Global and Stochastic Stability of SIQR epidemic model and the stability of the model with age

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October 7, 2015

Abstract

In this paper, addresses a time-delayed epidemiologic model by experiencing the disease; whenever the quarantine will return to the susceptible. First, the equilibrium and global stabilities of the endemic equilibrium. Second, Stochastic Stability. Finally, the equilibrium and stability of the epidemic model with age.

Keywords: Epidemic model, Global asymptotic stability, Lyaponov functional, Stochastic Stability.

1 Introduction

This paper considers the following epidemic model with temporary immunity:

$$\begin{cases} \dot{S}(t) = \rho - (\mu_1 + d) S(t) - \beta S(t)Q(t), \\ \dot{I}(t) = \beta S(t)Q(t) - (\mu_2 + d) I(t) - \gamma e^{-\mu_2 \tau} S(t-\tau)Q(t-\tau) + \nu, \\ \dot{Q}(t) = \gamma e^{-\mu_2 \tau} S(t-\tau)Q(t-\tau) - (\mu_3 + d + \delta) Q(t), \\ \dot{R}(t) = \delta Q(t) - (\mu_4 + d) R(t) \end{cases} \tag{1}$$

Consider a population of size $N(t)$ at time t, this population is divided into for subclasses, with $N(t) = S(t) + I(t) + Q(t) + R(t)$; where $S(t), I(t), Q(t)$ and $R(t)$ denote the sizes of the population susceptible to disease, infectious members, quarantine members with the possibility of infection through temporary immunity, and who were removed from the possibility of infection respectively. It is assumed that all new borns are susceptible.

The positive constants μ_1, μ_2, μ_3 and μ_4 represent the death rates of susceptible, infectious, quarantine and removed. Biologically, it is natural to assume that $\mu_1 \leq \min\{\mu_2, \mu_3, \mu_4\}$. The positive constant d is natural mortality rate. The positive constant γ represent the removed rate of infection. The positive constant β is the average numbers of contacts infective for S and I. ρ the positive constant is the parameter of represent the birth rate (from insidence) of the population. ν the positive constant is the parameter of emigration. The term $\gamma e^{-\mu_2\tau}S(t-\tau)Q(t-\tau)$ reflects the fact that an individual has recovered from infection and still are alive after infectious period τ, where τ is the length of immunity period.

The initial condition of (1) is given as.

$$S(\eta)=\Phi_1(\eta),\ I(\eta)=\Phi_2(\eta),\ Q(\eta)=\Phi_3(\eta),R(\eta)=\Phi_4(\eta),\qquad -\tau\leq\eta\leq 0, \tag{2}$$

Where $\Phi=(\Phi_1,\Phi_2,\Phi_3,\Phi_4)^T\in\mathbb{C}$ such that $S(\eta)=\Phi_1(\eta)=\Phi_1(0)\geq 0$, $I(\eta)=\Phi_2(\eta)=\Phi_2(0)\geq 0$, $Q(\eta)=\Phi_3(\eta)=\Phi_3(0)\geq 0$, $R(\eta)=\Phi_4(\eta)=\Phi_4(0)\geq 0$

Let C denote the Banach space $C([-\tau,0],\mathbb{R}^4)$ of continuous functions mapping the interval $[-\tau,0]$ into $\mathbb{R}^4$. With a biological meaning, we further assume that $\Phi_i(\eta)=\Phi_i(0)\geq 0$ for $i=1,\ 2,\ 3,\ 4$.

Consider the system without the parameter of emigrations. Hence system (1) can be rewritten as

$$\begin{cases}\dot{S}(t)=\rho-(\mu_1+d)S(t)-\beta S(t)Q(t),\\ \dot{I}(t)=\beta S(t)Q(t)-(\mu_2+d)I(t)-\gamma e^{-\mu_2\tau}S(t-\tau)Q(t-\tau),\\ \dot{Q}(t)=\gamma e^{-\mu_2\tau}S(t-\tau)Q(t-\tau)-(\mu_3+d+\delta)Q(t),\\ \dot{R}(t)=\delta Q(t)-(\mu_4+d)R(t)\end{cases} \tag{3}$$

With the same initial conditions in (2), where $\Phi_i(\eta)=\Phi_i(0)\geq 0$ for $i=1,\ 2,\ 3,\ 4$.

Since $\dot{N}(t)\leq\rho-(\mu_1+d)N(t)$, and $S(t)+I(t)+Q(t)+R(t)\leq N(t)$.

