Local Stability Analysis of epidemic models using a Corollary of Gershgorin's Circle Theorem

Agnes Adom-Konadu

July 2, 2020

Department of Mathematics, University of Cape Coast, Cape Coast, Ghana

Abstract

The processes that help to obtain necessary and sufficient conditions to determine the local stability of linearized systems is paramount. In this paper, a corollary of the Gershgorin's circle theorem was used to establish the local stability of different epidemic models with 3 states or more including, a Tuberculosis model, an SEIRS model, and malaria model. It was observed that no matter the state or the dimension of the system or matrix this corollary can be used to analyse local stability for both disease free and endemic equilibrium, by establishing that when $\mathcal{R}_0 < 1$, the Jacobian matrix will have negative eigenvalues or negative real part eigenvalues. Thus, disease free equilibrium is stable but when $\mathcal{R}_0 > 1$, the Jacobian matrix will have negative eigenvalues or negative real part eigenvalues making the endemic equilibrium is stable.

Keywords: Endemic, Gershgorin, local stability, malaria, Tuberculosis.

1 Introduction

Mathematical modelling plays an important role in epidemiology by providing better understanding of the underlying mechanisms for the spread of occurrence and reoccurrence of infectious diseases and suggesting effective control strategies [6]. It is an important tool that helps to understand the dynamics of infectious diseases and to support the development of control strategies [12]. Mathematical models for Infectious diseases is the representation of the dynamic transmission cycle, involving interactions between infected and susceptible hosts that are generally expressed as a set of ordinary differential equations (ODEs) [14].

The stability of an equilibrium point (stationary states) of a mathematical model for an infectious disease helps to determine whether the solutions remain near the equilibrium point or get further away or not. The equilibrium point can be either stable or unstable or a saddle point [7, 10].

The main method use to analyze the local stability of the equilibrium points of epidemic models is the Lyapunov's indirect method that is, to determine whether the eigenvalues of the Jacobian matrix evaluated at the equilibrium points of the system are negative or have negative real part (that is, equilibrium points lie in the left half of the complex plane). Since the characteristic equation for an $n-$ dimensional system is a polynomial equation of degree n for which it may be difficult or impossible to find all roots explicitly, different methods such as the Routh-Hurwitz criterion gives necessary and sufficient conditions for the eigenvalues to lie in the left half of the complex plane. In this case, the reproduction number can be obtained from the constant term. Whether the reproduction number is greater or less than 1 determines the sign of the constant term [8]. In most of these methods, it is complicated to apply in problem of many dimensions [2].

In this study, we investigate the local stability of some selected epidemic model using a corollary of Gershgorin's circle theorem. The Gershgorin's theorem also provides sufficient conditions for the eigenvalues to lie in the left half of the complex plane [1, 11, 9]. That is, the local stability can be established without the need to calculate the eigenvalues, instead the basic reproduction number which also gives a condition for an equilibrium point to be stable is used for the analysis. The Gershgorin circle theorem is a theorem which may be used to bound the size of the eigenvalues of a square matrix. It was first published by Belorussian mathematician Semyon Aranovich Gershgorin in 1931. Informally, the theorem says that if the off-diagonal entries of a square matrix over the complex numbers have small norms then its eigenvalues are similar in norm to the diagonal entries of the matrix. This theorem is a very useful tool in numerical analysis, particularly in perturbation theory [5].

Corollary 1.1 (Corollary of Gershgorin Circle Theorem) Let $\mathbf{A}$ be an $n \times n$ matrix with real entries. If the diagonal elements a_{ii} of $\mathbf{A}$ satisfy

$$a_{ii} < -r_i$$

where

$$r_i = \sum_{j=1, j\neq i}^{n} \mid a_{ij} \mid \tag{1}$$

for $i = 1, ..., n$, then the eigenvalues of $\mathbf{A}$ are negative or have negative real parts [1].

The paper is organized as follows: In Section 2, we established the local stability of a tuberculosis model . We establisheded the local stability of SEIRS model in Section 3. In Section 4, we investigated the local stability of a malaria model and gave a short conclusion in section 5.

2 Tuberculosis Model

Using the [3], the population under consideration is sub-divided into three epidemiological classes: susceptible S, latent or exposed E, and infectious I. The incidence rate given by βSI (using the mass action law). A portion $p\beta SI$ gives rise to immediate active cases (fast progression), while the rest $(1-p)\beta SI$ gives rise to latent-TB cases with a low risk of progressing to active TB (slow progression). The progression rate from latent TB to active TB is assumed to be proportional to the number of latent-TB cases, that is, it is given by κE. The total incidence rate is $p\beta SI + \kappa E$. The model is given by the following system:

$$\begin{aligned} \dot{S} &= \Lambda - \beta SI - \mu S \\ \dot{E} &= (1-p)\beta SI - \kappa E - \mu E \\ \dot{I} &= p\beta SI + \kappa E - \mu I - \delta I \end{aligned} \tag{2}$$

Let S be Susceptible individuals, E be Latently infected individuals or exposed individuals and I be Infectious individuals. Let Λ be recruitment rate of susceptible individuals, μ be natural death rate, β be transmission rate of active TB, κ be the progression rate from latent TB to active TB (Rate of slow progression), δ death rate due to TB infection and p rate of fast progression.

2.1 The Equilibrium Points

The equilibruim points of model (2) are

1. Disease free equilibrium point (P^0) given as
   $\left(S^0, E^0, I^0\right) = \left(\frac{\Lambda}{\mu}, 0, 0\right)$

2. Endemic equilibrium point(P^*) given as
   $\left(S^*, E^*, I^*\right) = \Big(\frac{(\delta\kappa + \delta\mu + \kappa\mu + \mu^2)}{(\beta(\mu p + \kappa))},$
   $-\frac{(p-1)(\Lambda\beta\mu p + \Lambda\beta\kappa - \delta\kappa\mu - \delta\mu^2 - \kappa\mu^2 - \mu^3)}{((\mu p + \kappa)(\mu + \kappa)\beta)},$
   $\frac{(\Lambda\beta\mu p + \Lambda\beta\kappa - \delta\kappa\mu - \delta\mu^2 - \kappa\mu^2 - \mu^3)}{(\beta(\delta\kappa + \delta\mu + \kappa\mu + \mu^2))}\Big)$

The basic reproduction number of the tuberculosis model is given as

$$\mathcal{R}_0 = \frac{\beta\Lambda(\kappa + \mu p)}{\mu(\kappa + \mu)(\gamma + \mu + \delta)}$$

The endemic equilibrium point (P^*) can now be expressed in terms of $\mathcal{R}_0$ as
$(P^*) = (S^*, E^*, I^*) = \left(\frac{\Lambda}{\mu\mathcal{R}_0}, -\frac{(p-1)(\mathcal{R}_0 - 1)\mu(\mu+\delta)}{(\mu p + \kappa)\beta}, \frac{(\mathcal{R}_0 - 1)\mu}{\beta}\right)$

2.2 Local Stability Analysis of the Disease free Equilibrium

We analyze the local stability of the disease free Equilibrium by applying the theorem which follows.

Theorem 2.1 The disease free equilibrium is locally asymptotically stable if $\mathcal{R}_0 < 1$.

