EXISTENCE AND SOME ESTIMATES OF HYPERSURFACES OF CONSTANT GAUSS CURVATURE WITH PRESCRIBED BOUNDARY

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Abstract. In [3], Guan and Spruck prove that if Γ in $\mathbb{R}^{n+1}$ $(n \geq 2)$ bounds a suitable locally convex hypersurface Σ with Gauss curvature K_Σ, then Γ bounds a locally convex K-hypersurface whose Gauss curvature is less than $\inf K_\Sigma$. In this article we are particularly interested in K-hypersurfaces which are not global graphs and will extend several results in [3]. The first main result is to establish the estimate $K_M \geq (\operatorname{diam} M/2)^{-n}$ for the Gauss curvature K_M of a K-hypersurface M which satisfies **Condition A** below. The second main task is that, in case Σ above is not a global graph, we construct a K-hypersurface $\widetilde{M}$ whose Gauss curvature $K_{\widetilde{M}}$ is slighter greater than $\inf K_\Sigma$. If, in addition, the hypersurface Σ satisfies **Condition B** below, then for each number K, $0 < K \leq (\operatorname{diam}\Sigma/2)^{-n}$, we show that there exists a locally convex immersed hypersurface M_1 in $\mathbb{R}^{n+1}$ with $\partial M_1 = \Gamma$ and the Gauss curvature $K_{M_1} \equiv K$.

1. Introduction

In the paper [3], Guan and Spruck are concerned with the problem of finding hypersurfaces of constant Gauss-Kronecker curvature (K-hypersurfaces) with prescribed boundary Γ in $\mathbb{R}^{n+1}$ $(n \geq 2)$. They prove that if Γ bounds a suitable locally convex hypersurface Σ, then Γ bounds a locally convex K-hypersurface. Here a surface Σ in $\mathbb{R}^{n+1}$ is said to be locally convex if at every point $p \in \Sigma$ there exists a neighborhood which is the graph of a convex function $x_{n+1} = u(x)$, $x \in \mathbb{R}^n$, for a suitable coordinate system in $\mathbb{R}^{n+1}$, such that locally the region $x_{n+1} \geq u(x)$ always lies on a fixed side of Σ. More precisely, they proved:

Theorem 1 (Theorem 1.1 in [3]). *Assume that there exists a locally convex immersed hypersurface Σ in $\mathbb{R}^{n+1}$ with $\partial\Sigma = \Gamma$ and the Gauss curvature K_Σ. Let $K_0 = \inf K_\Sigma$. Suppose, in addition, that, in a tubular neighborhood of its boundary Γ, Σ is C^2 and locally strictly convex. Then there exists a smooth (up to the boundary) locally strictly convex hypersurface M with $\partial M = \Gamma$ such that $K_M \equiv K_0$. Moreover, M is homeomorphic to Σ.*

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Note that a locally convex hypersurface is necessarily of class $C^{0,1}$ in the interior. For a locally convex hypersurface Σ which is not C^2, we refer to [5] the definition of Gauss curvature in weak sense.

As noted in [3], Theorem 1 is a huge jump in generality from the previous results in, e.g., [3], for it deals with general immersed K-hypersurfaces and not just graphs. In this article we are particularly interested in K-hypersurfaces which are not global graphs. We will extend several results in [3]. The first main result is concerning an estimate for the Gauss curvature K_M of a K-hypersurface M, which satisfies **Condition A** below. We shall establish the estimate $K_M \geq (\operatorname{diam} M/2)^{-n}$ for such a K-hypersurface M. To introduce **Condition A**, let $\mathbf{p}_i$, $1 \leq i \leq k$, be the vertices of the hypersurface M. Let D_i be the maximal domain (i.e. the largest simply connected region) on M containing $\mathbf{p}_i$ which, as a hypersurface in $\mathbb{R}^{n+1}$, can be represented as the graph of a convex function u_i defined in a domain Ω_i, $1 \leq i \leq k$.

Condition A. *There exists some number m, $1 \leq m \leq k$, such that the maximal domain D_m lies in the interior of M.*

We shall establish the following theorem, which is an immediate consequence of the proof of Theorem 3.5 in [3].

Theorem 2. *Assume that M is a smooth locally strictly convex K-hypersurface and also fulfills* **Condition A**. *Then there holds*

$$K_M \geq (\operatorname{diam} M/2)^{-n}. \tag{1}$$

We may notice that this result does not hold for proper subsets of a hemisphere, which does not fulfill **Condition A**. Also notice that the graph of any function does not fulfill **Condition A**.