The region Ω is positively invariant set of (3).

$$\Omega=\{(S,I,Q,R)\in\mathbb{R}_+^4, S+I+Q+R\leq N<\frac{\rho}{\mu_1+d}\} \tag{4}$$

This paper deals with the equilibrium and stability of system (3), precisely the global stability of endemic equilibrium by using Lyapunov functional technique, under certain conditions on the parameter this means that the disease persist in population.

Next, we introduce a Brownian motion to system (3) and we transform it into an Itô stochastic differential equation by using the contact rate a white noise. Finally study equilibrium of epidemic model with age.

The organization of this paper is as follows, in Section 1, Equilibrium and stability

of the model. In Section 2, Global asymptotic stability of endemic equilibrium. In

Section 3, stochastic stability. In Section 4, the model with age

2 Equilibrium and stability

An equilibrium point of system (3) satisfies

$$\begin{cases} \rho - (\mu_1 + d)\, S - \beta SQ = 0, \\ \beta SQ - (\mu_2 + d)\, I - \gamma e^{-\mu_2 \tau} S(t-\tau)Q(t-\tau) = 0, \\ \gamma e^{-\mu_2 \tau} S(t-\tau)Q(t-\tau) - (\mu_3 + d + \delta)\, Q = 0, \end{cases} \tag{5}$$

We calculate the points of equilibrium in the absence and presence of infection.

In the absence of infection $I = 0$, the system (5) has a disease-free equilibrium E_0:

$$E_0 = \left(\hat{S},\ \hat{I},\ \hat{Q}\right)^T = \left(\frac{\rho}{\mu_1 + d},\ 0,\ 0\right)^T.$$

The eigenvalues can be determined by solving the characteristic equation of the linearization of (3) near E_0 is

$$\det \begin{pmatrix} -(\mu_1 + d) - A & 0 & -\frac{\beta\rho}{\mu_1+d} \\ 0 & -(\mu_2 + d) - A & \frac{\rho\left(\beta - \gamma e^{-\mu_2\tau}\right)}{\mu_1+d} \\ 0 & 0 & \frac{\rho\gamma e^{-\mu_2\tau}}{\mu_1+d} - (\mu_3 + d + \delta) - A \end{pmatrix} = 0 \tag{6}$$

So the eigenvalues are

$$A_1 = -(\mu_1 + d)\,,\ A_2 = -(\mu_2 + d)\,.$$

In order for λ_1, λ_2, to be negative, it is required that.

$$\frac{\rho\gamma e^{-\mu_2\tau}}{\mu_1 + d} < (\mu_3 + d + \delta) \tag{7}$$

Then we define the basic reproduction number of the infection R_0 as follows.

$$R_0 = \frac{\rho\gamma e^{-\mu_2\tau}}{(\mu_1 + d)\,(\mu_3 + d + \delta)} \tag{8}$$

In the presence of infection $I \neq 0$, substituting in the system, Ω also contains a unique positive, endemic equilibrium $E_\tau^* = (S_\tau^*,\ I_\tau^*,\ Q_\tau^*)^T$ where

$$\begin{cases} S_\tau^* = \frac{\rho}{\mu_1+d} \times \frac{1}{R_0}, \\ I_\tau^* = \frac{R_0 - 1}{\mu_2+d}\left[\frac{\rho}{R_0} - \frac{(\mu_1+d)(\mu_3+d+\delta)}{\beta}\right], \\ Q_\tau^* = \frac{\mu_1+d}{\beta}\,(R_0 - 1) \end{cases} \tag{9}$$

So $E_\tau^* = (S_\tau^*,\ I_\tau^*,\ Q_\tau^*)^T$ is the unique positive endemic equilibrium point which exists if $R_0 > 1$.

Theorem 1 *The disease-free equilibrium E_0 is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.*

Theorem 2 *With $R_0 > 1$, system (3) has a unique non-trivial equilibrium $E_\tau^*$is locally asymptotically stable.*

3 Global asymptotic stability of endemic equilibrium

Consider system (3), with introducing the variables,

$$x(t) = S(t) - S^*_\tau, \qquad y(t) = I(t) - I^*_\tau, \qquad z(t) = Q(t) - Q^*_\tau, \tag{10}$$

System (3) is centered at the endemic equilibrium $E^*_\tau = (S^*_\tau,\ I^*_\tau,\ Q^*_\tau)^T$, then

$$\begin{cases} \dot{x}(t) = [-(\mu_1 + d) - \beta Q^*_\tau]\, x + [-\beta S^*_\tau]\, z, \\ \dot{y}(t) = [(\beta - \gamma e^{-\mu_2 \tau})\, Q^*_\tau]\, x + [-(\mu_2 + d)]\, y + [(\beta - \gamma e^{-\mu_2 \tau})\, S^*_\tau]\, z, \\ \dot{z}(t) = [\beta \gamma e^{-\mu_2 \tau} Q^*_\tau]\, x + [\gamma e^{-\mu_2 \tau} S^*_\tau - (\mu_3 + d + \delta)]\, z \end{cases} \tag{11}$$