Proof 2.1 The Jacobian matrix J of the system (2) is

$$J = \begin{bmatrix} -\beta I - \mu & 0 & -\beta S \\ (1-p)\beta I & -(\kappa+\mu) & (1-p)\beta S \\ p\beta I & \kappa & p\beta S - \delta - \mu \end{bmatrix} \tag{3}$$

Evaluating the matrix (3) at the disease free equilibrium gives

$$J_0 = \begin{bmatrix} -\mu & 0 & -\dfrac{\beta\Lambda}{\mu} \\ 0 & -(\kappa+\mu) & \dfrac{(1-p)\beta\Lambda}{\mu} \\ 0 & \kappa & \dfrac{p\beta\Lambda}{\mu} - (\mu+\delta) \end{bmatrix} \tag{4}$$

The disease free equilibrium point will be locally asymptotically stable if the eigenvalues of the Jacobian matrix are negative or have negative real parts. The matrix J_0 has one eigenvalue $-\mu$ which is negative. The remaining sub-matrix is given by

$$J_r = \begin{bmatrix} -(\kappa+\mu) & \dfrac{(1-p)\beta\Lambda}{\mu} \\ \kappa & \dfrac{p\beta\Lambda}{\mu} - (\mu+\delta) \end{bmatrix}$$

According to the corollary of Gershgorin's circle theorem, the matrix (J_r) will have negative eigenvalues if the following inequalities are satisfied:

$$(\kappa+\mu) > \frac{(1-p)\beta\Lambda}{\mu} \tag{5a}$$

$$-\frac{p\beta\Lambda}{\mu} + (\mu+\delta) > \kappa \tag{5b}$$

Dividing (5a) through by $(\kappa+\delta)$ yields

$$1 > \frac{(1-p)\beta\Lambda}{\mu(\kappa+\mu)} \tag{6}$$

Also dividing (5b) through by κ yields

$$-\frac{p\beta\Lambda}{\mu\kappa} + \frac{(\mu+\delta)}{\kappa} > 1$$

$$\frac{-p\beta\Lambda + \mu(\mu+\delta)}{\mu\kappa} > 1 \tag{7}$$

From (6) and (7) we have,

$$\begin{aligned} \frac{-p\beta\Lambda + \mu(\mu+\delta)}{\mu\kappa} &> 1 > \frac{(1-p)\beta\Lambda}{\mu(\kappa+\mu)} \\ \Rightarrow 1 &> \frac{(\kappa+\mu p)\beta\Lambda}{\mu(\delta+\mu)(\kappa+\mu)} \\ \Rightarrow 1 &> \mathcal{R}_0 \\ \mathcal{R}_0 &< 1 \end{aligned}$$

Epidemiologically, if $\mathcal{R}_0 < 1$, the epidemic is expected to be eliminated and should persist if $\mathcal{R}_0 > 1$. Therefore, we conclude that from the above proof the disease free equilibrium (E_0) is locally asymptotically stable.

2.3 Local Stability Analysis of the Endemic Equilibrium Point

We now investigate the local stability of the endemic equilibrium Point.

Theorem 2.2 The endemic equilibrium is locally asymptotically stable if $\mathcal{R}_0 > 1$.

Proof 2.2 Evaluating the matrix (3) at the endemic equilibrium gives

$$J_e^s = \begin{bmatrix} -(\mathcal{R}_0-1)\mu-\mu & 0 & -\dfrac{\beta\Lambda}{\mathcal{R}_0\mu} \\ -(1-p)(1-\mathcal{R}_0)\mu & -(\kappa+\mu) & \dfrac{(1-p)\beta\Lambda}{\mathcal{R}_0\mu} \\ -p(1-\mathcal{R}_0)\mu & \kappa & \dfrac{p\beta\Lambda}{\mathcal{R}_0\mu}-(\mu+\delta) \end{bmatrix} \tag{8}$$

According to equation the corollary of Gershgorin's circle theorem, the matrix (J_e^s) will have negative eigenvalues if the following inequalities are satisfied:

$$(\mathcal{R}_0-1)\mu+\mu > \frac{\beta\Lambda}{\mathcal{R}_0\mu} \tag{9a}$$

$$(\kappa+\mu) > (1-p)(1-\mathcal{R}_0)\mu + \frac{(1-p)\beta\Lambda}{\mathcal{R}_0\mu} \tag{9b}$$

$$-\frac{p\beta\Lambda}{\mathcal{R}_0\mu} + (\mu+\delta) - p(1-\mathcal{R}_0)\mu > \kappa \tag{9c}$$

Dividing (9b) through by $(\kappa+\mu)$ gives

$$1 > \frac{(1-p)(1-\mathcal{R}_0)\mu}{(\kappa+\mu)} + \frac{(1-p)\beta\Lambda}{\mathcal{R}_0\mu(\kappa+\mu)}.$$

Thus,

$$1 > \frac{(1-p)\mathcal{R}_0\mu(1-\mathcal{R}_0)\mu + (1-p)\beta\Lambda}{\mathcal{R}_0\mu(\kappa+\mu)} \tag{10}$$

Dividing (9c) through by κ gives

$$-\frac{p\beta\Lambda}{\mathcal{R}_0\mu\kappa} + \frac{(\mu+\delta)}{\kappa} - \frac{p(1-\mathcal{R}_0)\mu}{\kappa} > 1.$$

Thus,

$$\frac{-p\beta\Lambda + (\mu+\delta)\mathcal{R}_0\mu - p\mathcal{R}_0\mu(1-\mathcal{R}_0)\mu}{\mathcal{R}_0\mu\kappa} > 1 \tag{11}$$

From (10) and (11)

$$\frac{-p\beta\Lambda + (\mu+\delta)\mathcal{R}_0\mu - p\mathcal{R}_0\mu(1-\mathcal{R}_0)\mu}{\mathcal{R}_0\mu\kappa} > 1 > \frac{(1-p)\mathcal{R}_0\mu(1-\mathcal{R}_0)\mu + (1-p)\beta\Lambda}{\mathcal{R}_0\mu(\kappa+\mu)} \tag{12}$$

Expanding and simplifying (12) gives

$$\begin{aligned} 0 &> \mu(1-\mathcal{R}_0)(\mu p+\kappa) \\ \Rightarrow \mathcal{R}_0 &> 1 \end{aligned}$$

This shows that the Endemic Equilibrium point is locally asymptotically stable if $\mathcal{R}_0 > 1$.

3 SEIRS Model

The SEIRS model consists of four compartments, but the individual looses immunity after some time and moves back into the S class (that is, the individual becomes susceptible again).

