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A classification of three-dimensional real hypersurfaces in non-flat complex space forms in terms of their generalized Tanaka-Webster Lie derivative

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Abstract

On a real hypersurface M in a non-flat complex space form there exist the Levi-Civita and the k-th generalized Tanaka-Webster connections. The aim of the present paper is to study three dimensional real hypersurfaces in non-flat complex space forms, whose Lie derivative of the structure Jacobi operator with respect to the Levi-Civita connections coincides with the Lie derivative of it with respect to the k-th generalized Tanaka-Webster connection. The Lie derivatives are considered in direction of the structure vector field and in directions of any vecro field orthogonal to the structure vector field.

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1 Introduction

A *complex space form* is an n-dimensional Kähler manifold of constant holomorphic sectional curvature c. A complete and simply connected complex space form is analytically isometric to a complex projective space $\mathbb{C}P^n$ if $c > 0$, a complex Euclidean space $\mathbb{C}^n$ if $c = 0$, or a complex hyperbolic space $\mathbb{C}H^n$ if $c < 0$. Furthermore, the complex projective and complex hyperbolic spaces are called non-flat complex space forms and the symbol $M_n(c), n \geq 2$, is used to denote them when it is not necessary to distinguish them.

Let M be a *connected real hypersurface* of $M_n(c)$ without boundary. Let ∇ be the Levi-Civita connection on M and J the complex structure of $M_n(c)$. Take a locally defined unit normal vector field N on M and denote by $\xi = -JN$. This is a tangent vector field to M called the *structure vector field* on M . If it is an eigenvector of the shape operator A of M the real hypersurface is called a *Hopf hypersurface* and the corresponding eigenvalue is $\alpha = g(A\xi, \xi)$. Moreover, the complex structure J induces on M an almost contact metric structure (ϕ, ξ, η, g), where ϕ is the tangential component of J and η is an one-form given by $\eta(X) = g(X, \xi)$ for any X tangent to M .

The classification of homogeneous real hypersurfaces in $\mathbb{C}P^n, n \geq 2$ was obtained by Takagi and they were divided into six type of real hypersurfaces (see [14], [15], [16]). Among them the three dimensional real hypersurfaces in $\mathbb{C}P^2$ are geodesic hyperspheres of radius

r, $0 < r < \dfrac{\pi}{2}$, which are called real hypersurfaces of type (A) and tubes of radius r, $0 < r < \dfrac{\pi}{4}$, over the complex quadric, which are called real hypersurfaces of type (B). All of them are Hopf ones with constant principal curvatures (see [6]). In case of $\mathbb{C}H^n$, the study of Hopf hypersurfaces with constant principal curvatures, was initiated by Montiel in [8] and completed by Berndt in [1]. Such hypersurfaces in $\mathbb{C}H^2$ are open subsets of horospheres, geodesic hyperspheres, or tubes over totally geodesic complex hyperbolic hyperplane $\mathbb{C}H^1$ (type (A)), or tubes over totally geodesic real hyperbolic space $\mathbb{R}H^2$ (type (B)).

The *Jacobi operator* R_X of a Riemannian manifold $\tilde{M}$ with respect to a unit vector field X is given by $R_X = R(\cdot, X)X$, where R is the curvature tensor field on $\tilde{M}$. It is a self-adjoint endomorphism of the tangent space $T\tilde{M}$ and it is related to Jacobi vector fields, which are solutions of the second-order differential equation $\nabla_{\dot{\gamma}}(\nabla_{\dot{\gamma}}Y) + R(Y, \dot{\gamma})\dot{\gamma} = 0$ along a geodesic γ in $\tilde{M}$ (known as the Jacobi equation). In case of real hypersurfaces in $M_n(c)$ the Jacobi operator with respect to the structure vector field ξ, R_ξ, is called the *structure Jacobi operator* on M and it plays an important role in the study of them.