Lemma 3 *Let*

$$S(s) = S(0) > 0,\ I(s) = I(0) \geq 0 \textit{ for all } s \in [-\tau, 0] \textit{ and } Q(0) > 0.$$

$S(t)$, $I(t)$ and $Q(t)$ solutions of system (3) are positive for all $t > 0$.

Proof. For contraduction there exists the first time t_0, such that $S(t_0)\, Q(t_0) = 0$.

- Assume that $S(t_0) = 0$, then $Q(t) \geq 0$ for all $t \in [0, t_0]$.With Eq 1 in the system (1) we have
$$\dot{S}(t_0) = \rho > 0.$$
For $S(t_0) = 0$, $S_0 > 0$, we must have $\dot{S}(t_0) < 0$ which is contradiction.
- Assume that $I(t_0) = 0$, then with Eq 2 in the system (1) we have
$$\dot{I}(t_0) = -\gamma e^{-\mu_2 \tau} S(t - \tau) Q(t - \tau)$$
$\dot{I}(t_0)$ is positive because $S(t)$ and $Q(t)$ solutions of system (1)
are positive for all $t > 0$.
- For $I(t_0) = 0$, $I > 0$, we must have $\dot{I}(t_0) < 0$ which is contradiction.
- Assume that $Q(t_0) = 0$, then $S(t) \geq 0$ for all $t \in [0, t_0]$. with Eq 3 in the system (3) we have
$$\begin{aligned} \dot{Q}(t_0) &= \gamma e^{-\mu_2 \tau} S(t - \tau) Q(t - \tau), \\ Q(t_0) &= \gamma \int_{t_0 - \tau}^{t_0} e^{-\mu_2 (t_0 - s)} S(s) Q(s) ds. \end{aligned}$$
$S(s) > 0, S(s) > 0$ for all $t \in [0, t_0)$. We have $\gamma \int_{t_0-\tau}^{t_0} e^{-\mu_2 (t_0 - s)} S(s) Q(s) ds > 0$, and $Q(t_0) = 0$, which is contradiction.

■

Lemma 4 *Let*

$$S(s) = S_0 > 0,\ Q(s) = Q_0 > 0 \text{ for all } s \in [-\tau, 0] \text{ and } Q_0 > 0.$$

Then

$$S(t) \le \max\left\{\frac{\rho}{\mu_1 + d}, S_0 + I_0 + Q_0\right\} = M$$

Proof. We have

$$N(t) = S(t) + I(t) + Q(t),$$

For $R_0 < 1$ the solutions $S(t), I(t)$ and $Q(t)$ approach the disease free equilibrium as $t \to \infty$.

With Eq 2 in the system (3) we have $\dot{I} \le -(\mu_2 + d) I$, hence if $\mu_2 + d < 0$,

$$\lim_{t\to\infty} I(t) = 0,$$

With Eq 3 in the system (3) we have.

$$\lim_{t\to\infty} Q(t) = 0,$$

With Eq 1 in the system (3) we obtain $\dot{S} = \rho - (\mu_1 + d) S$.

$$\lim_{t\to\infty} S(t) = \frac{\rho}{\mu_1 + d},$$

Hence.

$$\lim_{t\to\infty} N(t) = \frac{\rho}{\mu_1 + d}.$$

From lemma1, $S(t)$, $I(t)$ and $Q(t)$ solutions of system (1) are positive.

$$S(t) \le \frac{\rho}{\mu_1 + d}, \text{for all} t \le 0.$$

Suppose that

$$N(0) \le \frac{\rho}{\mu_1 + d},\ \text{then } N(t) \le \frac{\rho}{\mu_1 + d}$$

On the contrary

If $N(0) > \frac{\rho}{\mu_1 + d}$ then $N(t) < N(0)$, and $S(t) < N(0)$ for all $t > 0$. ■