As a consequence of Theorem 2, we obtain:

Corollary 1. *Assume that M is a smooth locally strictly convex K-hypersurface and there holds*

$$K_M \leq (\operatorname{diam} M/2)^{-n},$$

then M does not satisfy **Condition A***; that is, each maximal domain $\overline{D}_i$, $1 \leq i \leq k$, meets ∂M.*

The second main task of this paper is to prove that, if Σ satisfies the hypotheses in Theorem 1, and if we assume, in addition, that Σ cannot globally be represented as the graph of any function, then we are able to construct a K-hypersurface $\widetilde{M}$ whose Gauss curvature $K_{\widetilde{M}}$ is slighter greater than $\inf K_\Sigma$. In order to prove this, it suffices, in view of Theorem 1, to establish Proposition 1 below.To put precisely, we let $\widehat{\mathbf{p}}_\ell \in \Sigma$, $\ell = 1, 2, \cdots, \widehat{k}$, be those vertices where K_Σ achieves the minimum value, i.e. $K(\widehat{\mathbf{p}}_\ell) = \inf_\Sigma K$, $1 \le \ell \le \widehat{k}$. Also, we let $\widehat{D}_\ell$ be the maximal domain on Σ which, as a hypersurface in $\mathbb{R}^{n+1}$, can be represented as the graph of the convex function $\widehat{u}_\ell$ defined in the domain $\widehat{\Omega}_\ell$, $1 \le \ell \le \widehat{k}$.

Proposition 1. *Suppose the hypersurface Σ satisfies the hypotheses of Theorem 1. Assume Σ is not a global graph and and K_Σ is not constant inside $\widehat{D}_\ell$ for any ℓ, $1 \le \ell \le \widehat{k}$. Then there exists a locally convex immersed hypersurface Σ_1 in $\mathbb{R}^{n+1}$ with $\partial\Sigma_1 = \Gamma$ and Gauss curvature $K_{\Sigma_1} > \inf K_\Sigma$ everywhere. Moreover, in a tubular neighborhood of its boundary Γ, Σ_1 is C^2 and locally strictly convex.*

From Proposition 1 and Theorem 1 we obtain the following result.

Theorem 3. *Suppose the hypersurface Σ satisfies the hypotheses of Proposition 1. Then there exists a number $K_1 > \inf K_\Sigma$ such that, for each number $0 < K < K_1$, there exists a smooth (up to the boundary) locally strictly convex hypersurface M with $\partial M = \Gamma$ and $K_M \equiv K$; moreover, M is homeomorphic to Σ.*

We will further improve Theorem 1 in case Σ satisfies **Condition B** below. We introduce:

Condition B. *For each ℓ, $1 \le \ell \le \widehat{k}$, the maximal domain $\widehat{D}_\ell$ lies in the interior of M.*

We shall show the following.

Proposition 2. *If the hypersurface Σ satisfies the hypotheses in Proposition 1 and* **Condition B**, *then there exists a locally convex immersed hypersurface Σ_2 in $\mathbb{R}^{n+1}$ with $\partial\Sigma_2 = \Gamma$ and $\inf K_{\Sigma_2} > \min_{1\le\ell\le\widehat{k}} (\mathrm{diam}\partial\widehat{D}_\ell/2)^{-n}$. Moreover, in a tubular neighborhood of its boundary Γ, Σ_2 is C^2 and locally strictly convex.*

From this and Theorem 1 we obtain:

Theorem 4. *Suppose the hypersurface Σ satisfies the hypotheses in Theorem 1 and* **Condition B**. *Then for each number K, $0 < K \le (\mathrm{diam}\Sigma/2)^{-n}$, there exists a locally convex immersed hypersurface M_1 in $\mathbb{R}^{n+1}$ with $\partial M_1 = \Gamma$ and the Gauss curvature $K_{M_1} \equiv K$. Moreover, in a tubular neighborhood of its boundary Γ, M_1 is C^2 and locally strictly convex.*

The key observation in proving Proposition 1 and Proposition 2 is that along $\partial \widehat{D}_\ell \setminus \Gamma$, the tangent hyperplane to Σ is vertical to the plane where $\widehat{\Omega}_\ell$ lies, and hence replacing $\widehat{D}_\ell$ by a graph "below" it while keeping $\Sigma \setminus \widehat{D}_\ell$ fixed we obtain another locally convex hypersurface.

2. Proofs of Theorems

2.1. Proof of Theorem 2.

We may first observe:

Lemma 1. *If M is a compact K-surface without boundary, then there holds*

$$K_M \geq (\,\mathrm{diam}\, M/2)^{-n}.$$

Indeed, let $\mathbf{a}$ and $\mathbf{b}$ be the points on M with $d :=\mathrm{dist}\,(\mathbf{a},\mathbf{b}) =\mathrm{diam}\, M$. Let $\mathbf{0}$ be the midpoint of the segment $\overline{\mathbf{ab}}$. Consider the ball $B := B_{d/2}(\mathbf{0})$ centered at $\mathbf{0}$ and of radius $d/2$, of which the segment $\overline{\mathbf{ab}}$ is a diameter. Then the sphere ∂B and the hypersurface M meet tangentially at the points $\mathbf{a}$ and $\mathbf{b}$. We treat two cases separetely.

Caes 1. M contacts ∂B from the inner side of $\overline{B}$ at $\mathbf{a}$ or $\mathbf{b}$; i.e. an open nighborhood of $\mathbf{a}$ or $\mathbf{b}$ on M lies in the inner side of $\overline{B}$. Therefore the Gauss curvature of M at $\mathbf{a}$ or $\mathbf{b}$ is greater than that of ∂B at $\mathbf{a}$ or $\mathbf{b}$, which is $(\,\mathrm{diam}\, M/2)^{-n}$.