Apart from the Levi-Civita connection on a non-degenerate, pseudo-Hermitian CR-manifold a canonical affine connection is defined and is called *Tanaka-Webster connection* (see [17], [19]). As a generalization of this connection, in [18] Tanno defined the *generalized Tanaka-Webster connection* for contact metric manifolds by

$$\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y.$$

Using the naturally extended affine connection of Tanno's generalized Tanaka-Webster connection, Cho defined the *k-th generalized Tanaka-Webster connection* $\hat{\nabla}^{(k)}$ on a real hypersurface M in $M_n(c)$ given by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{1.1}$$

for any vector fields X, Y tangent to M where k is a nonnull real number (see [2], [3]). Then the following relations hold

$$\hat{\nabla}^{(k)}\eta = 0, \quad \hat{\nabla}^{(k)}\xi = 0, \quad \hat{\nabla}^{(k)}g = 0, \quad \hat{\nabla}^{(k)}\phi = 0.$$

In particular, if the shape operator of a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, the generalized Tanaka-Webster connection coincides with the Tanaka-Webster connection.

The Lie derivative of a tensor field T of type (1,1) with respect to the generalized Tanaka-Webster connection is denoted by $\hat{\mathcal{L}}^{(k)}_X T$, called *k-th generalized Tanaka-Webster Lie derivative with respect to* X and is given by

$$(\hat{\mathcal{L}}^{(k)}_X T)Y = \hat{\nabla}^{(k)}_X TY - \hat{\nabla}^{(k)}_{TY} X - T\hat{\nabla}^{(k)}_X Y + T\hat{\nabla}^{(k)}_Y X,$$

where X, Y are tangent to M.

Many geometric conditions with respect to the k-th generalized Tanaka-Webster connection on real hypersurfaces have been studied. One of them is the classification of real hypersurfaces in $M_n(c)$, $n \geq 2$, whose k-th generalized Tanaka-Webster Lie derivative agrees with the ordinary Lie derivative when applied to the tensor field T of type (1,1), i.e. $(\hat{\mathcal{L}}_X^{(k)}T)Y = (\mathcal{L}_X T)Y$, for all X, Y tangent to M. The last relation because of (1.1) implies

$$g((\phi A + A\phi)X, TY)\xi - (\phi A - k\phi)(X \wedge TY)\xi = g((\phi A + A\phi)X, Y)T\xi - T(\phi A - k\phi)(X \wedge Y)\xi \quad (1.2)$$

and the wedge product is given by

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y,$$

for all X, Y Z tangent to M.

In [13] real hypersurfaces in $\mathbb{C}P^n, n \geq 3$, whose structure Jacobi operator satisfies relation $\hat{\mathcal{L}}_\xi^{(k)} R_\xi = \mathcal{L}_\xi R_\xi$ are classified. Furthermore, the non-existence of real hypersurfaces in $\mathbb{C}P^n, n \geq 3$, whose structure Jacobi operator satisfies relation $\hat{\mathcal{L}}_X^{(k)} R_\xi = \mathcal{L}_X R_\xi$, for any X orthogonal to ξ is proved.

The purpose of this paper is to extend the previous results to the case of three dimensional real hypersurfaces in $M_2(c)$. First, we study real hypersurfaces in $M_2(c)$ satisfying relation

$$\hat{\mathcal{L}}_\xi^{(k)} R_\xi = \mathcal{L}_\xi R_\xi \quad (1.3)$$

and the following Theorem is obtained

Theorem 1.1 *Every real hypersurface in $M_2(c)$, whose structure Jacobi operator satisfies relation (1.3) is a Hopf hypersurface. Moreover, M is locally congruent either to a real hypersurface of type (A), or to a Hopf hypersurface with $A\xi = 0$.*

Next we study three dimensional real hypersurfaces in $M_2(c)$, whose structure Jacobi operator satisfies relation

$$\hat{\mathcal{L}}_X^{(k)} R_\xi = \mathcal{L}_X R_\xi, \quad (1.4)$$

for all X orthogonal to ξ and the following Theorem is proved

Theorem 1.2 *There do not exist real hypersurfaces in $M_2(c)$, whose structure Jacobi operator satisfies relation (1.4).*

As an immediate consequence of the above Theorems we conclude that

Corollary 1.1 *There do not exist real hypersurfaces in $M_2(c)$ such that $\hat{\mathcal{L}}_X^{(k)} R_\xi = \mathcal{L}_X R_\xi$, for all $X \in TM$.*

This paper is organized as follows: In Section 2 basic results about real hypersurfaces in non-flat complex space forms are included. In Section 3 the proof of Theorem 1.1 is provided. Finally, in Section 4 the proof of Theorem 1.2 is given.

