Further, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by $d(x_{n+1},Tx_n) = d(A,B)$, converges to x.

Proof. Let $x_0 \in A_0$. Since $T(A_0) \subseteq B_0$, then by the definition of B_0, there exists $x_1 \in A_0$ such that

$$d(x_1,Tx_0) = d(A,B)$$

Again, $Tx_1 \in B_0$, it follows that there is $x_2 \in A_0$ such that

$$d(x_2,Tx_1) = d(A,B)$$

Continuing this process, we construct a sequence $\{x_n\}$ in A_0, such that $d(x_{n+1},Tx_n) = d(A,B)$, for every non-integer n because $T(A_0) \subseteq B_0$. Also, T is a generalized rational proximal contraction of the first kind, we have

$$\begin{aligned} d(x_n,x_{n+1}) \leq a_1\, d(x_1,x_2) &+ a_2\, \frac{\big(1+d(x_1,u_1)\big)}{1+d(x_1,x_2)}\, d(x_2,u_2) \\ &+ a_3\, \frac{\big(1+d(x_1,u_1)\big)}{1+d(x_1,x_2)}\, d(x_1,x_2) + a_4\, \big[d(x_1,u_1) + d(x_2,u_2)\big] \end{aligned}$$

$$
\begin{aligned}
&\quad + a_5 \left[d(x_1, u_2) + d(x_2, u_1)\right] \\
d(x_n, x_{n+1}) &\leq a_1\, d(x_{n-1}, x_n) + a_2 \frac{\big(1 + d(x_{n-1}, x_n)\big)}{1 + d(x_{n-1}, x_n)}\, d(x_n, x_{n+1}) \\
&\quad + a_3 \frac{\big(1 + d(x_{n-1}, x_n)\big)}{1 + d(x_{n-1}, x_n)}\, d(x_{n-1}, x_n) + a_4 \big[d(x_{n-1}, x_n) \\
&\quad + d(x_n, x_{n+1})\big] + a_5 \big[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)\big] \\
&\leq a_1\, d(x_{n-1}, x_n) + a_2\, d(x_n, x_{n+1}) + a_3\, d(x_{n-1}, x_n) \\
&\quad + a_4\, d(x_{n-1}, x_n) + a_4\, d(x_n, x_{n+1}) + a_5\, d(x_{n-1}, x_n) \\
&\quad + a_5 d(x_n, x_{n+1}) \\
&\leq \frac{(a_1 + a_3 + a_4 + a_5)}{1 - (a_2 + a_4 + a_5)}\, d(x_{n-1}, x_n)
\end{aligned}
$$

It follows that $d(x_n, x_{n+1}) \leq k\, d(x_{n-1}, x_n)$, where $k = (\frac{a_1+a_3+a_4+a_5}{1-(a_2+a_5+a_5)})$. Similarly, we will show that, $d(x_n, x_{n+1}) \leq k^2\, d(x_{n-2}, x_{n-1})$. By induction, we obtain, $d(x_n, x_{n+1}) \leq k^n\, d(x_0, x_1)$ Note that for $m, n \in \mathbb{N}$ such that $m > n$, we have,

$$
\begin{aligned}
d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\
&\leq (k^{m-1} + k^{m-2} + \cdots + k^n)\, d(x_0, x_1) \\
&= k^n (1 + k + k^2 + k^3 + \cdots + k^{m-n-1})\, d(x_0, x_1) \\
&\leq k^n \sum_{r=0}^{\infty} k^r\, d(x_0, x_1) \\
&\leq \frac{k^n}{1-k}\, d(x_0, x_1)
\end{aligned}
$$

Since $0 \leq k < 1$, then as $n \to \infty$, $k^n (1-k)^{-1} \to 0$ and $d(x_m, x_n) \to 0$ as $m, n \to \infty$. Therefore, $\{x_n\}$ is a Cauchy sequence in X and hence converges to some element $x \in A$.