Theorem 5 *Let $S(s) = S_0 > 0$, $Q(s) = Q_0 > 0$ for all $s \in [-\tau, 0]$ and $Q_0 > 0.E^*_\tau$ is globally asymptotically stable for all τ*

$$\tau > \max\left\{\begin{array}{c} \frac{1}{\gamma}\ln\frac{\omega M + 3\omega Q^*_\tau}{2\omega(\mu_1 + d)} \\ , \frac{1}{\gamma}\ln\frac{\omega M + 3\omega Q^*_\tau}{2\omega(\mu_2 + d) + (\mu_2 + d) - \beta M}, \\ \frac{1}{\gamma}\ln\frac{Q^*_\tau - 3\omega M}{2(\mu_3 + d + \delta) - \beta M} \end{array}\right\}$$

Where

$$
\begin{aligned}
M &= \max\left\{\frac{\rho}{\mu_1+d}, S_0+I_0+Q_0\right\}, \\
\omega &= \frac{\beta Q_\tau^*}{\mu_1+\mu_2+2d}
\end{aligned}
$$

Proof. We consider system (3).

Let us introduce the functional

$$
V(x,y,z)=\frac{1}{2}\omega(x+y)^2+\frac{1}{2}\left(y^2+z^2\right),
$$

The derivative $\dot{V}(x,y,z)$ is

$$
\begin{aligned}
\dot{V}(x,y,z) &= \omega(x+y)(\dot{x}+\dot{y})+y\dot{y}+z\dot{z} \\
&= \omega(x+y)\left[\left(-(\mu_1+d)-\beta Q_\tau^*\right)x-\beta S_\tau^* z+\left(\beta-\gamma e^{-\mu_2\tau}\right)Q_\tau^* x-(\mu_2+d)y+\left(\beta-\gamma e^{-\mu_2\tau}\right)S_\tau^* z\right] \\
&\quad +y\left[\left(\beta-\gamma e^{-\mu_2\tau}\right)Q_\tau^* x-(\mu_2+d)y+\left(\beta-\gamma e^{-\mu_2\tau}\right)S_\tau^* z\right]+ \\
&\quad +z\left[\beta\gamma e^{-\mu_2\tau}Q_\tau^* x+\left(\gamma e^{-\mu_2\tau}S_\tau^*-(\mu_3+d+\delta)\right)z\right] \\
&= -\omega(\mu_1+d)x^2-\left[(\omega+1)(\mu_2+d)\right]y^2-\left(\gamma e^{-\mu_2\tau}S_\tau^*-(\mu_3+d+\delta)\right)z^2 \\
&\quad +\left[\beta Q_\tau^*-\omega(\mu_1+d)-\omega(\mu_2+d)\right]xy+\beta S_\tau^* yz-\left[\omega Q_\tau^*\gamma e^{-\mu_2\tau}\right]xx(t-\tau) \\
&\quad -(\omega+1)Q_\tau^*\gamma e^{-\mu_2\tau}yx(t-\tau)+Q_\tau^*\gamma e^{-\mu_2\tau}zx(t-\tau)-\omega S_\tau^*\gamma e^{-\mu_2\tau}xz(t-\tau) \\
&\quad -(\omega+1)S_\tau^*\gamma e^{-\mu_2\tau}yz(t-\tau)+S_\tau^*\gamma e^{-\mu_2\tau}zz(t-\tau).
\end{aligned}
$$

By lemma 2 we have $S(t)\le M$ for all $t\ge 0$ and ω is an arbitrary real constant choosing as follows

$$
\omega=\frac{\beta Q_\tau^*}{\mu_1+\mu_2+2d}
$$

$$
\begin{aligned}
\dot{V}(x,y,z) &\le -\omega(\mu_1+d)x^2-\left[(\omega+1)(\mu_2+d)\right]y^2-(\mu_3+d+\delta)z^2 \\
&\quad +\beta Myz-\left[\omega Q_\tau^*\gamma e^{-\mu_2\tau}\right]xx(t-\tau)-(\omega+1)Q_\tau^*\gamma e^{-\mu_2\tau}yx(t-\tau) \\
&\quad +Q_\tau^*\gamma e^{-\mu_2\tau}zx(t-\tau)-\omega M\gamma e^{-\mu_2\tau}xz(t-\tau) \\
&\quad -(\omega+1)M\gamma e^{-\mu_2\tau}yz(t-\tau)+M\gamma e^{-\mu_2\tau}zz(t-\tau).
\end{aligned}
$$