Case 2. An open subset D_0 of M whose closure $\overline{D}_0$ contains $\mathbf{a}$ lies outside B. Since $d :=\mathrm{dist}\,(\mathbf{a},\mathbf{b})$ $=\mathrm{diam}\, M$, we know that some nonempty open subset of M lies in the interior of B. Therefore D_0 is included in a region D_0^* whose boundary ∂D_0^* is an $(n-2)$-dimensional closed subset of ∂B without boundary. A part of the region D_0^* and a part of ∂B including $\mathbf{p}$ can be respectively represented as the graphs of u_0 and a function u over a domain Ω_0^* such that $u_0 = u$ along $\partial\Omega_0^*$ and $u_0 < u$ in Ω_0^*. Were the Gauss curvature of D_0^* less than that of ∂B, the maximum principle would imply that $u_0 > u$ in Ω_0^*, which would not be the case. Therefore over some point $q \in \Omega_0^*$ the Gauss curvature of D_0 at $(q, u_0(q))$ is greater than that of ∂B at $(q, u(q))$. Thus again we conclude that $K_M \geq (\,\mathrm{diam}\, M/2)^{-n}$.

This result will not be used in the rest of this article. However, the reasoning which leads to this result will be used in the proof of Lemma 2 below, Proposition 1 in **2.2** and Proposition 2 in **2.3**.

Next we observe that the following result is essentially proved in the last paragraph of the proof of Theorem 3.5 in [3].

Proposition 3. *Assume that M is a smooth locally strictly convex K-hypersurface. Denote by $\kappa_{\max}[M]$ the maximum of all principal curvatures of M. If $\kappa_{\max}[M]$ is achieved at an interior point $\mathbf{p}$ of M, and we choose coordinates in $\mathbb{R}^{n+1}$ with origin at $\mathbf{p}$ such that the tangent hyperplane at $\mathbf{p}$ is given by $x_{n+1} = 0$ and M locally is written as a strictly convex graph $x_{n+1} = u(x_1, \cdots, x_n)$, then*

$$\kappa_{\max}(\mathbf{p}) \le C_0 K, \tag{1}$$

with

$$C_0 = (x^0_{n+1})^{n-1}; \tag{2}$$

here $\mathbf{x}^0 = (x^0_1, \cdots, x^0_n, x^0_{n+1}) \in \mathbb{R}^{n+1}$ is so chosen that the function $\widehat{\rho} := |\mathbf{x} - \mathbf{x}^0|$, $\mathbf{x} \in M$, achieves its local maximum value at $\mathbf{p}$.

Indeed, in the last paragraph of the proof of Theorem 3.5 in [3], this estimate of $\kappa_{\max}$ is obtained at a local maximum point of the function κe^{ρ}, the maximum being taken for all the normal curvatures κ over M, where $\rho = |\mathbf{x} - \mathbf{x}^0|^2$, $\mathbf{x} \in M$ and $\mathbf{x}^0 \in \mathbb{R}^{n+1}$ is a fixed point. However, in order to obtain an estimate of $\kappa_{\max}(\mathbf{p})$, the point $\mathbf{x}^0$ has to be so chosen that the function $\widehat{\rho} = |\mathbf{x} - \mathbf{x}^0|$, $\mathbf{x} \in M$, achieves its local maximum value at $\mathbf{p}$. Using the argument in [3] we are able to derive

$$0 \ge 2n\Big(\frac{\kappa_{\max}(\mathbf{p})}{K}\Big)^{\frac{1}{n-1}} - 2n x^0_{n+1},$$

from which follows (1). We notice that, in the fourth and fifth lines from the bottom in page 295 of [3], we should append the number n before the parentheses.

We are now able to formulate the following.

Corollary 2. *Under the hypotheses of Proposition 1 on M and $\mathbf{p}$, we have*

$$K = K(\mathbf{p}) \ge C_0^{-n/(n-1)},$$

where C_0 is the constant introduced in (2).

Indeed, from Proposition 1, we have

$$K(\mathbf{p}) = \kappa_1\kappa_2\cdots\kappa_n \leq (C_0K(\mathbf{p}))^n,$$

from which we obtain Corollary 1.

Instead of obtaining an estimate of the constant C_0, we make the following observation, from which and Corollary 2 we obtain Theorem 2.

Lemma 2. *Under the hypotheses of Proposition 1 on M and $\mathbf{p}$ and under* **Condition A** *with $\mathbf{p}_m = \mathbf{p}$, we have either*

$$C_0 \leq (\,\text{diam M}/2)^{n-1}, \tag{3}$$

or

$$K_M \geq (\,\text{diam M}/2)^{-n}. \tag{4}$$

Proof. As indicated in **Condition A**, $D_m \subset M$ is the maximal domain on M which can be represented as the graph of a convex function u_m defined in a domain Ω_m. Let P_m be the plane where Ω_m lies. We notice that the tangent hyperplane to M along ∂D_m is orthogonal to the plane P_m.

Let $\mathbf{a}$ and $\mathbf{b}$ be the points on ∂D_m such that $d_0 :=$dist$\,(\mathbf{a},\mathbf{b})$=diam$\,\partial D_m$. Let $\mathbf{0}$ be the midpoint of the segment $\overline{\mathbf{ab}}$, $d_1 :=$ dist$\,(\mathbf{0},\mathbf{p}_m)$ and $d := \max(d_1, d_0/2)$. Consider the ball $B := B_d(\mathbf{0})$ centered at $\mathbf{0}$ and of radius d. We treat two cases separately.