Let S be the proportion of susceptible individuals, E, be the proportion of exposed individuals, (infected but are not yet infectious), I, be the proportion of infectious individuals, and R, is the proportion of recovered individuals, (with temporary immunity). Furthermore, let the contact rate be given by β, Λ is the recruitment rate, μ is the birth rate (equal to the natural death rate), κ be the progression rate from E to I, γ the recovery rate, δ is the additional rate of disease-induced mortality, ρ is the rate of lost of immunity, α is the vaccination rate and N is the total population.

$$\begin{aligned} \dot{S} &= \Lambda - \frac{\beta IS}{N} - (\mu+\alpha)S + \rho R \\ \dot{E} &= \frac{\beta IS}{N} - (\kappa+\mu)E \\ \dot{I} &= \kappa E - (\mu+\gamma+\delta)I \\ \dot{R} &= \gamma I - (\mu+\rho)R + \alpha S \end{aligned} \tag{13}$$

The system (13) has two equilibrium points;

(i) a disease freee equilibrium point P^0, given by $P^0 = (S^0, E^0, I^0, R^0) = (\frac{\Lambda(\mu+\rho)}{\mu(\alpha+\mu+\rho)}, 0, 0, \frac{\Lambda\alpha}{\mu(\alpha+\mu+\rho)})$ and

(ii) and endemic equilibrium point $P^* = (S^*, E^*, I^*, R^*)$, where
$S^* = \frac{(\delta\kappa+\delta\mu+\gamma\kappa+\gamma\mu+\kappa\mu+\mu^2)}{(\beta\kappa)}$,
$E^* = (\mu+\gamma+\delta)(\Lambda\beta\kappa\mu+\Lambda\beta\kappa\rho-\alpha\delta\kappa\mu-\alpha\delta\mu^2-\alpha\gamma\kappa\mu-\alpha\gamma\mu^2-\alpha\kappa\mu^2-\alpha\mu^3-\delta\kappa\mu^2-\delta\kappa\mu\rho-\delta\mu^3-\delta\mu^2\rho-\gamma\kappa\mu^2-\gamma\kappa\mu\rho-\gamma\mu^3-\gamma\mu^2\rho-\kappa\mu^3-\kappa\mu^2\rho-\mu^4-\mu^3\rho)/(\beta(\delta\kappa\mu+\delta\kappa\rho+\delta\mu^2+\delta\mu\rho+\gamma\kappa\mu+\gamma\mu^2+\gamma\mu\rho+\kappa\mu^2+\kappa\mu\rho+\mu^3+\mu^2\rho)\kappa)$,
$I^* = (\Lambda\beta\kappa\mu+\Lambda\beta\kappa\rho-\alpha\delta\kappa\mu-\alpha\delta\mu^2-\alpha\gamma\kappa\mu-\alpha\gamma\mu^2-\alpha\kappa\mu^2-\alpha\mu^3-\delta\kappa\mu^2-\delta\kappa\mu\rho-\delta\mu^3-\delta\mu^2\rho-\gamma\kappa\mu^2-\gamma\kappa\mu\rho-\gamma\mu^3-\gamma\mu^2\rho-\kappa\mu^3-\kappa\mu^2\rho-\mu^4-\mu^3\rho)/(\beta(\delta\kappa\mu+\delta\kappa\rho+\delta\mu^2+\delta\mu\rho+\gamma\kappa\mu+\gamma\mu^2+\gamma\mu\rho+\kappa\mu^2+\kappa\mu\rho+\mu^3+\mu^2\rho))$,
and
$R^* = (\Lambda\beta\gamma\kappa^2+\alpha\delta^2\kappa^2+2\alpha\delta^2\kappa\mu+\alpha\delta^2\mu^2+\alpha\delta\gamma\kappa^2+3\alpha\delta\gamma\kappa\mu+2\alpha\delta\gamma\mu^2+2\alpha\delta\kappa^2\mu+4\alpha\delta\kappa\mu^2+2\alpha\delta\mu^3+\alpha\gamma^2\kappa\mu+\alpha\gamma^2\mu^2+\alpha\gamma\kappa^2\mu+3\alpha\gamma\kappa\mu^2+2\alpha\gamma\mu^3+\alpha\kappa^2\mu^2+2\alpha\kappa\mu^3+\alpha\mu^4-\delta\gamma\kappa^2\mu-\delta\gamma\kappa\mu^2-\gamma^2\kappa^2\mu-\gamma^2\kappa\mu^2-\gamma\kappa^2\mu^2-\gamma\kappa\mu^3)/(\beta\kappa(\delta\kappa\mu+\delta\kappa\rho+\delta\mu^2+\delta\mu\rho+\gamma\kappa\mu+\gamma\mu^2+\gamma\mu\rho+\kappa\mu^2+\kappa\mu\rho+\mu^3+\mu^2\rho))$.

E_0 and E_1 are the disease free and endemic equilibrium points respectively. The Basic Reproduction number $\mathcal{R}_0$was computed using the Next Generation Matrix approach given as

$$\mathcal{R}_0 = \frac{\Lambda}{\mu}\frac{\kappa}{(\kappa+\mu)}\frac{(\mu+\rho)}{(\alpha+\mu+\rho)}\frac{\beta}{(\gamma+\mu+\delta)}.$$

We now express the endemic equilibrium point in terms of $\mathcal{R}_0$ as

$$S^* = \frac{\Lambda(\mu+\rho)}{\mathcal{R}_0(\alpha)\mu(\alpha+\mu+\rho)},$$
$$E^* = \frac{(\mu+\gamma+\delta)(\mathcal{R}_0-1)\mu[(\mu+\kappa)(\alpha+\mu+\rho)(\gamma+\mu+\delta)]}{\alpha\kappa(\delta+\mu)+\mu((\alpha+\mu+\rho)(\gamma+\mu+\delta))},$$
$$I^* = \frac{(\mathcal{R}_0-1)\mu[(\mu+\kappa)(\alpha+\mu+\rho)(\gamma+\mu+\delta)]}{\alpha\kappa(\delta+\mu)+\mu((\alpha+\mu+\rho)(\gamma+\mu+\delta))},$$

and

$$\begin{aligned} R^* = & [\mu\mathcal{R}_0\kappa\gamma(\alpha+\mu+\rho)-(-\delta\kappa\rho-\delta\mu\rho+\gamma\kappa\mu-\gamma\mu\rho-\kappa\mu\rho-\mu^2\rho)]\mu(\mu+\kappa)(\alpha+\mu+\rho) \\ & (\gamma+\mu+\delta)/\mu(\alpha+\mu+\rho)[\alpha\kappa(\delta+\mu)+\mu((\alpha+\mu+\rho)(\gamma+\mu+\delta))]. \end{aligned}$$

3.1 Local Stability Analysis for the Disease free Equilibrium Point

The following theorem gives conditions for the disease free equilibrium point to belocally asymptotically stable.

Theorem 3.1 The disease free equilibrium point (P^0) is locally asymptotically stable if $\mathcal{R}_0 < 1$.