2 Preliminaries

Throughout this paper all manifolds, vector fields etc. are assumed to be of class C^∞ and all manifolds are assumed to be connected and the real hypersurfaces M are supposed to be without boundary. Furthermore, all the material mentioned in this Section is valid for all real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ without regard to the Lie derivative conditions.

Thus, let M be a real hypersurface immersed in a non-flat complex space form $(M_n(c), G)$ with complex structure J of constant holomorphic sectional curvature c and N be a locally defined unit normal vector field on M and $\xi = -JN$ be the structure vector field of M . For a vector field X tangent to M relation

$$JX = \phi X + \eta(X)N$$

holds, where ϕX and $\eta(X)N$ are respectively the tangential and the normal component of JX. The Riemannian connections $\overline{\nabla}$ in $M_n(c)$ and ∇ in M are related for any vector fields X, Y on M by

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N,$$

where g is the Riemannian metric induced from the metric G.
The *shape operator* A of the real hypersurface M in $M_n(c)$ with respect to N is given by

$$\overline{\nabla}_X N = -AX.$$

The real hypersurface M has an almost contact metric structure (ϕ, ξ, η, g) induced from J of $M_n(c)$, where ϕ is the *structure tensor*, which is a tensor field of type (1,1) and η is an 1-form such that

$$g(\phi X, Y) = G(JX, Y), \qquad \eta(X) = g(X, \xi) = G(JX, N).$$

Moreover, the following relations hold

$$\phi^2 X = -X + \eta(X)\xi, \qquad \eta \circ \phi = 0, \qquad \phi\xi = 0, \qquad \eta(\xi) = 1,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \phi Y) = -g(\phi X, Y).$$

The fact that J is parallel implies $\overline{\nabla} J = 0$ and this leads to

$$\nabla_X \xi = \phi AX, \qquad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{2.1}$$

The ambient space $M_n(c)$ is of constant holomorphic sectional curvature c and this results in the Gauss and Codazzi equations are respectively given by

$$R(X, Y)Z = \frac{c}{4}[g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X \tag{2.2}$$
$$-g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z] + g(AY, Z)AX - g(AX, Z)AY,$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi],$$

where R denotes the Riemannian curvature tensor on M and X, Y, Z are any vector fields on M.

The tangent space T_PM at every point $P \in M$ can be decomposed as

$$T_PM = span\{\xi\} \oplus \mathbb{D},$$

where $\mathbb{D} = \ker\eta = \{X \in T_PM : \eta(X) = 0\}$ and is called (*maximal*) *holomorphic distribution* (*if* $n \geq 3$). Due to the above decomposition the vector field $A\xi$ can be written

$$A\xi = \alpha\xi + \beta U,$$

where $\beta = |\phi\nabla_\xi\xi|$ and $U = -\frac{1}{\beta}\phi\nabla_\xi\xi \in \ker(\eta)$ is a unit vector field, provided that $\beta \neq 0$.

Next, the following results concern any non-Hopf real hypersurface M in $M_2(c)$ with local orthonormal basis $\{U, \phi U, \xi\}$ at a point P of M.