Case 1. If $d = d_0/2 \geq d_1$, then the segment $\overline{\mathbf{ab}}$ is a diameter of the ball B. Since the tangent hyperplane to M along ∂D_m is vertical to the plane P_m, we know that the sphere ∂B and the hypersurface M meet tangentially at the points $\mathbf{a}$ and $\mathbf{b}$. Since d_0 =(diam$\,\partial D_m)/2 \geq d_1$:=dist$\,(\mathbf{0},\mathbf{p}_m)$, an open subset of the boundary ∂D_m, together with the vertex $\mathbf{p}_m$, lies inside the ball $\overline{B}$. The reasoning leading to Lemma 1 can be applied here to conclude that one of the following holds:

(i) M contacts ∂B from the inner side of $\overline{B}$ at $\mathbf{a}$ or $\mathbf{b}$ and therefore (4) holds.

(ii) An open subset D_m^0 of D_m whose closure contains $\mathbf{a}$ lies outside B. Since $\mathbf{p}_m$ lies inside B, we know that D_m^0 is included in a region D_m^* whose boundary ∂D_m^* is an $(n-2)$-dimensional subset of ∂B without boundary. The reasoning in **Case 2** in the proof of Lemma 1 again enables us to conclude (4).

Case 2. If $d = d_1 \geq d_0/2$, then the sphere ∂B meets the hypersurface M tangentially at the point $\mathbf{p}_m$. We shall treat two possibilities separately.

(i) If the function $\widehat{\rho}_0 := |\mathbf{x} - \mathbf{0}|$, $\mathbf{x} \in M$, achieves its local maximum value at $\mathbf{p}_m$, then we are allowed to take $\mathbf{x}^0 = \mathbf{0}$ in Proposition 3, from which we obtain $|\mathbf{x}^0| = \sqrt{(x_1^0)^2 + \cdots + (x_{n+1}^0)^2} = d_1$ and hence (3).

(ii) If the function $\widehat{\rho}_0 = |\mathbf{x} - \mathbf{0}|$, $\mathbf{x} \in M$, fails to take its local maximum value at $\mathbf{p}_m$, then, since M meets ∂B tangentially, an open subset $\widehat{D}'_m$ of D_m whose closure contains $\mathbf{p}_m$ lies outside B. However, since $d_1 \geq d_0/2$, we know that some open subset of ∂D_m lies in the interior of B. Therefore $\widehat{D}'_m$ is included in a region $\widehat{D}''_m$ whose boundary $\partial \widehat{D}''_m$ is an $(n-2)$-dimensional subset of ∂B without boundary. The reasoning in **Case 2** in the proof of Lemma 1 again yields (4). $\diamondsuit$

2.2. Proof of Proposition 1 and Theorem 3.

We first recall the approach taken in [3]. Namely, according to [1], if Σ is the graph of a locally convex function $x_{n+1} = u(x)$ over a domain Ω in $\mathbb{R}^n$, then $K_\Sigma = K$ if and only if u is a viscosity solution of the Gauss curvature equation

$$\det(u_{ij}) = K(1 + |\nabla u|^2)^{\frac{n+2}{2}} \quad \text{in } \Omega. \tag{5}$$

A major difficulty in proving Theorem 1 lies in the lack of global coordinate systems to reduce the problem to solving certain boundary value problem for this Monge-Ampère type equation. To overcome the difficulty, Guan and Spruck [3] adopted a Perron method to deform Σ into a K-hypersurface by solving the Dirichlet problem for the equation (5) locally. They consider a disk on Σ which can be represented as the graph of a function and use the following existence result to replace such a disk by another graph of less curvature.

Lemma 3 (Theorem 1.1. [2], Theorem 2.1 [3]). *Let Ω be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^{0,1}$. Suppose there exists a locally convex viscosity subsolution $\underline{u} \in C^{0,1}(\overline{\Omega})$ of (5), i.e.*

$$\det(\underline{u}_{ij}) \geq K(1 + |\nabla \underline{u}|^2)^{\frac{n+2}{2}} \quad \text{in } \Omega, \tag{6}$$

where $K \geq 0$ is a constant. Then there exists a unique locally convex viscosity solution $u \in C^{0,1}(\overline{\Omega})$ of (5) satisfying $u = \underline{u}$ on $\partial\Omega$.

Motivated by the approach taken in [3], we now proceed to establish Proposition 1. We consider a disk on Σ which can be represented as the graph of a function and contains a point at which the Gauss curvature takes the value $\inf K_\Sigma$ and then, instead of using Lemma 3, we shall replace such a disk by a graph whose Gauss curvature is everywhere greater than $\inf K_\Sigma$. Namely, as introduced before, we let $\widehat{\mathbf{p}}_\ell \in \Sigma$, $\ell = 1, 2, \cdots, \widehat{k}$, be those vertices where K_Σ achieves the minimum value, i.e. $K(\widehat{\mathbf{p}}_\ell) = \inf_\Sigma K$, $1 \leq \ell \leq \widehat{k}$, and let $\widehat{D}_\ell$ be the maximal domain on Σ which, as a hypersurface in $\mathbb{R}^{n+1}$, can be represented as the graph of a convex function $\widehat{u}_\ell$ defined in a domain $\widehat{\Omega}_\ell$, $1 \leq \ell \leq \widehat{k}$. Then the tangent hyperplane to M along $\partial\widehat{D}_\ell \setminus \Gamma$ is vertical to the plane P_ℓ.