Applying Cauchy-Chwartz inequality; we obtain:

$$\begin{aligned}
\dot{V}(x,y,z) \leq\ & -\omega(\mu_1+d)x^2-[(\omega+1)(\mu_2+d)]y^2-(\mu_3+d+\delta)z^2 \\
& -\frac{1}{2}\omega Q_\tau^*\gamma e^{-\mu_2\tau}\left[x^2+x^2(t-\tau)\right]-\frac{1}{2}(\omega+1)Q_\tau^*\gamma e^{-\mu_2\tau}\left[y^2+x^2(t-\tau)\right] \\
& +\frac{1}{2}Q_\tau^*\gamma e^{-\mu_2\tau}\left[z^2+x^2(t-\tau)\right]-\frac{1}{2}\omega M\gamma e^{-\mu_2\tau}\left[x^2+z^2(t-\tau)\right] \\
& -\frac{1}{2}(\omega+1)M\gamma e^{-\mu_2\tau}\left[y^2+z^2(t-\tau)\right]+\frac{1}{2}M\gamma e^{-\mu_2\tau}\left[z^2+z^2(t-\tau)\right] \\
& +\frac{1}{2}\beta M\left[y^2+z^2\right], \\
\leq\ & \left[-\omega(\mu_1+d)-\frac{1}{2}\omega\gamma e^{-\mu_2\tau}(M+Q_\tau^*)\right]x^2 \\
& +\left[\frac{1}{2}\beta M-(\omega+1)(\mu_2+d)-\frac{1}{2}(\omega+1)\gamma e^{-\mu_2\tau}(M+Q_\tau^*)\right]y^2 \\
& +\left[\frac{1}{2}\beta M-(\mu_3+d+\delta)+\frac{1}{2}\gamma e^{-\mu_2\tau}(M+Q_\tau^*)\right]z^2 \\
& -\left[\omega Q_\tau^*\gamma e^{-\mu_2\tau}\right]x^2(t-\tau)-\left[\frac{1}{2}(3\omega+1)\gamma M e^{-\mu_2\tau}\right]z^2(t-\tau)
\end{aligned}$$

Choose the Lyapunov functional

$$V(x_t,y_t,z_t)=V(x,y,z)-\omega Q_\tau^*\gamma e^{-\mu_2\tau}\int_{t-\tau}^{t}x^2(\theta)d\theta-\frac{1}{2}(3\omega+1)\gamma M e^{-\mu_2\tau}\int_{t-\tau}^{t}z^2(\theta)d\theta$$

Then

$$
\begin{aligned}
\dot{V}\left(x_t, y_t, z_t\right) &= \dot{V}(x, y, z) - \omega Q_\tau^* \gamma e^{-\mu_2 \tau} x^2(t) + \omega Q_\tau^* \gamma e^{-\mu_2 \tau} x^2(t-\tau) \\
&\quad - \frac{1}{2}(3\omega+1)\gamma M e^{-\mu_2\tau} z^2(t) + \frac{1}{2}(3\omega+1)\gamma M e^{-\mu_2\tau} z^2(t-\tau) \\
&\leq \left[-\omega\left(\mu_1+d\right) - \frac{1}{2}\omega\gamma e^{-\mu_2\tau}\left(M+Q_\tau^*\right)\right] x^2 \\
&\quad + \left[\frac{1}{2}\beta M - (\omega+1)\left(\mu_2+d\right) - \frac{1}{2}(\omega+1)\gamma e^{-\mu_2\tau}\left(M+Q_\tau^*\right)\right] y^2 \\
&\quad + \left[\frac{1}{2}\beta M - \left(\mu_3+d+\delta\right) + \frac{1}{2}\gamma e^{-\mu_2\tau}\left(M+Q_\tau^*\right)\right] z^2 \\
&\quad - \left[\omega Q_\tau^* \gamma e^{-\mu_2\tau}\right] x^2 + \omega Q_\tau^* \gamma e^{-\mu_2\tau} x^2(t-\tau) \\
&\quad - \left[\frac{1}{2}(3\omega+1)\gamma M e^{-\mu_2\tau}\right] z^2 + \frac{1}{2}(3\omega+1)\gamma M e^{-\mu_2\tau} z^2(t-\tau) \\
&\leq \left[-\omega\left(\mu_1+d\right) - \frac{1}{2}\omega\gamma e^{-\mu_2\tau}\left(M+Q_\tau^*\right)\right] x^2 \\
&\quad + \left[\frac{1}{2}\beta M - (\omega+1)\left(\mu_2+d\right) - \frac{1}{2}(\omega+1)\gamma e^{-\mu_2\tau}\left(M+Q_\tau^*\right)\right] y^2 \\
&\quad + \left[\frac{1}{2}\beta M - \left(\mu_3+d+\delta\right) + \frac{1}{2}\gamma e^{-\mu_2\tau}\left(M+Q_\tau^*\right)\right] z^2 \\
&\quad - \left[\omega Q_\tau^* \gamma e^{-\mu_2\tau}\right] x^2 - \left[\frac{1}{2}(3\omega+1)\gamma M e^{-\mu_2\tau}\right] z^2
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\dot{V}\left(x_t, y_t, z_t\right) &\leq -\left[\omega\left(\mu_1+d\right) + \frac{1}{2}\omega\gamma e^{-\mu_2\tau}\left(M+3Q_\tau^*\right)\right] x^2 \\
&\quad - \left[(\omega+1)\left(\mu_2+d\right) - \frac{1}{2}\beta M + \frac{1}{2}(\omega+1)\gamma e^{-\mu_2\tau}\left(M+Q_\tau^*\right)\right] y^2 \\
&\quad - \left[\left(\mu_3+d\right) - \frac{1}{2}\beta M - \frac{1}{2}\gamma\left(Q_\tau^*+M\right) + \omega M\gamma e^{-\mu_2\tau}\right] z^2 .
\end{aligned}
$$