Lemma 2.1 *Let M be a non-Hopf real hypersurface in $M_2(c)$. The following relations hold on M*

$$\begin{array}{lll} AU = \gamma U + \delta\phi U + \beta\xi, & A\phi U = \delta U + \mu\phi U, & A\xi = \alpha\xi + \beta U \\ \nabla_U\xi = -\delta U + \gamma\phi U, & \nabla_{\phi U}\xi = -\mu U + \delta\phi U, & \nabla_\xi\xi = \beta\phi U, \\ \nabla_U U = \kappa_1\phi U + \delta\xi, & \nabla_{\phi U}U = \kappa_2\phi U + \mu\xi, & \nabla_\xi U = \kappa_3\phi U, \\ \nabla_U\phi U = -\kappa_1 U - \gamma\xi, & \nabla_{\phi U}\phi U = -\kappa_2 U - \delta\xi, & \nabla_\xi\phi U = -\kappa_3 U - \beta\xi, \end{array} \quad (2.3)$$

where $\alpha, \beta, \gamma, \delta, \mu, \kappa_1, \kappa_2, \kappa_3$ are smooth functions on M and $\beta \neq 0$.

Remark 2.1 *The proof of Lemma 2.1 is included in [12].*

The Codazzi equation for $X \in \{U, \phi U\}$ and $Y = \xi$ because of Lemma 2.1 implies the following relations

$$\xi\delta = \alpha\gamma + \beta\kappa_1 + \delta^2 + \mu\kappa_3 + \frac{c}{4} - \gamma\mu - \gamma\kappa_3 - \beta^2 \quad (2.4)$$

$$\xi\mu = \alpha\delta + \beta\kappa_2 - 2\delta\kappa_3 \quad (2.5)$$

$$(\phi U)\alpha = \alpha\beta + \beta\kappa_3 - 3\beta\mu \quad (2.6)$$

$$(\phi U)\beta = \alpha\gamma + \beta\kappa_1 + 2\delta^2 + \frac{c}{2} - 2\gamma\mu + \alpha\mu \quad (2.7)$$

and for $X = U$ and $Y = \phi U$

$$U\delta - (\phi U)\gamma = \mu\kappa_1 - \kappa_1\gamma - \beta\gamma - 2\delta\kappa_2 - 2\beta\mu \quad (2.8)$$

Furthermore, combination of the Gauss equation (2.2) with the formula of Riemannian curvature $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, taking into account relations of Lemma 2.1, implies

$$U\kappa_2 - (\phi U)\kappa_1 = 2\delta^2 - 2\gamma\mu - \kappa_1^2 - \gamma\kappa_3 - \kappa_2^2 - \mu\kappa_3 - c, \quad (2.9)$$

Relation (2.2) implies that the structure Jacobi operator R_ξ is given by

$$R_\xi(X) = \frac{c}{4}[X - \eta(X)\xi] + \alpha AX - \eta(AX)A\xi, \tag{2.10}$$

for any vector field X tangent to M, where $\alpha = \eta(A\xi) = g(A\xi, \xi)$.

Moreover, the structure Jacobi operator for $X = U$, $X = \phi U$ and $X = \xi$ due to (2.3) is given by

$$R_\xi(U) = (\frac{c}{4} + \alpha\gamma - \beta^2)U + \alpha\delta\phi U, \;\; R_\xi(\phi U) = \alpha\delta U + (\frac{c}{4} + \alpha\mu)\phi U \;\text{ and }\; R_\xi(\xi) = 0. \tag{2.11}$$

The following Theorem in case of $\mathbb{C}P^n$ is owed to Maeda [7] and in case of $\mathbb{C}H^n$ is owed to Ki and Suh [5] (also Corollary 2.3 in [10]).

Theorem 2.1 *Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 2$, with $A\xi = \alpha\xi$. Then*
i) α is constant.
ii) If W is a vector field which belongs to $\mathbb{D}$ such that $AW = \lambda W$, then

$$(\lambda - \frac{\alpha}{2})A\phi W = (\frac{\lambda\alpha}{2} + \frac{c}{4})\phi W.$$

iii) If the vector field W satisfies $AW = \lambda W$ and $A\phi W = \nu\phi W$ then

