

Study of Partitioned Operators and Its Applications

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Abstract

In this paper, we find new properties of Partitioned matrices, and some new relations between Positive semidefinite matrices and Hermitian block, and we give necessary and sufficient conditions for a partitioned operator matrix to have the Drazin inverse with Banachiewicz–Schur.

Mathematics Subject Classification:

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1 Introduction

The concepts of partitioned matrices and Block matrices are ubiquitous in physics and applied mathematics, appearing naturally in the description of systems with multiple discrete variables (e.g., quantum spin, quark color and flavor).

The need to calculate determinants and the multiplication of the higher matrices is very time consuming and sometime impossible since the memory in computer to store the matrices are very large.

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Let H be a Hilbert space. the set of all pounded linear operator on H denoted by $B(H)$, Every operator $M \in B(H)$

can be written in a block-form $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, Let. If there

exists an operator $K^D \in B(H)$ such that

$$KK^D = K^D K, \quad K^D K K^D = K^D \quad K^{k+1} K^D = K^k$$

Where k the index of K , then K^D is called drazin inverse.

Let M_n^+ denote the set of positive and semidefinite $n \times n$ complex matrix and M be any positive block-matrices; that is ,
 $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in M_{n+m}^+$ where $A \in M_n^+$ and $B \in M_m^+$, a positive block-matrix can be partitioned into a small number of Hermitian blocks by a decomposition lemmas.

2 Preliminary Notes

Definition 2.1 [1]. (Block matrix)

In mathematics, a block matrix or a partitioned matrix is a matrix which is interpreted as having been broken into sections called blocks or submatrices.

A partitioned $m \times n$ matrix is an $m \times n$ matrix $A = \{a_{ij}\}$ that has been reexpressed in the general form

$$A_{ij} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1c} \\ A_{21} & A_{22} & \cdots & A_{2c} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rc} \end{bmatrix}$$

Here A_{ij} is an $m_i \times n_i$ matrix ($i = 1, \dots, r$, $j = 1, \dots, c$), where m_1, \dots, m_r and n_1, \dots, n_c are positive integers such that

$$m_1 + m_2 + \dots + m_r = m \text{ and } n_1 + n_2 + \dots + n_c = n$$

Theorem 2. 1 [3]. let

$$B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1v} \\ B_{21} & B_{22} & \cdots & B_{2v} \\ \vdots & \vdots & \ddots & \vdots \\ B_{u1} & B_{u2} & \cdots & B_{uv} \end{bmatrix}$$

represent a partitioned $p \times q$ matrix whose ij th block B_{ij} is of dimensions $p_i \times q_j$.

B partitioned in the same way as those of A , Then

$$AB = \begin{bmatrix} F_{11} & F_{12} & \cdots & F_{1v} \\ F_{21} & F_{22} & \cdots & F_{2v} \\ \vdots & \vdots & \ddots & \vdots \\ F_{r1} & F_{r2} & \cdots & F_{rv} \end{bmatrix}$$

where $F_{ij} = \sum_{k=1}^c A_{ik} B_{kj} = A_{i1} B_{1j} + \dots + A_{ic} B_{cj}$.

Definition 2.2 [3].

$$A_{ij} = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{rc} \end{bmatrix}.$$

Then A is called a **block-diagonal matrix**, and sometimes

$\text{diag}(A_{11}, A_{22}, \dots, A_{rr})$ is written for A .

Definition 2.3 [3].

If $A_{ij} = 0$ for $j < i = 1, \dots, r$, that is if

, then A is called a **lower block- $A = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rc} \end{bmatrix}$**

triangular matrix.

Theorem 2.2 [2]. (Matrix Inversion in Block form)

Let a $m \times n$ matrix M be partitioned into a block form:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \text{ where the } n \times n \text{ matrix } A \text{ and } D \text{ are invertible}$$

$$\text{,Then } M^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}$$

Theorem 2.3 [5]. If

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \text{ where } A, B, C, D \text{ are } n \times n \text{ matrices over } F,$$

$$\text{and } D \text{ is invertible, then } \det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - BD^{-1}CD)$$

Lemma 2.1 [5].