$d(x_{n+1}, x_n) = d(A, B)$, for any non-negative integer n. Since T is a generalized rational proximal contraction of the second kind, we obtain,

$$
\begin{aligned}
d(Tx_n, Tx_{n+1}) &\leq a_1\, (Tx_{n-1}, Tx_n) + a_2 \frac{\big(1 + d(Tx_{n-1}, Tx_n)\big)}{1 + d(Tx_{n-1}, Tx_n)}\, d(Tx_n, Tx_{n+1}) \\
&\quad + a_3 \frac{\big(1 + d(Tx_{n-1}, Tx_n)\big)}{1 + d(Tx_{n-1}, Tx_n)}\, d(Tx_{n-1}, Tx_n) \\
&\quad + a_4 \left[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})\right] \\
&\quad + a_5 \left[d(Tx_{n-1}, Tx_{n+1}) + d(Tx_n, Tx_n)\right] \\
&\leq a_1\, d(Tx_{n-1}, Tx_n) + a_2\, d(Tx_n, Tx_{n+1}) \\
&\quad + a_3\, d(Tx_{n-1}, Tx_n) + a_4\, d(Tx_{n-1}, Tx_n) \\
&\quad + a_4\, d(Tx_n, Tx_{n+1}) + a_5\, d(Tx_{n-1}, Tx_{n+1}) \\
&\leq a_1\, d(Tx_{n-1}, Tx_n) + a_2\, d(Tx_n, Tx_{n+1}) \\
&\quad + a_3\, d(Tx_{n-1}, Tx_n) + a_4\, d(Tx_{n-1}, Tx_n) \\
&\quad + a_4\, d(Tx_n, Tx_{n+1}) + a_5\, d(Tx_{n-1}, Tx_n)
\end{aligned}
$$

$$+ a_5\, d(Tx_n, Tx_{n+1}).$$

It follows that $d(Tx_n, Tx_{n+1}) \leq k\, d(Tx_{n-1}, Tx_n)$, where $k = (\frac{a_1+a_3+a_4+a_5}{1-(a_2+a_4+a_5)})$, we can show that $\{Tx_n\}$ is a Cauchy sequence. Since X is complete, then the sequence $\{Tx_n\}$ converges to some $y \in B$. Therefore, $d(x, y) = \lim_{n\to\infty} d(x_{n_1}, Tx_n) = d(A, B)$. Clearly, x must be in A_0. Since $T(A_0) \subseteq B_0$, then, $d(u, Tx) = d(A, B)$,for some $u \in A$. Since T is a generalized rational proximal contraction of the first kind, we obtain,

$$\begin{aligned} d(u, x_{n+1}) \leq a_1\, d(x, x_n) + a_2\, &\frac{\big(1 + d(x, u)\big)}{1 + d(x, x_n)}\, d(x_n, x_{n+1}) \\ &+ a_3\, \frac{\big(1 + d(x, u)\big)}{1 + d(x, x_n)}\, d(x, x_n) + a_4\, \big[d(x, u) + d(x_n, x_{n+1})\big] \\ &+ a_5\, \big[d(x, x_{n+1}) + d(x_n, u)\big] \end{aligned}$$

Taking the limit as $n \to \infty$, we have, $d(u, x) \leq (a_4 + a_5)\, d(x, u)$ which yields $x = u$, since $a_4+a_5 < 1$. Thus, it follows that $d(x, Tx) = d(u, Tx) = d(A, B)$.
Uniqueness: Let y be the other best proximity points of T. So that $d(y, Ty) = d(A, B)$ Since T is a generalized rational proximity contraction of the first kind, we have,

$$\begin{aligned} d(x, y) \leq a_1\, d(x, y) + a_2\, &\frac{\big(1 + d(x, x)\big)}{1 + d(x, y)}\, d(y, y) \\ &+ a_3\, \frac{\big(1 + d(x, x)\big)}{1 + d(x, y)}\, d(x, y) + a_4\, \big[d(x, x) + d(y, y)\big] \\ &+ a_5\, \big[d(x, y) + d(x, y)\big] \end{aligned}$$

which yields $d(x, y) < (a_1 + a_3 + 2a_5)\, d(x, y)$. It follows that $x = y$, since $a_1 + a_3 + 2a_5 < 1$.
Hence T has a unique best proximal points.This completes the proof. ∎

Example 2.4. *Let (X, d) be complete metric space.*
Let $X = \mathbb{R}$ endowed with usual metric $d(x, y) = |x - y|$, for all $x, y \in X$. Let $A = [-2, 2]$ and $B = [-4, -3] \bigcup [3, 4]$
Define $T : A \to B$ by

$$\begin{aligned} Tx = &\quad 3 \quad if \quad x \in \mathbb{Q} \\ = &\quad 4 \quad if \quad x \notin \mathbb{Q} \end{aligned}$$

Indeed, $d(A, B) = 1$, $A_0 = \{-2, 2\}$, $B_0 = \{-3, 3\}$ and $T(A_0) \subseteq B_0$.
Hence, T is a generalized rational proximal contraction of the first and second kinds. All the hypothesis of Theorem 2.3 are satisfies and $d\big(2, T(2)\big) = d(A, B)$.

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