Proof 3.1 The Jacobian matrix J for the system (13) is

$$J = \begin{bmatrix} -\beta I-(\mu+\alpha) & 0 & -\beta S & \rho \\ \beta I & -(\kappa+\mu) & \beta S & 0 \\ 0 & \kappa & -(\gamma+\mu+\delta) & 0 \\ \alpha & 0 & \gamma & -(\mu+\rho) \end{bmatrix} \tag{14}$$

Evaluating the matrix J at the disease free equilibrium gives

$$J_0 = \begin{bmatrix} -(\mu+\alpha) & 0 & -\dfrac{\beta\Lambda(\mu+\rho)}{\mu(\alpha+\mu+\rho)} & \rho \\ 0 & -(\kappa+\mu) & \dfrac{\beta\Lambda(\mu+\rho)}{\mu(\alpha+\mu+\rho)} & 0 \\ 0 & \kappa & -(\gamma+\mu+\delta) & 0 \\ \alpha & 0 & \gamma & -(\mu+\rho) \end{bmatrix} \tag{15}$$

According to the corollary of Gershgorin's circle theorem, the matrix (J_0) will have negative eigenvalues if the following inequalities are satisfied

(i)
$$(\mu+\alpha) > \frac{\beta\Lambda(\mu+\rho)}{\mu(\alpha+\mu+\rho)} + \rho,$$

(ii)
$$(\kappa+\mu) > \frac{\beta\Lambda(\mu+\rho)}{\mu(\alpha+\mu+\rho)},$$

(iii)
$$(\gamma+\mu+\delta) > \kappa,$$

(iv)
$$(\mu+\rho) > (\alpha+\gamma).$$

Combining (ii) and (iii) gives

$$1 > \frac{\beta\Lambda(\mu+\rho)}{\mu(\alpha+\mu+\rho)(\mu+\kappa)}, \tag{16}$$

and

$$\frac{(\gamma+\mu+\delta)}{\kappa} > 1, \tag{17}$$

From (16) and (17) we have,

$$\frac{(\gamma+\mu+\delta)}{\kappa} > 1 > \frac{\beta\Lambda(\mu+\rho)}{\mu(\alpha+\mu+\rho)(\mu+\kappa)}$$

which implies that

$$\frac{(\gamma+\mu+\delta)}{\kappa} > \frac{\beta\Lambda(\mu+\rho)}{\mu(\alpha+\mu+\rho)(\mu+\kappa)}.$$

It follows that

$$1 > \frac{\beta\Lambda(\mu+\rho)\kappa}{\mu(\alpha+\mu+\rho)(\mu+\kappa)(\gamma+\mu+\delta)} = \mathcal{R}_0.$$

3.2 Local Stability Analysis for the Endemic Equilibrium

Theorem 3.2 The Endemic equilibrium (P^*) is locally asymptotically stable if $\mathcal{R}_0 > 1$.

Proof 3.2 The Jacobian matrix J evaluated at the endemic equilibrium gives

$$J_1 = \begin{bmatrix} b-(\alpha+\mu) & 0 & -\frac{(\mu+\kappa)(\mu+\gamma+\delta)}{\kappa} & \rho \\ -b & -(\kappa+\mu) & \frac{(\mu+\kappa)(\mu+\gamma+\delta)}{\kappa} & 0 \\ 0 & \kappa & -(\gamma+\mu+\delta) & 0 \\ \alpha & 0 & \gamma & -(\mu+\rho) \end{bmatrix} \tag{18}$$

where $b = \frac{(\alpha+\mu+\rho)(\mu(\mu+\kappa))(\mu+\gamma+\delta)(1-\mathcal{R}_0)}{\mu(\mu+\kappa)(\mu+\gamma+\delta)+\rho(\mu(\mu+\gamma+\delta)+\kappa(\mu+\delta))}$

According to the corollary of Gershgorin's circle theorem, the matrix (J_0) will have negative eigenvalues if the following inequalities are satisfied

(i) $-\frac{(\alpha+\mu+\rho)(\mu(\mu+\kappa))(\mu+\gamma+\delta)(\mathcal{R}_0-1)}{\mu(\mu+\kappa)(\mu+\gamma+\delta)+\rho(\mu(\mu+\gamma+\delta)+\kappa(\mu+\delta))} - (\alpha+\mu) < -\Big(\frac{(\mu+\kappa)(\mu+\gamma+\delta)}{\kappa}+\rho\Big)$

(ii) $-(\kappa+\mu) < -\Big(\frac{(\alpha+\mu+\rho)(\mu(\mu+\kappa))(\mu+\gamma+\delta)(1-\mathcal{R}_0)}{\mu(\mu+\kappa)(\mu+\gamma+\delta)+\rho(\mu(\mu+\gamma+\delta)+\kappa(\mu+\delta))} + \frac{(\mu+\kappa)(\mu+\gamma+\delta)}{\kappa}\Big)$

(iii) $-(\mu+\delta+\gamma) < -\kappa$

(iv) $-(\mu+\rho) < -(\alpha+\gamma)$

The inequalities (i) to (iv) can be rewritten as

(i^*) $\frac{(\alpha+\mu+\rho)(\mu(\mu+\kappa))(\mu+\gamma+\delta)(\mathcal{R}_0-1)}{\mu(\mu+\kappa)(\mu+\gamma+\delta)+\rho(\mu(\mu+\gamma+\delta)+\kappa(\mu+\delta))} + (\alpha+\mu) > \frac{(\mu+\kappa)(\mu+\gamma+\delta)}{\kappa}+\rho$

(ii^*) $(\kappa+\mu) > \frac{(\alpha+\mu+\rho)(\mu(\mu+\kappa))(\mu+\gamma+\delta)(1-\mathcal{R}_0)}{\mu(\mu+\kappa)(\mu+\gamma+\delta)+\rho(\mu(\mu+\gamma+\delta)+\kappa(\mu+\delta))} + \frac{(\mu+\kappa)(\mu+\gamma+\delta)}{\kappa}$

(iii^*) $(\mu+\delta+\gamma) > \kappa$

(iv^*) $(\mu+\rho) > \alpha+\gamma$

Dividing (ii^*) through by $(\kappa+\mu)$ gives

$$1 > \frac{(\alpha+\mu+\rho)(\mu)(\mu+\gamma+\delta)(1-\mathcal{R}_0)}{\mu(\mu+\kappa)(\mu+\gamma+\delta)+\rho(\mu(\mu+\gamma+\delta)+\kappa(\mu+\delta))} + \frac{(\mu+\gamma+\delta)}{\kappa} \tag{19}$$

and dividing (iii^*) through by κ gives

$$\frac{(\mu+\delta+\gamma)}{\kappa} > 1 \tag{20}$$

From inequalities (19) and (20)

$$0 > \frac{(\alpha+\mu+\rho)(\mu)\kappa(1-\mathcal{R}_0)}{\mu(\mu+\kappa)(\mu+\gamma+\delta)+\rho(\mu(\mu+\gamma+\delta)+\kappa(\mu+\delta))}.$$

The above inequality holds if

$$0 > 1-\mathcal{R}_0.$$

that is, if

$$\mathcal{R}_0 > 1.$$

This means that the endemic equilibrium point is stable if $\mathcal{R}_0 > 1$.

Even though the disease malaria, falls under the infectious diseases that can be modelled using the SEIRS compartment, the next section looks at transmission dynamic model between the Host (human) and vector (mosquito).