Let S be a complex block matrix of the form

$$S = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1N} \\ S_{21} & S_{22} & \cdots & S_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ S_{N1} & S_{N2} & \cdots & S_{NN} \end{bmatrix} \quad (1)$$

and let us define the set of block matrices $\{\alpha^{(0)}, \alpha^{(1)}, \dots, \alpha^{(N-1)}\}$,

where $\alpha^{(k)}$ is an $(N-k) \times (N-k)$ block matrix with blocks

$$\alpha_{ij}^{(0)} = S_{ij}$$

$$\alpha_{ij}^{(k+1)} = \alpha_j^{(k)}i - \alpha_{i, N-k}^{(k)} \left(\alpha_{N-k, N-k}^{(k)} \right)^{-1} \alpha_{N-k, j}^{(k)}, \quad k \geq 1. \quad (2)$$

Then the determinants of consecutive $\alpha^{(k)}$ are related via

$$\det \left(\alpha^{(k)} \right) = \det \left(\alpha^{(k+1)} \right) \det \left(\alpha_{N-k, N-k}^{(k)} \right)$$

Theorem 2.4 [5]. Given a complex block matrix of the form (1),

and the matrices $\alpha_{ij}^{(k)}$ defined in Eq. (2), the determinant of S is

$$\text{given by} \quad \det(S) = \prod_{k=1}^N \det \left(\alpha_{kk}^{(N-k)} \right).$$

Lemma 2.2 [7].

There exist a nonsingular matrix Q such that $J = Q^{-1}AQ$.

Where,

$Q = [v_s \ v_{s-1} \ \cdots \ v_1]$, and v_i is an eigenvector corresponding to eigenvalue λ_i .

Lemma 2.3 [4].

For every matrix in M_{n+m}^+ partitioned into blocks, we have a decomposition

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} = U \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} V^*$$

for some unitaries $U, V \in M_{n+m}^+$.

Corollary 2.1 [4].

For every matrix in M_{2n}^+ written in blocks of the same size, we have the decomposition:

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} = U \begin{bmatrix} \frac{A+B}{2} + I(X) & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} - I(X) \end{bmatrix} V^*$$

for some unitaries $U, V \in M_{n+m}$.

Theorem 2.5 [3]. Let

$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and $S = (A - BD^{-1}C)$ be the generalized Schur

complement of A in M . Then,

$$M^D = \begin{bmatrix} (A - BD^D C)^{-1} & -(A - BD^D C)^D B D^D \\ -S^D C A^D & S^D \end{bmatrix} \quad (3.3)$$

if and only if

$$(I - A A^D) B S^D = A^D B (I - S^D S), \quad (I - S S^D) C A^D = S^D C (I - A^D)$$

and

$$\begin{bmatrix} A(I - A^D A) & (I - A A^D) B \\ (I - S S^D) C I - A^D A & S(I - S^D S) \end{bmatrix} \text{ is a nilpotent operator}$$

3 Main Results

Theorem 3.1. *Let*

$$B = \begin{bmatrix} B_{11} & 0 & \cdots & 0 \\ B_{21} & B_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_{c1} & B_{c2} & \cdots & B_{cc} \end{bmatrix}$$

represent an $n \times n$ lower block-triangular matrix whose ij th block B_{ij} is of dimensions $n_i \times n_j$ ($j \geq i = 1, \dots, c$). Then B is lower triangular if and only if each of its diagonal blocks $B_{11}, B_{22}, \dots, B_{cc}$ is lower triangular

proof :

let B be a lower triangular matrix then

Diagonal of $B = \text{dig}(B_{11}, B_{22}, \dots, B_{cc})$ and B_{ij} can be written as

$$B_{ij} = \begin{bmatrix} b_{n_1+\dots+n_{i-1}+1, n_1+\dots+n_{j-1}+1} & \cdots & b_{n_1+\dots+n_{i-1}, n_1+\dots+n_j} \\ \vdots & \vdots & \vdots \\ b_{n_1+\dots+n_i+1, n_1+\dots+n_{j-1}+1} & \cdots & b_{n_1+\dots+n_i, n_1+\dots+n_j} \end{bmatrix} \cdot \text{where } b_{ij} \in B$$

since B lower triangular matrix,

thus, $b_{n_1+\dots+n_{i-1}+1, n_1+\dots+n_{i-1}+1}, \dots, b_{n_1+\dots+n_i, n_1+\dots+n_i}$ ($i = 1, \dots, c$) represent

diagonal of B , Thus, $B_{11}, B_{22}, \dots, B_{cc}$ will be written as

$$B_{ii} = \begin{bmatrix} b_{n_1+\dots+n_{i-1}+1, n_1+\dots+n_{i-1}+1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ b_{n_1+\dots+n_i+1, n_1+\dots+n_{i-1}+1} & \cdots & b_{n_1+\dots+n_i, n_1+\dots+n_i} \end{bmatrix}, \text{ where } i = 1, \dots, c$$