For $1 \leq \ell \leq \widehat{k}$, let $\widehat{\Omega}_{\ell,\delta}$ be the tubular neighborhood with width δ along $\partial\widehat{\Omega}_\ell$, i.e.

$$\widehat{\Omega}_{\ell,\delta} = \{x \in \Omega_\ell : \operatorname{dist}(x, \partial\widehat{\Omega}_\ell) \leq \delta\}.$$

We shall construct a convex function $\widetilde{u}_\ell$ defined over $\widehat{\Omega}_\ell$ with $\widetilde{u}_\ell = \widehat{u}_\ell$ along $\partial\widehat{\Omega}_\ell$ and $\widetilde{u}_\ell < \widehat{u}_\ell$ in $\widehat{\Omega}_{\ell,\delta} \setminus \partial\widehat{\Omega}_\ell$ for some $\delta > 0$. The graph of the function $\widetilde{u}_\ell$ over $\widehat{\Omega}_\ell$ is then a convex hypersurface $\widetilde{D}_\ell$. This naturally induces a $C^{0,1}$-diffeomorphism $\Psi_{\widetilde{\Sigma}} : \Sigma \to \widetilde{\Sigma} := \cup\widetilde{D}_\ell \cup (\Sigma \setminus \cup\widehat{D}_\ell)$ which is fixed on $\Sigma \setminus \cup\widehat{D}_\ell$. Since the tangent hyperplane to $\widetilde{D}_\ell$ along $\partial\widehat{D}_\ell \setminus \Gamma$ is vertical to the plane P_ℓ, $\widetilde{u}_\ell = \widehat{u}_\ell$ over $\partial\widehat{\Omega}_\ell$ and $\widetilde{u}_\ell < \widehat{u}_\ell$ in $\widehat{\Omega}_{\ell,\delta} \setminus \partial\widehat{\Omega}_\ell$, $1 \leq \ell \leq \widehat{k}$, we know that the hypersurface $\widetilde{\Sigma}$ is locally convex with $\partial\widetilde{\Sigma} = \partial\Sigma$.

In order to obtain the inequality $\inf K_{\widetilde{\Sigma}} > \inf K_\Sigma$, we choose the coordinate system with $\mathbf{p}_\ell = u_\ell(0, \cdots, 0)$, and then, letting $\widetilde{\mathbf{p}}_\ell = \widetilde{u}_\ell(0, \cdots, 0)$, we choose the function $\widetilde{u}_\ell$ to be strictly convex and to have $\inf K_{\widetilde{\Sigma}} = K_{\widetilde{\Sigma}}(\widetilde{\mathbf{p}}_\ell) > K_\Sigma(\mathbf{p}_\ell)$. For this, we observe that, since $K_\Sigma(\mathbf{p}_\ell) = \inf K_\Sigma < \sup_{\widehat{D}_\ell} K_\Sigma$, the equality $\inf K_{\widetilde{\Sigma}} = K_{\widetilde{\Sigma}}(\widetilde{\mathbf{p}}_\ell)$ can be achieved by choosing $v_\ell := \widehat{u}_\ell - \widetilde{u}_\ell$ defined over $\overline{\widehat{\Omega}_\ell}$ to be nonnegative and small enough. In order to obtain the strict convexity of $\widetilde{u}_\ell$, we make $v_\ell(x)$ strictly decreasing as the distance from x to $(0, \cdots, 0)$ increases. This also yields the inequality $K_{\widetilde{\Sigma}}(\widetilde{\mathbf{p}}_\ell) > K_\Sigma(\mathbf{p}_\ell)$. Indeed, let $\mathbf{e}_{n+1}$ be the unit vector pointing in the direction of positive x_{n+1} axis and move the surface $\widetilde{D}_\ell$ in the direction of $\mathbf{e}_{n+1}$ and in the distance $v_\ell(0, \cdots, 0)$ to obtain the parallel surface $\widetilde{D}_\ell + v_\ell(0, \cdots, 0)\mathbf{e}_{n+1}$, which is the graph of the function $\widetilde{u}_\ell(x) + v_\ell(0, \cdots, 0)$ inside $\widehat{\Omega}_\ell$. Because v_ℓ achieves its maximum value at $(0, \cdots, 0)$, the surface

$\widetilde{D}_\ell + v_\ell(0,\cdots,0)\mathbf{e}_{n+1}$ meets the surface $\widehat{D}_\ell$ tangentially at $\mathbf{p}_\ell$ and $\widetilde{u}_\ell(x) + v_\ell(0,\cdots,0) > u_\ell(x)$ inside $\widehat{\Omega}_\ell$. This yields the inequality $K_{\widetilde{\Sigma}}(\widetilde{\mathbf{p}}_\ell) = K_{\widetilde{D}_\ell}(\widetilde{\mathbf{p}_\ell}) = K_{\widetilde{D}_\ell+v_\ell(0,\cdots,0)\mathbf{e}_{n+1}}(\mathbf{p}_\ell) > K_{\widehat{D}_\ell}(\mathbf{p}_\ell) = K_\Sigma(\mathbf{p}_\ell)$. We therefore obtain Proposition 1 by taking $\Sigma_1 = \cup\widetilde{D}_\ell \cup (\Sigma \setminus \cup\widehat{D}_\ell)$, from which follows Theorem 3.