4 Malaria model

In this section, a malaria model as an *SEIRS* for the host population and *SEI* for the vector population similar to that of [4] is analyzed. The model divides the human population into 4 classes: Susceptible, S_h, (People enter the susceptible class, either through birth or immigration at a constant rate); then comes the Exposed, L_h, the Latent or exposed. When an infectious mosquito bites a susceptible human, symptoms usually appear 10-15 days after the bite [13] (people become infectious and progress to infectious class). I_h, people who have been infected and are capable of spreading the disease to those in the susceptible class and finally, the recovered (immune), R_h, people who recover from the infection through clinical treatment with temporary immunity . The recovered humans have some immunity to the disease and do not get clinically ill, but after some period of time, they lose their immunity and return to the susceptible class. These humans can not transmit the infection to the vector because we assume that they have no plasmodium parasites in their bodies.
The vector population is divided into 3 classes: Susceptible, S_v, Latent or Exposed, L_v and Infectious, I_v.
Let Λ_h be Humans birth rate, μ_h be Humans death rate, κ be Transition rate from Latent class to infectious class at time t, γ be recovery rate of human, β_h be Transmission rate of host (bite rate plus probability of transmission of disease), δ be Disease-induced death rate for humans, ρ be Rate of loss of immunity for humans, λ_v be Vector birth rate = vector death rate.

$$
\begin{aligned}
\dot{S_h} &= \Lambda_h - \beta_h S_h I_v - \alpha S_h + \rho R_h - S_h \mu_h \\
\dot{L_h} &= \beta_h S_h I_v - (\kappa + \mu_h) L_h \\
\dot{I_h} &= \kappa L_h - (\gamma + \mu + \delta) I_h \\
\dot{R_h} &= \gamma I_h - \mu R_h + \alpha S_h - \rho R_h \\
\dot{S_v} &= \lambda_v - \beta_v S_v I_h - \lambda_v S_v \\
\dot{L_v} &= \beta_v S_v I_h - (\theta + \lambda_v) L_v \\
\dot{I_v} &= \theta L_v - \lambda_v I_v
\end{aligned} \tag{21}
$$

4.1 Equilibrium Points of the malaria model

Steady state solutions or equilibrium points are the roots or solutions of the system of equations when the right-hand side of a nonlinear system is set to zero. That is, using the nonlinear system (21), we have

$$
\begin{aligned}
\Lambda_h - \beta_h S_h I_v - \alpha S_h + \rho R_h - S_h \mu_h &= 0 \\
\beta_h S_h I_v + (\kappa + \mu_h) L_h &= 0 \\
\kappa L_h + (\gamma + \mu + \delta) I_h &= 0 \\
\gamma I_h - \mu R_h + \alpha S_h - \rho R_h &= 0 \\
\lambda_v - \beta_v S_v I_h - \lambda_v S_v &= 0 \\
\beta_v S_v I_h - (\theta + \lambda_v) L_v &= 0 \\
\theta L_v - \lambda_v I_v &= 0
\end{aligned} \tag{22}
$$

Let $(S_h^*, L_h^*, I_h^*, R_h^*, S_v^*, L_v^*, I_v^*)$ be the steady state of (22) which can be obtained by solving. The system (21) has two equilibrium, namely;

(a) Disease Free equilibrium $(E_0) = (S_h^0, L_h^0, I_h^0, R_h^0, S_v^0, L_v^0, I_v^0)$ given by
$\left(\quad S_h^0 = \dfrac{(\rho + \mu_h)\Lambda_h}{((\alpha + \rho + \mu_h)\mu_h)}, \quad L_h^0 = 0, \quad I_h^0 = 0, \quad R_h^0 = \dfrac{\alpha \Lambda_h}{((\alpha + \rho + \mu_h)\mu_h)}, \right.$
$\left. S_v^0 = 1, \quad L_v^0 = 0, \quad I_v^0 = 0 \right)$

(b) Endemic equilibrium $(E_1) = (S_h^*, L_h^*, I_h^*, R_h^*, S_v^*, L_v^*, I_v^*)$ see equation (23)

The Basic Reproduction number $\mathcal{R}_0(0)$ and $\mathcal{R}_0(\alpha)$ were computed using the Next Generation Matrix approach where $\mathcal{R}_0(\alpha)$ is the basic reproduction number with effective prevention strategy, the basic reproduction number without effective prevention strategy is $\mathcal{R}_0(0)$

$$\mathcal{R}_0(\alpha) = \sqrt{\frac{\beta_h \beta_v \theta(\mu_h + \rho)\Lambda_h \kappa}{\lambda_v(\kappa+\mu_h)(\mu_h+\gamma+\delta)(\theta+\lambda_v)(\alpha+\mu_h+\rho)\mu_h}}$$

If $\alpha = 0$, then we have

$$\mathcal{R}_0(0) = \sqrt{\frac{\beta_h \beta_v \theta \Lambda_h \kappa}{\lambda_v(\kappa+\mu_h)(\mu_h+\gamma+\delta)(\theta+\lambda_v)\mu_h}}$$

The threshold parameter $\mathcal{R}_0$ can be defined as square roots of the product of number of humans one mosquito infects during its infectious lifetime $\mathcal{R}_{0h}$ and number of mosquitoes one human infects during the duration of the infectious period $\mathcal{R}_{0v}$ provided all humans and mosquitoes are susceptible. Therefore,

$$\mathcal{R}_0(0) = \sqrt{\frac{\beta_h(\mu_h+\rho)\Lambda_h\kappa}{(\alpha+\mu_h+\rho)(\kappa+\mu_h)(\gamma+\mu_h+\delta+\mu_h)} \times \frac{\beta_v\theta}{(\theta+\lambda_v)\lambda_v}}$$

$$\mathcal{R}_0(0) = \sqrt{\mathcal{R}_{0h} \times \mathcal{R}_{0v}}$$

Where:

$\frac{\kappa}{(\kappa+\mu_h)}$ means the probability that a human will survive the exposed state to become infectious.

$\frac{\theta}{\theta+\lambda_v)}$ is the probability that a vector will survive the exposed state to become infectious.

$\frac{\beta_h\theta}{(\theta+\lambda_v)\lambda_v}$ is the number of humans that one vector infects during its infectious lifetime, provided all humans are susceptible.

$\frac{\kappa\beta_v}{(\kappa+\mu_h)(\gamma+\mu_h+\delta)}$ is the number of vectors that one human infects during the duration of the infectious period, provided all vectors are susceptible.

The $(L_h^*, I_h^*, L_v^*, I_v^*)$ and (S_h^*, R_h^*, S_v^*) of the Endemic equilibrium (E_1) can be expressed in terms the basic Reproduction number $(\mathcal{R}_0^2(\alpha))$ and simplified as

$$(E_1) = (S_h^*, L_h^*, I_h^*, R_h^*, S_v^*, L_v^*, I_v^*) \tag{23}$$

given $(S_h^*, L_h^*, I_h^*, R_h^*, S_v^*, L_v^*, I_v^*)$ by

$S_h^* = \frac{a}{b}$

$a = (\mu_h+\kappa)(\lambda_v+\theta)(\gamma+\mu_h+\delta)\Big(\kappa\Lambda_h\beta_v(\rho+\mu_h) + \lambda_v(\delta\kappa\rho + \delta\kappa\mu_h + \delta\rho\mu_h + \delta\mu_h^2 + \gamma\kappa\mu_h + \gamma\rho\mu_h + \gamma\mu_h^2 + \kappa\rho\mu_h + \kappa\mu_h^2 + \rho\mu_h^2 + \mu_h^3)\Big)$

$b = \kappa\beta_v\Big(\mu_h(\mu_h+\kappa)(\mu_h+\alpha+\rho)(\lambda_v+\theta)(\gamma+\mu_h+\delta) + \beta_h\theta(\delta\kappa\rho + \delta\kappa\mu_h + \delta\rho\mu_h + \delta\mu_h^2 + \gamma\kappa\mu_h + \gamma\rho\mu_h + \gamma\mu_h^2 + \kappa\rho\mu_h + \kappa\mu_h^2 + \rho\mu_h^2 + \mu_h^3)\Big)$