Then $B_{11}, B_{22}, \dots, B_{cc}$ is lower triangular.

Now, let $B_{11}, B_{22}, \dots, B_{cc}$ be lower triangular, thus, B_{ii} can be written as above. Then B is lower triangular. \square

Theorem 3.2: Let

$$A = \begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix} \text{ where } A_{11} \text{ is } k \times k \text{ and } A_{22} \text{ is } (n-k) \times (n-k).$$

If A_{11} and A_{22} are both invertible, then

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & O \\ O & A_{22}^{-1} \end{bmatrix}.$$

Proof:

Since A_{11} and A_{22} are both invertible, then

$$\begin{bmatrix} A_{11}^{-1} & O \\ O & A_{22}^{-1} \end{bmatrix} \begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11}^{-1}A_{11} & O \\ O & A_{22}^{-1}A_{22} \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix} = I$$

and

$$\begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix} \begin{bmatrix} A_{11}^{-1} & O \\ O & A_{22}^{-1} \end{bmatrix} = \begin{bmatrix} A_{11}A_{11}^{-1} & O \\ O & A_{22}A_{22}^{-1} \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix} = I.$$

Thus A is invertible and $A^{-1} = \begin{bmatrix} A_{11}^{-1} & O \\ O & A_{22}^{-1} \end{bmatrix}$. \square

Theorem 3.3 Let

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \text{ where } A, B, C, D \in {}^nR^n, \text{ then,}$$

$$\det_R \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det_R \begin{bmatrix} -C & -D \\ A & B \end{bmatrix}.$$

Proof:

Assume that $\det_F I_n = 1$, where I_n is the $n \times n$ identity matrix, and observe that

$$\begin{bmatrix} I_n & -I_n \\ O & I_n \end{bmatrix} \begin{bmatrix} I_n & O \\ I_n & I_n \end{bmatrix} \begin{bmatrix} I_n & -I_n \\ O & I_n \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} -C & -D \\ A & B \end{bmatrix}$$

By theorem 2.3 and since the first three matrices on the left are unitriangular, thus,

$$\det_R \begin{bmatrix} I_n & -I_n \\ O & I_n \end{bmatrix} = \det_R \begin{bmatrix} I_n & O \\ I_n & I_n \end{bmatrix} = \det_R \begin{bmatrix} I_n & -I_n \\ O & I_n \end{bmatrix} = \det_R I_n = 1$$

it follows from this that $\det_{\mathbb{R}} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det_{\mathbb{R}} \begin{bmatrix} -C & -D \\ A & B \end{bmatrix}$. \square

Theorem 3.4: Let

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \text{ be } 3 \times 3 \text{ upper-block triangular matrix,}$$

Then $\det A = \det(A_{11}A_{22}A_{33} - A_{12}A_{33})$.

Proof:

By theorem 2.4 $\det(A) = \det(\alpha_{11}^{(2)}) \det(\alpha_{22}^{(1)}) \det(\alpha_{33}^{(0)})$,

and By lemma 2.1

$$\alpha_{ij}^{(0)} = A_{ij}, \quad \alpha_{ij}^{(1)} = A_{ij} - A_{i3}A_{33}^{-1}A_{3j}$$

$$\alpha_{ij}^{(2)} = \left[A_{ij} - A_{i3}A_{33}^{-1}A_{3j} \right] - \left[A_{i2} - A_{i3}A_{33}^{-1}A_{32} \right] \left[A_{22} - A_{23}A_{33}^{-1}A_{32} \right]^{-1} \left[A_{2j} - A_{23}A_{33}^{-1}A_{3j} \right],$$