2.3. Proof of Proposition 2 and Theorem 4.

We now proceed to prove Proposition 2. It suffices to construct, for each ℓ, $1 \le \ell \le \widehat{k}$, a strictly convex hypersurface $\widetilde{D}_\ell$ with $\partial\widetilde{D}_\ell = \partial\widehat{D}_\ell$ and $\inf K_{\widetilde{D}_\ell} \ge (\operatorname{diam}\partial\widehat{D}_\ell/2)^{-n}$, for we can then take $\Sigma_2 = \cup\widetilde{D}_\ell\cup(\Sigma\setminus\cup\widehat{D}_\ell)$ to complete the proof of Proposition 2. For this purpose, we fix ℓ, $1 \le \ell \le \widehat{k}$. Let $\mathbf{a}_\ell$ and $\mathbf{b}_\ell$ be the points on $\partial\widehat{D}_\ell$ such that $d_\ell :=\operatorname{dist}(\mathbf{a}_\ell, \mathbf{b}_\ell) =\operatorname{diam}\partial\widehat{D}_\ell$. Let $\mathbf{0}_\ell$ be the midpoint of the segment $\overline{\mathbf{a}_\ell\mathbf{b}_\ell}$. Consider the ball $B_\ell := B_{d_\ell/2}(\mathbf{0}_\ell)$ centered at $\mathbf{0}_\ell$ and of radius $d_\ell/2$, of which the segment $\overline{\mathbf{a}_\ell\mathbf{b}_\ell}$ is a diameter. Since the tangent hyperplane to $\widehat{D}_\ell$ along $\partial\widehat{D}_\ell$ is vertical to the plane P_ℓ, the sphere ∂B_ℓ and the hypersurface $\widehat{D}_\ell$ meet tangentially at the points $\mathbf{a}_\ell$ and $\mathbf{b}_\ell$. We claim

Lemma 4. *The whole $\partial\widehat{D}_\ell$ lies inside $\overline{B}_\ell$.*

Proof. It suffices to claim that each curve which is cut from $\partial\widehat{D}_\ell$ by a plane containing $\mathbf{a}_\ell$ and $\mathbf{b}_\ell$ lies in $\overline{B}_\ell$. Indeed, consider such a curve Γ_0. Since $d_\ell :=\operatorname{dist}(\mathbf{a}_\ell, \mathbf{b}_\ell) =\operatorname{diam}\partial\widehat{D}_\ell$, an open subset $\widetilde{\Gamma}_0$ of Γ_0 lies in B_ℓ. Suppose another open subset of Γ_0 does not lie in B_ℓ. We shall derive respective contradictions in two cases below and finish the proof.

Case i. Suppose the curvature of Γ_0 is increasing from $\mathbf{a}_\ell$ to a point $\mathbf{c} \in \Gamma_0$ and then decreasing from $\mathbf{c}$ to $\mathbf{b}_\ell$. Then near $\mathbf{a}_\ell$ and $\mathbf{b}_\ell$ the curvature of Γ_0 is less than $(\operatorname{diam}\partial\widehat{D}_\ell/2)^{-1}$, and hence this part of Γ_0 lies outside B_ℓ. Since $\widetilde{\Gamma}_0$ lies in B_ℓ, Γ_0 intersects ∂B_ℓ at points $\mathbf{c}_1$ and $\mathbf{c}_2$ such that $\mathbf{a}_\ell$ is nearer to $\mathbf{c}_1$ than $\mathbf{c}_2$. The maximum principle produces two points with curvature greater than $(\operatorname{diam}\partial\widehat{D}_\ell/2)^{-1}$ one of which is between $\mathbf{a}_\ell$ and $\mathbf{c}_1$, and the other is between $\mathbf{b}_\ell$ and $\mathbf{c}_2$. Therefore the part of Γ_0 between $\mathbf{c}_1$ and $\mathbf{c}_2$, which lies inside B_ℓ, has curvature greater than $(\operatorname{diam}\partial\widehat{D}_\ell/2)^{-1}$ everywhere, contradicting the maximum principle.