While the above inequality is always negative provided that

$$
\tau > \max\left\{\begin{array}{c}
\frac{1}{\gamma}\ln\frac{M+3Q_\tau^*}{2(\mu_1+d)}, \\
\frac{1}{\gamma}\ln\frac{(\omega+1)(M+Q_\tau^*)}{\beta M-2(\omega+1)(\mu_2+d)}, \\
\frac{1}{\gamma}\ln\frac{23\omega M}{(M+Q_\tau^*)+\beta M-2(\mu_3+d)}
\end{array}\right\}
$$

With application of the Lyapunove-LaSalle type theorem in [10]

$$
\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} y(t) = 0, \quad \lim_{t\to\infty} z(t) = 0.
$$

■

4 Stochastic Stability

We limit ourselves here to perturbing only the contact rate so we replace β by $\beta + \sigma W(t)$,

where$W(t)$ is white noise (Brownian motion). The system (3) is transformed to the following Itô stochastic differential equations, with $\gamma_0 = \gamma e^{-\mu_2 \tau}$

$$\begin{cases} dS = [\rho - (\mu_1 + d)\,S - \beta SQ] - \sigma SQdW, \\ dI = [\beta SQ - (\mu_2 + d)\,I - \gamma_0 SQ] + \sigma SQdW, \\ dQ = [\gamma_0 SQ - (\mu_3 + d)\,Q]\,, \end{cases} \tag{12}$$

In this section, we will proof, under some conditions, that E_0 is globally exponentially mean square and almost surely stable, and for this purpose, we need the following Theorem

Theorem 6 *The set Ω is almost surely invariant by the stochastic system* (12). *Thus if $(S_0, I_0, Q_0) \in \Omega$, then $P\ [(S, I, Q) \in \Omega] = 1$.*

Proof. The system (12) implies that $dN \leq [\rho - (\mu_1 + d)\,N]\,dt$, then we have

$$N\,(t) \leq \frac{\rho}{\mu_1 + d} + \left(N_0 - \frac{\rho}{\mu_1 + d}\right), \text{ for all } t \geq 0.$$

Since $(S_0, I_0, Q_0) \in \Omega$, then

$$N\,(t) \leq \frac{\rho}{\mu_1 + d}, \text{ for all } t \geq 0. \tag{13}$$

There exist $\varepsilon_0 > 0$, such that $S_0 > \varepsilon_0 > 0$, $I_0 > \varepsilon_0 > 0$ and $Q_0 > \varepsilon_0 > 0$. Considering

$$\begin{aligned} \upsilon_\varepsilon &= \inf\{t \geq 0,\ S\,(t) \leq \varepsilon \text{ or } I\,(t) \leq \varepsilon \text{ or } Q\,(t) \leq \varepsilon, \}\,, \text{ for } \varepsilon \leq \varepsilon_0, \quad (14) \\ \upsilon &= \lim_{t \to 0} \upsilon_\varepsilon = \inf\{t \geq 0,\ S\,(t) \leq 0 \text{ or } I\,(t) \leq 0 \text{ or } Q\,(t) \leq 0, \} \end{aligned}$$

Let

$$V\,(t) = \log \frac{\rho}{(\mu_1 + d)\,S\,(t)} + \log \frac{\rho}{(\mu_1 + d)\,I\,(t)} + \log \frac{\rho}{(\mu_1 + d)\,Q\,(t)}.$$