$$\begin{aligned} \det(A) &= \det \left(\left[A_{11} - A_{13}A_{33}^{-1}A_{31} \right] \right. \\ &\quad \left. - \left[A_{12} - A_{13}A_{33}^{-1}A_{32} \right] \left[A_{22} - A_{23}A_{33}^{-1}A_{32} \right]^{-1} \left[A_{21} - A_{23}A_{33}^{-1}A_{31} \right] \right) \\ &\quad \times \det \left(A_{22} - A_{23}A_{33}^{-1}A_{32} \right) \det(A_{33}). \\ &= \det(A_{11} - A_{12}A_{22}) \times \det(A_{22}) \det(A_{33}) \\ &= \det(A_{11}A_{22}A_{33} - A_{12}A_{33}). \square \end{aligned}$$

Theorem 3.5

Let B any $n \times m$ matrix, and $\begin{bmatrix} I & 0 \\ B & I \end{bmatrix}$ are non singular, then

$$\text{Let } \begin{bmatrix} I & 0 \\ B & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -B & I \end{bmatrix}$$

Proof:

$$\begin{aligned} \begin{bmatrix} I & 0 \\ B & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -B & I \end{bmatrix} &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \text{ and} \\ \begin{bmatrix} I & 0 \\ -B & I \end{bmatrix} \begin{bmatrix} I & 0 \\ B & I \end{bmatrix} &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \end{aligned}$$

Theorem 3.6;

let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, and $A_{11}, A_{12}, A_{21}, A_{22}$ are square matrix,

if A_{22} is nonsingular, Then

$$BAB^T = \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ 0 & A_{22} \end{bmatrix}$$

Proof:

By theorem 2.1

$$BA = \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ A_{22} & A_{22} \end{bmatrix}$$

Implies that

$$\begin{aligned} BAB^T &= \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ A_{22} & A_{22} \end{bmatrix} \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ 0 & A_{22} \end{bmatrix} \\ &= \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ 0 & A_{22} \end{bmatrix} \end{aligned}$$

Theorem 3.7

For every matrix in M_{2n}^+ written in blocks of the same size, we have the decomposition:

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} = U \begin{bmatrix} \frac{A+B}{2} + R(X) & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} - R(X) \end{bmatrix} V^*$$

for some unitaries $U, V \in M_{2n}$

Proof: Let $J = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -I & I \end{bmatrix}$

where I is the identity of M_n , J is a unitary matrix, and we have:

$$J \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} J^* = \underbrace{\begin{bmatrix} \frac{A+B}{2} + R(X) & \frac{B-A}{2} + \frac{X-X^*}{2} \\ \frac{B-A}{2} - \frac{X-X^*}{2} & \frac{A+B}{2} - R(X) \end{bmatrix}}_C$$

Since $R(X) = \frac{X+X^*}{2}$

$$J^*J \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} J^*J = J^* \underbrace{\begin{bmatrix} \frac{A+B}{2} + R(X) & \frac{B-A}{2} + \frac{X-X^*}{2} \\ \frac{B-A}{2} - \frac{X-X^*}{2} & \frac{A+B}{2} - R(X) \end{bmatrix}}_C J$$

Factorize C as a square of positive matrices, that is

$$M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} = J^* Q^2 J$$

Now, decompose Q^2 as in Lemma 3.1.1, then

$$M = J^* (T^* T + S^* S) J = J^* (T^* T) J + J^* (S^* S) J$$

$$\text{Where } TT^* = \begin{bmatrix} \frac{A+B}{2} + R(X) & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and } SS^* = \begin{bmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} - R(X) \end{bmatrix}$$

Corollary 3.1 For every matrix in M_{2n}^+ written in blocks of the same size, we have

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \leq \frac{1}{2} \left\{ U \begin{bmatrix} A+B + |X+X^*| & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & 0 \\ 0 & A+B + |X+X^*| \end{bmatrix} V^* \right\}$$

Proof. By the fact $R(X) \leq |R(X)|$.