$L_h^* = \frac{-c}{d}$

$c = \Big((1-\mathcal{R}_0^2(\alpha))\lambda_v\mu_h(\mu_h+\kappa)(\lambda_v+\theta)(\gamma+\mu_h+\delta)(\mu_h+\alpha+\rho)\Big)(\gamma+\mu_h+\delta)$

$d = \kappa\beta_v\Big(\mu_h(\mu_h+\kappa)(\mu_h+\alpha+\rho)(\lambda_v+\theta)(\gamma+\mu_h+\delta) + \beta_h\theta(\delta\kappa\rho + \delta\kappa\mu_h + \delta\rho\mu_h + \delta\mu_h^2 + \gamma\kappa\mu_h + \gamma\rho\mu_h + \gamma\mu_h^2 + \kappa\rho\mu_h + \kappa\mu_h^2 + \rho\mu_h^2 + \mu_h^3)\Big)$

$I_h^* = \frac{-e}{f}$

$-e = (1-\mathcal{R}_0^2(\alpha))\lambda_v(\mu_h+\kappa)(\lambda_v+\theta)(\gamma+\mu_h+\delta)(\mu_h+\alpha+\rho)$

$f = \beta_v\Big(\kappa\Lambda_h\beta_v(\rho+\mu_h)+\lambda_v(\delta\kappa\rho+\delta\kappa\mu_h+\delta\rho\mu_h+\delta\mu_h^2+\gamma\kappa\mu_h+\gamma\rho\mu_h+\gamma\mu_h^2+\kappa\rho\mu_h+\kappa\mu_h^2+\rho\mu_h^2+\mu_h^3)\Big)$

$S_v^* = \dfrac{g}{h}$

$g = \lambda_v\Big((\mu_h(\mu_h+\kappa)(\lambda_v+\theta)(\gamma+\mu_h+\delta)(\mu_h+\alpha+\rho)+\beta_h\theta(\delta\kappa\rho+\delta\kappa\mu_h+\delta\rho\mu_h+\delta\mu_h^2+\gamma\kappa\mu_h+\gamma\rho\mu_h+\gamma\mu_h^2+\kappa\rho\mu_h+$
$\kappa\mu_h^2+\rho\mu_h^2+\mu_h^3)\Big)$

$h = \beta_h\theta\Big(\kappa\Lambda_h\beta_v(\rho+\mu_h)+\lambda_v(\delta\kappa\rho+\delta\kappa\mu_h+\delta\rho\mu_h+\delta\mu_h^2+\gamma\kappa\mu_h+\gamma\rho\mu_h+\gamma\mu_h^2+\kappa\rho\mu_h+\kappa\mu_h^2+\rho\mu_h^2+\mu_h^3)\Big)$

$L_v = \dfrac{-n}{m}$

$n = \lambda_v(1-\mathcal{R}_0^2(\alpha))\lambda_v\mu_h(\mu_h+\kappa)(\lambda_v+\theta)(\gamma+\mu_h+\delta)(\mu_h+\alpha+\rho)$

$m = \beta_h(\theta+\lambda_v)\Big(\kappa\Lambda_h\beta_v(\rho+\mu_h)+\lambda_v(\delta\kappa\rho+\delta\kappa\mu_h+\delta\rho\mu_h+\delta\mu_h^2+\gamma\kappa\mu_h+\gamma\rho\mu_h+\gamma\mu_h^2+\kappa\rho\mu_h+\kappa\mu_h^2+\rho\mu_h^2+\mu_h^3)\Big)$

$I_v = \dfrac{-q}{r}$

$q = (1-\mathcal{R}_0^2(\alpha))\lambda_v\mu_h(\mu_h+\kappa)(\lambda_v+\theta)(\gamma+\mu_h+\delta)(\mu_h+\alpha+\rho)$

$r = \beta_h(\theta+\lambda_v)\Big(\kappa\Lambda_h\beta_v(\rho+\mu_h)+\lambda_v(\delta\kappa\rho+\delta\kappa\mu_h+\delta\rho\mu_h+\delta\mu_h^2+\gamma\kappa\mu_h+\gamma\rho\mu_h+\gamma\mu_h^2+\kappa\rho\mu_h+\kappa\mu_h^2+\rho\mu_h^2+\mu_h^3)\Big)$

4.1.1 The Relationship between $\mathcal{R}_0(\alpha)$ and $\mathcal{R}_0(0)$

From

$$\mathcal{R}_0^2(\alpha) = \frac{\kappa\beta_h\Lambda_h\theta\beta_m(\mu_h+\rho)}{\mu_h\lambda_v(\theta+\lambda_v)(\alpha+\mu_h+\rho)(\kappa+\mu_h)(\mu_h+\gamma+\delta)}$$

It can be seen that

$$\frac{(\mu_h+\rho)}{(\alpha+\mu_h+\rho)} < 1$$

and it is obvious that

$$\frac{\kappa\beta_h\Lambda_h\theta\beta_v(\mu_h+\rho)}{\mu_h\lambda_v(\theta+\lambda_v)(\alpha+\mu_h+\rho)(\kappa+\mu_h)(\mu_h+\gamma+\delta)} < \frac{\kappa\beta_h\Lambda_h\theta\beta_v}{\mu_h\lambda_v(\theta+\lambda_v)(\kappa+\mu_h)(\mu_h+\gamma+\delta)} \tag{24}$$

4.2 Local Stability Analysis of the malaria model at the Disease Free Equilibrium Point

Theorem 4.1 The disease free equilibrium (E_0) with $\alpha = 0$ is locally asymptotically stable if $\mathcal{R}_0(0) < 1$.

Proof 4.1

$$J = \begin{bmatrix} \lambda^* & 0 & 0 & \rho & 0 & 0 & -\beta_h S_h \\ \beta_h I_v & -(\kappa+\mu_h) & 0 & 0 & 0 & 0 & \beta_h S_h \\ 0 & \kappa & -(\mu_h+\gamma+\delta) & 0 & 0 & 0 & 0 \\ \alpha & 0 & \gamma & -(\mu_h+\rho) & 0 & 0 & 0 \\ 0 & 0 & -\beta_v S_v & 0 & -I_h\beta_v-\lambda_v & 0 & 0 \\ 0 & 0 & \beta_v S_v & 0 & I_h\beta_v & -(\theta+\lambda_v) & 0 \\ 0 & 0 & 0 & 0 & 0 & \theta & -\lambda_v \end{bmatrix} \tag{25}$$

where $\lambda^* = -\beta_h I_v - \alpha - \mu_h$

If $\alpha = 0$, then the Jacobian matrix evaluating at the disease free equilibrium (E_0) gives

$$J = \begin{bmatrix} -\mu_h & 0 & 0 & \rho & 0 & 0 & -\frac{\beta_h\Lambda_h}{\mu_h} \\ 0 & -(\kappa+\mu_h) & 0 & 0 & 0 & 0 & \frac{\beta_h\Lambda_h}{\mu_h} \\ 0 & \kappa & -(\mu_h+\gamma+\delta) & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & -(\mu_h+\rho) & 0 & 0 & 0 \\ 0 & 0 & -\beta_v & 0 & -\lambda_v & 0 & 0 \\ 0 & 0 & \beta_v & 0 & 0 & -(\theta+\lambda_v) & 0 \\ 0 & 0 & 0 & 0 & 0 & \theta & -\lambda_v \end{bmatrix} \tag{26}$$