Case ii. Suppose the curvature of Γ_0 is decreasing from $\mathbf{a}_\ell$ to a point $\mathbf{c}_0 \in \Gamma_0$ and then increasing from $\mathbf{c}_0$ to $\mathbf{b}_\ell$. We first claim that in this case near $\mathbf{a}_\ell$ and $\mathbf{b}_\ell$ the curve Γ_0 lies inside $\overline{B}_\ell$ and the curvatures of Γ_0 at $\mathbf{a}_\ell$ and $\mathbf{b}_\ell$ are greater than $(\operatorname{diam}\partial\widehat{D}_\ell/2)^{-1}$. Indeed, would a part of Γ_0 between $\mathbf{a}_\ell$ and some point $\mathbf{c}_3$ lie outside B_ℓ, then the maximum principle would produce a point with curvature greater than $(\operatorname{diam}\partial\widehat{D}_\ell/2)^{-1}$

in this part of Γ_0. Therefore the curvature at $\mathbf{a}_\ell$ would be greater than $(\operatorname{diam}\partial\widehat{D}_\ell/2)^{-1}$, contradicting the assumption that near $\mathbf{a}_\ell$ the curve Γ_0 lies outside B_ℓ. Hence near $\mathbf{a}_\ell$ the curve Γ_0 lies inside $\overline{B}_\ell$ and hence the curvature of Γ_0 at $\mathbf{a}_\ell$ is greater than $(\operatorname{diam}\partial\widehat{D}_\ell/2)^{-1}$. The behavior of the curve Γ_0 near $\mathbf{b}_\ell$ can be understood analogously.

If Γ_0 intersects ∂B_ℓ at some points $\mathbf{c}_4$ other than $\mathbf{a}_\ell$ and $\mathbf{b}_\ell$, then the part of Γ_0 between $\mathbf{c}_4$ and some other point $\mathbf{c}_5$ lies outside B_ℓ, which provides us with a point with curvature greater than $(\operatorname{diam}\partial\widehat{D}_\ell/2)^{-1}$ by the maximum principle. This implies that the part of Γ_0 between $\mathbf{a}_\ell$ and $\mathbf{c}_4$, which lies inside B_ℓ, has curvature greater than $(\operatorname{diam}\partial\widehat{D}_\ell/2)^{-1}$ everywhere, contradicting the maximum principle. $\diamondsuit$

To proceed further, we consider two cases separately.

Case I. The point $\mathbf{p}_\ell$ lies inside $\overline{B}_\ell$.

We proceed to prove the following.

Lemma 5. *In* **Case I**, *the whole* $\widehat{D}_\ell$ *lies in* $\overline{B}_\ell$.

Proof. Consider the plane $\widetilde{P}_\ell$ containing $\overline{\mathbf{a}_\ell\mathbf{b}_\ell}$ and the point $\mathbf{p}_\ell$. Let $\Gamma_\ell := \widetilde{P}_\ell \cap B_\ell$ and $\widehat{\Gamma}_\ell := \widetilde{P}_\ell \cap \widehat{D}_\ell$. We first observe that in Case I the curve $\widehat{\Gamma}_\ell$ in $\overline{D}_\ell$ lies inside $\overline{B}_\ell$; in other words, $\widehat{\Gamma}_\ell$ situates "above" Γ_ℓ. Indeed, would some part of $\widehat{\Gamma}_\ell$ lie outside $\overline{B}_\ell$, then we would, analogously to the proof of Lemma 4, derive respective contradictions in two cases. From this observation, Lemma 4 and the assumption that $\mathbf{p}_\ell \in \overline{B}_\ell$, we conclude that each curve in $\widehat{D}_\ell$ which is cut by a plane containing $\overline{\mathbf{0}_\ell\mathbf{p}_\ell}$ lies inside $\overline{B}_\ell$. This enables us to conclude that the whole $\widehat{D}_\ell$ lies in $\overline{B}_\ell$. $\diamondsuit$

In view of Lemma 5, it is easy to construct a $C^{0,1}$ convex surface $D_{0,\ell}$ passing through Γ_ℓ as well as $\partial\widehat{D}_\ell$, which situates "below" $\widehat{D}_\ell$ and "above" ∂B_ℓ in the sense that $D_{0,\ell}$ and a portion of ∂B_ℓ can be represented respectively as the graphs of functions $u_{0,\ell}$ and v_ℓ in $\widehat{\Omega}_\ell$ such that $v_\ell \le u_{0,\ell} \le \widehat{u}_\ell$ in $\widehat{\Omega}_\ell$. We may replace $\widehat{D}_\ell$ by $D_{0,\ell}$ while fixing $\Sigma \setminus \widehat{D}_\ell$. This provides us with a $C^{0,1}$ hypersurface $\widetilde{\Sigma}_0$. Since the tangent hyperplane to Σ along $\partial\widehat{D}_\ell$ is vertical to the plane P_ℓ, the hypersurface $\widetilde{\Sigma}_0$ is locally strictly convex. By approximation, we may assume without loss of generality that $D_{0,\ell}$ is C^2.

Let $\mathbf{p}_{0,\ell}$ be the "lowest" point of $D_{0,\ell}$. Each curve on $D_{0,\ell}$ which is cut by a plane containing $\overline{\mathbf{0}_\ell\mathbf{p}_{0,\ell}}$ lies in $\overline{B}_\ell$ and hence has the curvature at $\mathbf{p}_{0,\ell}$ greater than or equal to $(\operatorname{diam}\partial\widehat{D}_\ell/2)^{-1}$. Therefore the hypersurface

$\widetilde{\Sigma}_0$ has the Gauss curvature $K_{\widetilde{\Sigma}_0}(\mathbf{p}_{0,\ell}) \geq (\operatorname{diam}(\partial\widehat{D}_\ell))^{-n}$.