$$\lambda\nu = \frac{\alpha}{2}(\lambda + \nu) + \frac{c}{4}. \tag{2.12}$$

Remark 2.2 *In case of three dimensional Hopf hypersurfaces we can always consider a local orthonormal basis $\{W, \phi W, \xi\}$ at some point $P \in M$ such that $AW = \lambda W$ and $A\phi W = \nu\phi W$. Thus, relation (2.12) is satisfied. Furthermore, the structure Jacobi operator of Hopf hypersurfaces, whose shape operator is given by the previous relations for $X = W$ and $X = \phi W$ is given by*

$$R_\xi(W) = (\frac{c}{4} + \alpha\lambda)W \;\text{ and }\; R_\xi(\phi W) = (\frac{c}{4} + \alpha\nu)\phi W. \tag{2.13}$$

We also mention the following Theorem, which plays an important role in the study of real hypersurfaces in $M_n(c)$, which is due to Okumura in case of $\mathbb{C}P^n$ (see [11]) and to Montiel and Romero in case of $\mathbb{C}H^n$ (see [9]). It provides the classification of real hypersurfaces in $M_n(c)$, $n \geq 2$, whose shape operator A commutes with the structure tensor field ϕ.

Theorem 2.2 *Let M be a real hypersurface of $M_n(c)$, $n \geq 2$. Then $A\phi = \phi A$, if and only if M is locally congruent to a homogeneous real hypersurface of type (A). More precisely:*
In case of $\mathbb{C}P^n$
(A_1) *a geodesic hypersphere of radius r, where $0 < r < \frac{\pi}{2}$,*
(A_2) *a tube of radius r over a totally geodesic $\mathbb{C}P^k$, $(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$.*
In case of $\mathbb{C}H^n$
(A_0) *a horosphere in $\mathbb{C}H^n$, i.e a Montiel tube,*
(A_1) *a geodesic hypersphere or a tube over a totally geodesic complex hyperbolic hyperplane $\mathbb{C}H^{n-1}$,*
(A_2) *a tube over a totally geodesic $\mathbb{C}H^k$ $(1 \leq k \leq n-2)$.*

Remark 2.3 *In case of three dimensional real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ type (A_2) hypersurfaces do not occur.*

Finally, we mention the following Proposition (see [4]), which is used in the proof of the present Theorems.

Proposition 2.1 *There do not exist real hypersurfaces in $M_2(c)$, whose structure Jacobi operator vanishes.*

3 Proof of Theorem 1.1

Let M be a non-Hopf real hypersurface in $M_2(c)$ whose structure Jacobi operator satisfies relation (1.3). More analytically, the previous relation due to (1.2) for $T = R_\xi$ and $X = \xi$ and since $R_\xi(\xi) = 0$ implies

$$g(\phi A\xi, R_\xi(Y))\xi - (\phi A - k\phi)(\xi \wedge R_\xi(Y))\xi = -R_\xi(\phi A - k\phi)(\xi \wedge Y)\xi, \tag{3.1}$$

for all Y tangent to M.

We consider $\mathcal{N}$ the open subset of M such that

$$\mathcal{N} = \{P \in M : \beta \neq 0, \text{ in a neighborhood of } P.\}$$

On $\mathcal{N}$ Lemma 2.1 holds and the inner product of relation (3.1) for $Y = U$ with ξ due to the first of (2.11) yields

$$\alpha\delta = 0.$$

Suppose that $\alpha \neq 0$ then the above relation implies $\delta = 0$ and relations (2.3) and (2.11) become respectively

$$AU = \gamma U + \beta\xi, \quad A\phi U = \mu\phi U \quad \text{and} \quad A\xi = \alpha\xi + \beta U, \tag{3.2}$$

$$R_\xi(U) = (\frac{c}{4} + \alpha\gamma - \beta^2)U, \quad R_\xi(\phi U) = (\frac{c}{4} + \alpha\mu)\phi U \quad \text{and} \quad R_\xi(\xi) = 0. \tag{3.3}$$

The inner product of (3.1) for $Y = \phi U$ with ξ because of (3.2) and the second of (3.3) implies

$$\mu = -\frac{c}{4\alpha} \Rightarrow R_\xi(\phi U) = 0.$$

Moreover, relation (3.1) for $Y = \phi U$ taking into account that $R_\xi(\phi U) = 0$ and the first of (3.3) results in

$$(\mu - k)R_\xi(U) = 0.$$

If $\mu \neq k$ then $R_\xi(U) = 0$. So the structure Jacobi operator vanishes identically, which is impossible because of Proposition 2.1 in Section 2.