Applying the corollary of Gershgorin's circle theorem gives the following inequalities,

$$\mu_h > (\rho + \frac{\beta_h \Lambda_h}{\mu_h}) \tag{27a}$$

$$(\kappa + \mu_h) > \frac{\beta_h \Lambda_h}{\mu_h} \tag{27b}$$

$$(\mu_h + \gamma + \delta) > \kappa \tag{27c}$$

$$(\mu_h + \rho) > \gamma \tag{27d}$$

$$\lambda_v > \beta_v \tag{27e}$$

$$\theta + \lambda_v > \beta_v \tag{27f}$$

$$\theta > \lambda_v \tag{27g}$$

from (27b) we have,

$$1 > \frac{\beta_h \Lambda_h}{\mu_h (\kappa + \mu_h)}. \tag{28}$$

And from (27c) we obtain,

$$\frac{\mu_h + \gamma + \delta}{\kappa} > 1. \tag{29}$$

It can be seen from equation (28) and (29) that

$$1 > \frac{\kappa \beta_h \Lambda_h}{\mu_h (\kappa + \mu_h)(\mu_h + \gamma + \delta)} \tag{30}$$

Also from (27f) and (27g) we obtain,

$$1 > \frac{\beta_v}{\theta + \lambda_v} \tag{31}$$

and

$$\frac{\lambda_v}{\theta} > 1 \tag{32}$$

It can be seen from equations (31) and (32) that,

$$\frac{\lambda_v (\theta + \lambda_v)}{\theta \beta_v} > 1 \tag{33}$$

From equations (30) and (33) we get,

$$\begin{aligned} \frac{\lambda_v (\theta + \lambda_v)}{\theta \beta_v} &> 1 > \frac{\kappa \beta_h \Lambda_h}{\mu_h (\kappa + \mu_h)(\mu_h + \gamma + \delta)} \\ 1 &> \mathcal{R}_0^2(0) \end{aligned} \tag{34}$$

Since

$$\mathcal{R}_0^2(0) = \frac{\kappa \beta_h \Lambda_h \theta \beta_v}{\mu_h \lambda_v (\theta + \lambda_v)(\kappa + \mu_h)(\mu_h + \gamma + \delta)} < 1.$$

Then, from (24) we conclude that

$$\mathcal{R}_0^2(\alpha) < \mathcal{R}_0^2(0) < 1.$$

Thus,

$$\mathcal{R}_0(\alpha) < 1$$

This implies

$$\mathcal{R}_0(\alpha) < \mathcal{R}_0^2(0) < 1$$

This satisfies the condition that the disease free equilibrium is stable if $\mathcal{R}_0(\alpha) < 1$

4.3 Local Stability Analysis of the malaria Model at the Endemic Equilibrium Point

Theorem 4.2 The Endemic equilibrium (E_1) is locally asymptotically stable if $\mathcal{R}_0(\alpha) > 1$.

Proof 4.2 The Jacobian matrix evaluating at the Endemic equilibrium (E_0) gives

$$J = \begin{bmatrix} a^* - \alpha - \mu_h & 0 & 0 & \rho & 0 & 0 & -b^* \\ -a^* & -(\kappa+\mu_h) & 0 & 0 & 0 & 0 & b^* \\ 0 & \kappa & -(\mu_h+\gamma+\delta) & 0 & 0 & 0 & 0 \\ \alpha & 0 & \gamma & -(\mu_h+\rho) & 0 & 0 & 0 \\ 0 & 0 & -c^* & 0 & d^*-\lambda_v & 0 & 0 \\ 0 & 0 & c^* & 0 & -d^* & -\theta-\lambda_v & 0 \\ 0 & 0 & 0 & 0 & 0 & \theta & -\lambda_v \end{bmatrix} \tag{35}$$

$a^* = \dfrac{a_1}{b_1}$

$a_1 = (1-\mathcal{R}_0^2(\alpha))\lambda_v\mu_h(\mu_h+\kappa)(\lambda_v+\theta)(\gamma+\mu_h+\delta)(\mu_h+\alpha+\rho)$

$b_1 = \beta_h(\theta+\lambda_v)\Big(\kappa\Lambda_h\beta_v(\rho+\mu_h)+\lambda_v(\delta\kappa\rho+\delta\kappa\mu_h+\delta\rho\mu_h+\delta\mu_h^2+\gamma\kappa\mu_h+\gamma\rho\mu_h+\gamma\mu_h^2+\kappa\rho\mu_h+\kappa\mu_h^2+\rho\mu_h^2+\mu_h^3)\Big)$

$b^* = \dfrac{a_2}{b_2}$

$a_2 = \beta_h(\mu_h+\kappa)(\lambda_v+\theta)(\gamma+\mu_h+\delta)\Big(\kappa\Lambda_h\beta_v(\rho+\mu_h)+\lambda_v(\delta\kappa\rho+\delta\kappa\mu_h+\delta\rho\mu_h+\delta\mu_h^2+\gamma\kappa\mu_h+\gamma\rho\mu_h+\gamma\mu_h^2+\kappa\rho\mu_h+$
$\kappa\mu_h^2+\rho\mu_h^2+\mu_h^3)\Big)$

$b_2 = \kappa\beta_v\Big(\mu_h(\mu_h+\kappa)(\mu_h+\alpha+\rho)(\lambda_v+\theta)(\gamma+\mu_h+\delta)+\beta_h\theta(\delta\kappa\rho+\delta\kappa\mu_h+\delta\rho\mu_h+\delta\mu_h^2+\gamma\kappa\mu_h+\gamma\rho\mu_h+\gamma\mu_h^2+\kappa\rho\mu_h+$
$\kappa\mu_h^2+\rho\mu_h^2+\mu_h^3)\Big)$

$c^* = \dfrac{a_3}{b_3}$

$a_3 = \beta_v\lambda_v\Big((\mu_h(\mu_h+\kappa)(\lambda_v+\theta)(\gamma+\mu_h+\delta)(\mu_h+\alpha+\rho)+\beta_h\theta(\delta\kappa\rho+\delta\kappa\mu_h+\delta\rho\mu_h+\delta\mu_h^2+\gamma\kappa\mu_h+\gamma\rho\mu_h+\gamma\mu_h^2+$
$\kappa\rho\mu_h+\kappa\mu_h^2+\rho\mu_h^2+\mu_h^3)\Big)$

$b_3 = \beta_h\theta\Big(\kappa\Lambda_h\beta_v(\rho+\mu_h)+\lambda_v(\delta\kappa\rho+\delta\kappa\mu_h+\delta\rho\mu_h+\delta\mu_h^2+\gamma\kappa\mu_h+\gamma\rho\mu_h+\gamma\mu_h^2+\kappa\rho\mu_h+\kappa\mu_h^2+\rho\mu_h^2+\mu_h^3)\Big)$

$d^* = \dfrac{a_4}{b_4}$

$a_4 = (1-\mathcal{R}_0^2(\alpha))\lambda_v(\mu_h+\kappa)(\lambda_v+\theta)(\gamma+\mu_h+\delta)(\mu_h+\alpha+\rho)$