And theorem 3.7

$$\begin{aligned} \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} &= U \begin{bmatrix} \frac{A+B}{2} + R(X) & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} - R(X) \end{bmatrix} V^* \\ &\leq U \begin{bmatrix} \frac{A+B}{2} + |R(X)| & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} + |R(X)| \end{bmatrix} V^* \quad \square \end{aligned}$$

Theorem 3.8

Let $A \in \mathbb{C}^{n \times n}$ then There exist a nonsingular matrix Q such that

$$A^D = Q \begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$$

Proof:

let A be of index K then ,by Lemma 2.2

There exist a nonsingular matrix Q such that $J = Q^{-1}AQ$, Where $Q = [v_s \ v_{s-1} \ \cdots \ v_1]$ and v_i is an eigenvector corresponding to eigenvalue of A

Implies that $A = QJQ^{-1}$.

Now, let $M = Q \begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$ Then

$$A^K M A = A \ , M A M = M \ , A M = M A$$

Which implies that M is drizn inverse of A . \square

Theorem 3.9

Let M be a given matrix of form (3) and $S = D - CA^D B$ be the generalized Schur complement of A in M . then

$$M^D = \begin{bmatrix} (A - BD^D C)^{-1} & -(A - BD^D C)^D BD^D \\ -S^D CA^D & S^D \end{bmatrix}$$

if $C(I - A^D A) = (I - SS^D)C = 0$ and $(I - AA^D)BS^D = A^D B(I - S^D S)$

Proof:

$A(I - A^D A)$ and $(I - S^D S)C$ are nilpotent operators

And since $C(I - A^D A) = (I - SS^D)C = 0$

$(I - SS^D)C(I - A^D A)$ are nilpotent operators ,this implies

$$\begin{bmatrix} A(I - A^D A) & (I - AA^D)B \\ (I - SS^D)C(I - A^D A) & S(I - S^D S) \end{bmatrix} \text{ are nilpotent operators}$$

Then by Theorem 2.5 $M^D = \begin{bmatrix} (A - BD^D C)^{-1} & -(A - BD^D C)^D BD^D \\ -S^D CA^D & S^D \end{bmatrix} \square$

Theorem 3.10

Let A be a nonsingular matrix, then

$$\det \begin{bmatrix} B & A \\ A^T & 1 \end{bmatrix} = \det \begin{bmatrix} B & A \\ A^T & 1 \end{bmatrix} = \det(B)(1 - A^T B^{-1}A)$$

Proof: By theorem 2.4

$$\det \begin{bmatrix} B & A \\ A^T & 1 \end{bmatrix} = \det(B)(1 - A^T B^{-1}A) \quad \text{and}$$

$$\det \begin{bmatrix} B & A \\ A^T & 1 \end{bmatrix} = \det(B)(1 - A^T B^{-1}A) \quad \square$$

Theorem 3.11

let A be a square matrix and a block-diagonal with the form

$$A = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{rr} \end{bmatrix}$$

Where A_{ii} is a square matrix, Then

A is nonsingular if and only if $A_{11}, A_{22}, \dots, A_{rr}$ are nonsingular

Proof: Let A is nonsingular, Then A^{-1} exist but

$$\begin{aligned} & \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{rr} \end{bmatrix} \begin{bmatrix} A_{11}^{-1} & 0 & \cdots & 0 \\ 0 & A_{22}^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{rr}^{-1} \end{bmatrix} \\ &= \begin{bmatrix} A_{11}A_{11}^{-1} & 0 & \cdots & 0 \\ 0 & A_{22}A_{22}^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{rr}A_{rr}^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} A_{11}^{-1} & 0 & \cdots & 0 \\ 0 & A_{22}^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{rr}^{-1} \end{bmatrix} \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{rr} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}^{-1}A_{11} & 0 & \cdots & 0 \\ 0 & A_{22}^{-1}A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{rr}^{-1}A_{rr} \end{bmatrix} = \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}$$

Then inverse of A is

$$\begin{bmatrix} A_{11}^{-1} & 0 & \cdots & 0 \\ 0 & A_{22}^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{rr}^{-1} \end{bmatrix} \text{ This implies } A_{11}^{-1}, A_{22}^{-1}, \dots, A_{rr}^{-1} \text{ exists then}$$

$A_{11}, A_{22}, \dots, A_{rr}$ nonsingular

Now, Let $A_{11}, A_{22}, \dots, A_{rr}$ are nonsingular then A^{-1} exists and A

Nonsingular

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