We now consider two possibilities separately.

(i) If $K_\Sigma(\mathbf{x}) > (\operatorname{diam}\partial\widehat{D}_\ell/2)^{-n}$ at each point $\mathbf{x} \in \partial\widehat{D}_\ell$, then by choosing $\widehat{u}_\ell - u_{0,\ell}$ small enough, there still holds $K_{D_{0,\ell}}(\mathbf{x}) > (\operatorname{diam}\partial\widehat{D}_\ell/2)^{-n}$ at each point $\mathbf{x} \in \partial\widehat{D}_\ell$. Then, since there holds also $K_{D_{0,\ell}}(\mathbf{p}_{0,\ell}) > (\operatorname{diam}\partial\widehat{D}_\ell/2)^{-n}$ and $D_{0,\ell}$ is C^2, we have $K_{D_{0,\ell}}(\mathbf{x}) > (\operatorname{diam}\partial\widehat{D}_\ell/2)^{-n}$ at every point $\mathbf{x} \in D_{0,\ell}$. Therefore in this case we take $\widetilde{D}_\ell = D_{0,\ell}$ to complete the proof of Proposition 2.

(ii) Suppose $K_\Sigma(\mathbf{x}) \leq (\operatorname{diam}\partial\widehat{D}_\ell/2)^{-n}$ at some points $\mathbf{x} \in \partial\widehat{D}_\ell$. Then we consider a small neighborhood of $\partial\widehat{D}_\ell$ on Σ

$$D_{\ell,\delta} = \{\mathbf{x} \in \Sigma; \operatorname{dist}(\mathbf{x}, \partial\widehat{D}_\ell) < \delta\}$$

and replace $D_{\ell,\delta}$ by a C^2 hypersurfacce $\widetilde{D}_{\ell,\delta}$ with $K_{\widetilde{D}_{\ell,\delta}} > (\operatorname{diam}\partial\widehat{D}_\ell/2)^{-n}$ everywhere and $\partial\widetilde{D}_{\ell,\delta} = \partial D_{\ell,\delta}$, while keeping $\Sigma \setminus D_{\ell,\delta}$ fixed. Let $\widetilde{D}^*_{\ell,\delta}$ be the largest region in $\widetilde{D}_{\ell,\delta}$ which can be represented as the graph of some function and has $\partial D_{\ell,\delta} \cap \widehat{D}_\ell$ as one component of its boundary. We then apply the previous construction to the hypersurface $\widetilde{D}^*_{\ell,\delta} \cup (\widehat{D}_\ell \setminus \widetilde{D}_{\ell,\delta})$, instead of $\widehat{D}_\ell$, and obtain the desired hypersurface $\widetilde{D}_\ell$ to complete the proof of Proposition 2.

Case II. The point $\mathbf{p}_\ell$ lies outside B_ℓ.

In this case, to prove Proposition 2 it suffices to prove the following lemma and then take $\widetilde{D}_\ell = \widehat{D}_\ell$.

Lemma 6. *In* **Case II**, *the Gauss curvature $K_\Sigma(\mathbf{p}_\ell)$ of Σ at $\mathbf{p}_\ell$ is greater than $(\operatorname{diam}\partial D_\ell/2)^{-n}$ at $\mathbf{p}_\ell$.*

Indeed, in this case we choose the coordinate system whose origin 0 is at $\mathbf{p}_\ell$ and whose x_{n+1}-axis points in the normal direction of D_ℓ from $\mathbf{p}_\ell$ to ∂B_ℓ. Then a portion of $\widehat{D}_\ell$ and a portion of ∂B_ℓ can be represented as the graphs of functions $\widetilde{u}$ and $\widetilde{v}$ respectively over a neighborhood E of 0. Consider the nonnegative function $w := \widetilde{v} - \widetilde{u}$ over E. In view of Lemma 4, the function w achieves its maximum value at 0. We now use the reasoning used at the last paragraph in the proof of Proposition 1. Namely, Let $\mathbf{e}_{n+1}$ be the unit vector in the direction of the x_{n+1}-axis. By moving the hypesurface $\widehat{D}_\ell$ in the direction of $\mathbf{e}_{n+1}$ and in the distance of $w(0)$, we obtain the parallel hypersurface $\widehat{D}_\ell + w(0)\mathbf{e}_{n+1}$, which meets ∂B_ℓ tangentially at $\mathbf{p}_{0,\ell}$

and has greater curvature than ∂B_ℓ at $\mathbf{p}_\ell + w(0)\mathbf{e}_{n+1}$. That is, $K_\Sigma(\mathbf{p}_\ell) = K_{\widehat{D}_\ell + w(0)\mathbf{e}_{n+1}}(\mathbf{p}_\ell + w(0)\mathbf{e}_{n+1}) > K_{\partial B_\ell}(\mathbf{p}_\ell + w(0)\mathbf{e}_{n+1})$. Since $K_{\widehat{D}_\ell}(\mathbf{p}_\ell) = \inf K_{\widehat{D}_\ell}$, we conclude that $K_\Sigma \geq (\mathrm{diam}\,\partial\widehat{D}_\ell/2)^{-n}$. $\diamondsuit$

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