Thus, $\mu = k$. Furthermore, the inner product of (3.1) for $Y = U$ with ϕU due to the first of (3.3) and $R_\xi(\phi U) = 0$ implies

$$(\gamma - k)g(R_\xi(U), U) = 0.$$

If $\gamma \neq k$ then $g(R_\xi(U), U) = 0$ and this results in $R_\xi(U) = 0$, which implies that the structure Jacobi operator vanishes identically, which is impossible due to Proposition 2.1.

So $\gamma = k$. Differentiation of the last relation with respect to ϕU yields $(\phi U)\gamma = 0$. Thus, relation (2.8) since $\delta = 0$ and $\mu = \gamma = k$ implies $k = 0$, which is a contradiction.

Therefore, on M we have $\alpha = 0$ and relation (2.11) becomes

$$R_\xi(U) = (\frac{c}{4} - \beta^2)U, \quad R_\xi(\phi U) = \frac{c}{4}\phi U \text{ and } R_\xi(\xi) = 0. \tag{3.4}$$

The inner product of relation (3.1) for $Y = \phi U$ with ξ because of the second relation of (3.4) gives $c = 0$, which is a contradiction.

Thus, $\mathcal{N}$ is empty and the following Proposition is proved

Proposition 3.1 *Every real hypersurface in $M_2(c)$ whose structure Jacobi operator satisfies relation (1.3) is a Hopf hypersurface.*

Due to the above Proposition, relations in Theorem 2.1 and remark 2.2 hold. Relation (3.1) for $Y = W$ and $Y = \phi W$ taking into account (2.13) implies respectively

$$k\alpha(\lambda - \nu) = \lambda\alpha(\lambda - \nu) \quad \text{and} \quad k\alpha(\lambda - \nu) = \nu\alpha(\lambda - \nu). \tag{3.5}$$

If there is a point where $\lambda \neq \nu$ relation (3.5) yields $k\alpha = \alpha\lambda$ and $k\alpha = \nu\alpha$, which implies $\alpha(\lambda - \nu) = 0$. So, $\alpha = 0$.

If $\lambda = \nu$ at all points this implies

$$(A\phi - \phi A)X = 0$$

for any X tangent to M. So due to Theorem 2.2 M is locally congruent to a real hypersurface of type (A) and this completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

Relation (1.4) because of (1.2), since $T = R_\xi$ and $X \in \mathbb{D}$ because of $R_\xi(\xi) = 0$ implies

$$g((\phi A + A\phi)X, R_\xi(Y))\xi = -R_\xi(\phi A - k\phi)(X \wedge Y)\xi, \tag{4.1}$$

for all X orthogonal to ξ and for all vectors Y tangent to M.

First we prove the following Proposition

Proposition 4.1 *There do not exist Hopf hypersurfaces in $M_2(c)$ whose structure Jacobi operator satisfies relation (1.4).*

Proof: Let M be a Hopf hypersurface. Then we have $A\xi = \alpha\xi$, where α is constant, and remark 2.2 holds. Relation (4.1) for (X, Y) being (W, ξ), $(\phi W, \xi)$, $(W, \phi W)$ and $(\phi W, W)$

taking into account relation (2.13) implies respectively

$$(\lambda - k)(\alpha\nu + \frac{c}{4}) = 0 \quad (4.2)$$

$$(\nu - k)(\alpha\lambda + \frac{c}{4}) = 0 \quad (4.3)$$

$$(\lambda + \nu)(\alpha\nu + \frac{c}{4}) = 0 \quad (4.4)$$

$$(\lambda + \nu)(\alpha\lambda + \frac{c}{4}) = 0 \quad (4.5)$$

There are three possibilities to consider:

1. Suppose $\alpha = 0$. Then relations (4.2) and (4.3) give $\lambda = \nu = k$. So, relation (4.4) implies $k = 0$, which is a contradiction.