Then, using Itô formula we have, for all $t \geq 0$ and $T \in [0, t \wedge \upsilon_\varepsilon]$,

$$\begin{aligned} dV\,(T) &= \left[-\frac{\rho}{S} + (\mu_1 + d) + \beta Q + \frac{1}{2} I^2\right] dT + \sigma Q dW \\ &\quad + \left[(\gamma_0 - \beta)\,\frac{SQ}{I} + (\mu_2 + d) + \frac{1}{2} S^2\right] dT + \sigma \frac{SQ}{I} dW \\ &\quad + [-\gamma_0 S + (\mu_3 + d)]\,dT, \end{aligned}$$

$$dV\,(T) \leq \left[\mu_1 + \mu_2 + \mu_3 + 3d + \beta Q + \frac{1}{2} I^2 + \frac{1}{2} S^2\right] dT + \sigma \frac{Q}{I}\,(I - S)\,dW \tag{15}$$

With (13), we have S, I and $Q \in \left[0, \frac{\rho}{(\mu_1+d)}\right]$
Let

$$L = \mu_1 + \mu_2 + \mu_3 + 3d + \beta\frac{\rho}{\mu_1+d} + \left(\frac{\rho}{\mu_1+d}\right)^2, \quad (16)$$
$$f(I) = \frac{Q}{I},$$

We remplace (16) into (15), we obtain

$$dV(T) \le LdT + \sigma(I(T) - S(T)) f(I(T)) dW, \quad (17)$$

Then

$$V(T) \le LT + \sigma\int_0^T f(I(u))(I(u) - S(u)) dW(u), \quad (18)$$

With proposition 7.6 in [6], $\sigma\int_0^T f(I(u))(I(u) - S(u)) dW(u)$ is mean zero process then,

$$E(V(T)) \le LT \quad (19)$$

for all $t \ge 0$ and $T \in [0, t \wedge \upsilon_\varepsilon]$,

$$S(t \wedge \upsilon_\varepsilon), I(t \wedge \upsilon_\varepsilon), \text{ and } Q(t \wedge \upsilon_\varepsilon) \in \left[0, \frac{\rho}{(\mu_1+d)}\right],$$

Then

$$E(V(t \wedge \upsilon_\varepsilon)) \le L(t \wedge \upsilon_\varepsilon) \le Lt,$$

$V(t \wedge \upsilon_\varepsilon) \ge 0$,

$$E(V(t \wedge \upsilon_\varepsilon)) \ge E(V(t)) \times \varkappa_{[\upsilon_\varepsilon \le t]} \ge P(\upsilon_\varepsilon \le t)\log\frac{\rho}{(\mu_1+d)\varepsilon} \quad (20)$$

Where $\varkappa_{[\upsilon_\varepsilon \le t]}$ is the indicator function of a subset $[\upsilon_\varepsilon \le t]$,
Combining (19), and (20), we obtain

$$P(\upsilon_\varepsilon \le t) \le \frac{Lt}{\log\frac{\rho}{(\mu_1+d)\varepsilon}}, \text{ for all } t \ge 0; \quad (21)$$

for all $t \ge 0$, and $\varepsilon \to 0$, we obtain $P(\upsilon \le t) = 0$;
From where

$$P(\upsilon \le \infty) = 0$$

■

5 The Model with Age

The age distributions of the numbers in the classes are denoted by $S(a,t)$, $I(a,t)$, and $Q(a,t)$, denote the sizes of the population susceptible to disease, and infectious members, quarantine members with the possibility of infection through temporary immunity, respectively of age a, at time t, $d(a)$ is the age-specific death rate,

The system of partial equations for the age distributions is

$$\begin{cases} \frac{\partial S}{\partial t} + \frac{\partial S}{\partial a} = -\left(\mu_1 + d(a)\right) S(a,t) + \beta_1(t) S(a,t), \\ \frac{\partial I}{\partial t} + \frac{\partial I}{\partial a} = -\beta_1(t) S(a,t) - \left(\mu_2 + d(a)\right) I(a,t) + \gamma_1(t-\tau) S(a,t-\tau), \\ \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial a} = -\gamma_1(t-\tau) S(a,t-\tau) - \left(\mu_3 + d(a)\right) Q(a,t), \end{cases} \tag{22}$$

With

$$\begin{aligned} \beta_1(t) &= -\beta \int Q(a,t) da \\ \gamma_1(t-\tau) &= -\gamma \int e^{-\mu_2 \tau} Q(a,t-\tau) da \end{aligned} \tag{23}$$

5.1 Equilibrium and stability

Assume that sub population does not depend on the time when the system (22) is written as follows

$$\begin{cases} \frac{dS}{da} = \left(\beta_1 - \mu_1 - d(a)\right) S(a), \\ \frac{dI}{da} = \left(\gamma_1 - \beta_1\right) S(a) - \left(\mu_2 + d(a)\right) I(a), \\ \frac{dQ}{da} = -\gamma_1 S(a) - \left(\mu_3 + d(a)\right) Q(a), \end{cases} \tag{24}$$