$b_4 = \Big(\kappa\Lambda_h\beta_v(\rho+\mu_h)+\lambda_v(\delta\kappa\rho+\delta\kappa\mu_h+\delta\rho\mu_h+\delta\mu_h^2+\gamma\kappa\mu_h+\gamma\rho\mu_h+\gamma\mu_h^2+\kappa\rho\mu_h+\kappa\mu_h^2+\rho\mu_h^2+\mu_h^3)\Big)$

Applying a corollary of gershgorin's circle theorem yields

$$\mu_h+\alpha > \rho+a^*+b^* \tag{36a}$$

$$1 > \frac{a^*}{\mu_h+\kappa}+\frac{b^*}{\mu_h+\kappa} \tag{36b}$$

$$\frac{\delta+\mu_h+\gamma}{\kappa} > 1 \tag{36c}$$

$$\frac{\alpha+\gamma}{\mu_h+\rho} > 1 \tag{36d}$$

$$\lambda_v > d^*+c^* \tag{36e}$$

$$\theta+\lambda_v > -d^*+c^* \tag{36f}$$

$$\lambda_v > \theta \tag{36g}$$

From equation (36b) and (36c) we get

$$\begin{aligned}\frac{\delta+\mu_h+\gamma}{\kappa} &> 1 > \frac{a^*}{\mu_h+\kappa}+\frac{b^*}{\mu_h+\kappa}\\ \frac{\delta+\mu_h+\gamma}{\kappa} &> \frac{a^*}{\mu_h+\kappa}+\frac{b^*}{\mu_h+\kappa}\\ 1 &> \frac{a^*\kappa}{(\mu_h+\kappa)(\delta+\mu_h+\gamma)}+\frac{b^*\kappa}{(\mu_h+\kappa)(\delta+\mu_h+\gamma)} \qquad (37)\end{aligned}$$

Let $a^{**}=\dfrac{a_1\kappa}{(\mu_h+\kappa)(\delta+\mu_h+\gamma)}$ and $b^{**}=\dfrac{b_1\kappa}{(\mu_h+\kappa)(\delta+\mu_h+\gamma)}$

Then equation (37) becomes

$$1 > a^{**}+b^{**} \qquad (38)$$

Adding equation (36e) and equation (36f) yields

$$\theta+2\lambda_v > 2c^* \qquad (39)$$

but

$$\begin{aligned}2(\theta+\lambda_v) &> 2c^*\\ \Rightarrow 1 &> \frac{c^*}{\theta+\lambda_v} \qquad (40)\end{aligned}$$

also from (36g) we get

$$\frac{\lambda_v}{\theta} > 1 \qquad (41)$$

From equation (40) and (41) gives

$$1 > \frac{c^*\theta}{(\theta+\lambda_v)\lambda_v} \qquad (42)$$

But $\dfrac{c_1\theta}{(\theta+\lambda_v)\lambda_v}=\dfrac{1}{b^{**}}$

This implies equation (42) becomes

$$\begin{aligned}1 &> \frac{1}{b^{**}}\\ b^{**} &> 1\\ -1 &> -b^{**} \qquad (43)\end{aligned}$$

Adding equation (38) and (43) gives

$a^{**}=\dfrac{a_{11}}{b_{11}}$

$a_{11}=\kappa(1-\mathcal{R}_0^2(\alpha))\lambda_v\mu_h(\mu_h+\kappa)(\lambda_v+\theta)(\gamma+\mu_h+\delta)(\mu_h+\alpha+\rho)$

$b_{11}=\beta_h(\theta+\lambda_v)\Big(\kappa\Lambda_h\beta_v(\rho+\mu_h)+\lambda_v(\delta\kappa\rho+\delta\kappa\mu_h+\delta\rho\mu_h+\delta\mu_h^2+\gamma\kappa\mu_h+\gamma\rho\mu_h+\gamma\mu_h^2+\kappa\rho\mu_h+\kappa\mu_h^2+\rho\mu_h^2+\mu_h^3\Big)(\mu_h+\kappa)(\delta+\mu_h+\gamma))$

$$\begin{aligned}0 &> a^{**},\\ 0 &> (1-\mathcal{R}_0^2(\alpha)),\\ \mathcal{R}_0^2(\alpha) &> 1. \qquad (44)\end{aligned}$$

Thus,

$$\mathcal{R}_0(\alpha) > 1$$

This satisfies the condition that the endemic equilibrium is stable if $\mathcal{R}_0(\alpha) > 1$

5 Conclusion

We investigated the local stability of both the disease free and endemic equilibria of SEIRS, a Tuberculosis models and a malaria model. It was observed that no matter the state or the dimension of the system or the matrix, this corollary can be used to analyze the local stability for both disease free and endemic equilibrium, by establishing that if $\mathcal{R}_0 < 1$, the Jacobian matrix will have negative or negative real part eigenvalues. Thus, disease free equilibrium is stable but if $\mathcal{R}_0 > 1$, the endemic equilibrium is stable.

REFERENCES

[1] Linda J. S. Allen. *An Introduction to Mathematical Biology*. New Jersey, Pearson Education Inc., Pearson Prentice Hall, 2007.

[2] Fred Brauer and Carlos Castillo-Chaves. *Mathematical Models in population Biology and Epidemiology*. Springer-Verlag, New York, 2000.

[3] Carlos Castillo-Chavez and Baojun Song. Dynamical models of tuberculosis and their applications. *Mathematical biosciences and engineering*, 1(2):361–404, 2004.

[4] Nakul Rashmin Chitnis. *Using mathematical models in controlling the spread of malaria*. PhD thesis, The University of Arizona., 2005.

[5] Danny Gómez. A more direct proof of gerschgorin´ s theorem. *Matemáticas: Enseñanza Universitaria*, 14 (2), 2006.

[6] Herbert W Hethcote. The mathematics of infectious diseases. *SIAM review*, 42(4):599–653, 2000.

[7] J. A Karl and R. M. Murray. *An introduction for Scientists and Engineers*. Princeton University Press, Princeton and Oxford, 2016.

[8] Maia Martcheva. *An Introduction to Mathematical Epidemiology*, volume 61. Springer, 2015. ISBN 978-1-4899-7611-6.

[9] B. Noble. *Applied Linear Algebra*. Prentice Hall, Englewood Cliffs, N.J, 1969.

[10] J. K. Nthiiri. Global stability of equilibrium points of typhoid fever model with protection. *British Journal of Mathematics & Computer Science*, 21(5)(BJMCS.32690):1–6, April 2017. ISSN 2231-0851.

[11] J. M. Ortega. *Matrix Theory. A Second Course*. Plenum Press, New York., 1987.

[12] Roman Ullah, Gul Zaman, and Saeed Islam. Stability analysis of a general sir epidemic model. *VFAST Transactions on Mathematics*, 1(1), 2013.

[13] WHO. Malaria fact sheets, March 2019. URL `http://www.who.int/ith/diseases/malaria/en`.

[14] Jianyong Wu, Radhika Dhingra, Manoj Gambhir, and Justin V Remais. Sensitivity analysis of infectious disease models: methods, advances and their application. *Journal of The Royal Society Interface*, 10(86): 20121018, 2013.