2. Suppose $\alpha \neq 0$ and there is a point where $\lambda \neq \nu$. If $\lambda \neq k$ then relation (4.2) implies $\alpha\nu + \frac{c}{4} = 0$. So $\alpha\lambda + \frac{c}{4} \neq 0$ and relation (4.3) yields $\nu = k$. Furthermore, relation (4.5) gives $\lambda + \nu = 0$. So, $\lambda = -k$ and the Hopf hypersurface has three constant principal curvatures and must be an open subset of a type (B) hypersurface. But type (B) hypersurfaces satisfy $\lambda\nu + \frac{c}{4} = 0$ and substitution of the last in (2.12) leads to a contradiction.

3. $\alpha \neq 0$ and $\lambda = \nu$. This implies because of (4.4) that either $\lambda = 0$ or $\alpha\lambda = \frac{c}{4}$. Substitution of the previous in (2.12) leads to a contradiction and this completes the proof of the Proposition.

□

Next we examine non-Hopf hypersurfaces in $M_2(c)$ whose structure Jacobi operator satisfies relation (4.1). Since M is a non-Hopf hypersurface we have that $\beta \neq 0$ and relation (2.3) holds. Relation (4.1) for $X = Y = U$, $X = U$ and $Y = \phi U$ and for $X = \phi U$ and $Y = U$ implies respectively

$$(\gamma + \mu)g(R_\xi(U), \phi U) = 0 \quad (4.6)$$

$$(\gamma + \mu)g(R_\xi(\phi U), \phi U) = 0 \quad (4.7)$$

$$(\gamma + \mu)g(R_\xi(U), U) = 0. \quad (4.8)$$

If $\gamma + \mu \neq 0$ then relations (4.6), (4.7) and (4.8) result in

$$g(R_\xi(U), \phi U) = g(R_\xi(\phi U), \phi U) = g(R_\xi(U), U).$$

The above relation leads to the conclusion that the structure Jacobi operator R_ξ vanishes identically and because of Proposition 2.1 this is impossible.

Thus on M, relation $\gamma + \mu = 0$ holds. Moreover, relation (1.4) for $X = U$ and $Y = \xi$

and for $X=\phi U$ and $Y=\xi$ due to (2.11) and $\gamma+\mu=0$ implies

$$\begin{aligned}\delta(\frac{c}{4}-\beta^2+\alpha k) &= 0, && (4.9)\\ (\mu+k)(\frac{c}{4}+\alpha\mu) &= -\alpha\delta^2, && (4.10)\\ (\mu-k)(\frac{c}{4}-\alpha\mu-\beta^2) &= \alpha\delta^2, && (4.11)\\ \delta(\frac{c}{4}+\alpha k) &= 0. && (4.12)\end{aligned}$$

Suppose that $\delta\neq 0$ then combination of relations (4.9) and (4.12) yields $\beta=0$, which is a contradiction.

So, on M we have $\delta=0$ and $\gamma=-\mu$ and relations (4.10) and (4.11) become

$$(\mu+k)(\frac{c}{4}+\alpha\mu)=0 \text{ and } (\mu-k)(\frac{c}{4}-\alpha\mu-\beta^2)=0. \tag{4.13}$$