The initial condition of (24) is given as

$$S(0) = S_1, \quad I(0) = I_1, \quad Q(0) = Q_1 \tag{25}$$

Differential equations of the system (24) are solved with different methods of resolutions and with (25), so

$$S(a) = S_1 e^{-(\mu_1 - \beta_1)a}\ \Phi(a), \tag{26}$$

$$I(a) = I_1 \Phi(a) e^{-\mu_2 a} - \frac{\left(\gamma_1 - \beta_1\right) S_1 \Phi(a)}{\mu_1 - \beta_1 - \mu_2} \left(e^{-(\mu_1 - \beta_1)a} - e^{-\mu_2 a}\right), \tag{27}$$

$$Q(a) = Q_1 \Phi(a) e^{-\mu_3 a} - \frac{\gamma_1 S_1 \Phi(a)}{\mu_1 - \beta_1 - \mu_3} \left(e^{-(\mu_1 - \beta_1)a} - e^{-\mu_3 a}\right) \tag{28}$$

Where

$$\Phi(a) = \exp\left(-\int d(a) da\right) \tag{29}$$

The system (24) has the unique positive equilibrium point P_1,

$$P_1 = \left(\hat{S}_1,\ \hat{I}_1,\ \hat{Q}_1\right)^T = (0,\ 0,\ 0)^T .$$

We calculate the Jacobian matrix according to the system (24) with P_1

$$J\left(P_1\right)=\left[\begin{array}{ccc}\beta_1-\mu_1-d(a) & 0 & 0\\ \lambda-\gamma_0 & -\left(\mu_2+d(a)\right) & 0\\ -\gamma_0 & 0 & -\left(\mu_3+d(a)\right)\end{array}\right]$$

The epidemic is locally asymptotically stable if and only if all eigenvalues of the Jacobian matrix $J\left(P_1\right)$ have negative real part. The eigenvalues can be determined by solving the characteristic equation of the linearization of (25) near P_1 is

$$\det\left(\begin{array}{ccc}\beta_1-\mu_1-d(a)-A & 0 & 0\\ \lambda-\gamma_0 & -\left(\mu_2+d(a)\right)-A & 0\\ -\gamma_0 & 0 & -\left(\mu_3+d(a)\right)-A\end{array}\right)=0 \tag{31}$$

So the eigenvalues are

$$A_1=\beta_1-\mu_1-d(a),\quad A_2=-\left(\mu_2+d(a)\right),\quad A_3=-\left(\mu_3+d(a)\right)$$

In order to A_1, A_2,and A_3 will be negative, it is required that

$$\beta_1<\mu_1+d(a)$$

The basic reproduction number R_0 is defined as the total number of infected population in the resulting sub-infected population where almost all of the uninfected. The basic reproduction number of the infection R_0 is defined as follows:

$$R_0=\frac{\beta_1}{\mu_1+d(a)} \tag{32}$$

The time during which people remain infective is defined as

$$T=\frac{1}{\mu_1+d(a)}$$

The doubling time t_d of the epidemic can be obtained as

$$t_d=\frac{(\ln 2)\,T}{R_0-1} \tag{33}$$

Theorem 1 The disease-free equilibrium P_1 is locally asymptotically stable if $R_0<1$ and unstable if $R_0>1$.

Let (26), so if $R_0<1$ then $\mu_1-\beta_1>0$, so $S(a)$ converges to zero.
Let (27), so

$$I(a)\leq\left[I_1\Phi(a)-\frac{\left(\gamma_1-\beta_1\right)S_1\Phi(a)}{\mu_1-\beta_1-\mu_2}\right]e^{-m_1 a},\ m_1=\min\left\{\mu_1-\beta_1,\mu_2\right\} \tag{34}$$

If $R_0 < 1$, $i(a)$ converges to zero.
Let (28) , so

$$Q(a) = \left[Q_1\Phi(a) - \frac{\gamma_1 S_1 \Phi(a)}{\mu_1 - \beta_1 - \mu_3} \right] e^{-m_2 a},\ m_1 = \min\{\mu_1 - \beta_1, \mu_3\} \qquad (35)$$

If $R_0 < 1$, $Q(a)$ converges to zero.

This paper addresses a SIQ model with temporary immunity, whenever the quarantine individuals will return to the susceptible. The endemic equilibrium is globally asymptotically stable, then under some conditions, study the stochastic stability. Finally, the equilibrium and stability of the epidemic model with age.

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