If $k+\mu\neq 0$ then $\frac{c}{4}+\alpha\mu=0$ and the second of the above relation gives $\mu=k$, because if $\frac{c}{4}-\alpha\mu-\beta^2=0$ then relation (2.11) implies that the structure Jacobi operator R_ξ vanishes identically, which is impossible. Since $k=\mu$ we obtain $\xi\mu=0$ and relation (2.5) implies $\kappa_2=0$. Furthermore, differentiation of $\gamma=-\mu$ with respect to ϕU gives

$$(\phi U)\mu=(\phi U)\gamma=0.$$

Furthermore, differentiation of $\frac{c}{4}+\alpha\mu=0$ with respect to ϕU because of the above relation and relation (2.6) gives $\kappa_3=3\mu-\alpha$. Since $(\phi U)\gamma=0$ relation (2.8) implies $\kappa_1=\frac{\beta}{2}$. So relation (2.4) bearing in mind all the previous relations gives $\frac{\beta^2}{2}=c+7\mu^2$. Differentiating the last relation with respect to ϕU yields $(\phi U)\beta=0$ and relation (2.7) implies $\frac{\beta^2}{2}+\frac{c}{2}+2\mu^2=0$. Moreover, since $\kappa_1=\frac{\beta}{2}$ and $(\phi U)\beta=0$ we conclude that $(\phi U)\kappa_1=0$ and relation (2.9) due to $\gamma=-\mu$, $\kappa_1=\frac{\beta}{2}$, $\kappa_3=3\mu-\alpha$ and $\kappa_2=0$ results in $\frac{\beta^2}{2}=4\mu^2-2c$. Combination of the last one with $\frac{\beta^2}{2}=c+7\mu^2$ implies $c=-\mu^2$. Substitution of the latter in $\frac{\beta^2}{2}+2\mu^2+\frac{c}{2}=0$ due to $\frac{\beta^2}{2}=4\mu^2-2c$ leads to $c=0$, which is a contradiction.

Thus, on M we have $\mu+k=0$. Summarizing on M the following relations hold

$$\delta=0 \text{ and } \gamma=-\mu=k.$$

The second of (4.13) implies that $k\alpha=\beta^2-\frac{c}{4}$.

Moreover, relation (2.5) due to $\mu=-k$ implies $\kappa_2=0$ and relation (2.8) bearing in mind all the previous relations results in $\beta=2\kappa_1$. Furthermore, relation (2.7) because of $\gamma=-\mu$, $\beta=2\kappa_1$ and $\mu=-k$ implies $(\phi U)\beta=\frac{\beta^2}{2}+\frac{c}{2}+2k^2$ and relation (2.9) taking into account

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$\gamma+\mu=0$, $\kappa_2=0$ and $\beta=2\kappa_1$ yields $(\phi U)\beta=-4k^2+\frac{\beta^2}{2}+2c$. Combination of the last two relations of $(\phi U)\beta$ results in $c=4k^2$. The last relation leads to a contradiction when the ambient space is $\mathbb{C}H^2$. So it remains to examine the case when the ambient space is $\mathbb{C}P^2$.

Since $c=4k^2$ and $k\neq 0$ relation $k\alpha=\beta^2-\frac{c}{4}$ implies $\alpha=\frac{\beta^2}{k}-k$. Differentiation of the latter with respect to ϕU taking into account relations (2.6) and (2.7) yields $\kappa_3=6k$. Furthermore, relation (2.4) because of the last one and $\beta=2\kappa_1$ results in $\beta^2=22k^2$. So relation (2.7) because of the above relations implies $\beta^2+2c=0$. The last relation due to $c=4k^2$ and $\beta^2=22k^2$ results in $k=0$, which is impossible and this completes the proof of Theorem 1.2.

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References

[1] J. Berndt, Real hypersurfaces with constant principal curvatures in complex hyperbolic space, *J. Reine Angew. Math.* **395** (1989), 132-141.

[2] J.T. Cho, CR-structures on real hypersurfaces of a complex space form, *Publ. Math. Debrecen* **54** (1999), 473-487.

[3] J.T. Cho, Pseudo-Einstein CR-structures on real hypersurfaces in a complex space form, *Hokkaido Math. J.* **37** (2008), 1-17.

[4] T. A. Ivey and P. J. Ryan, The structure Jacobi operator for real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$, *Results Math.* **56** (2009), 473-488.

[5] U.- H. Ki and Y. J. Suh, On real hypersurfaces of a complex space form, *Math. J. Okayama Univ.*, **32** (1990), 207-221.

[6] M. Kimura, Real hypersurfaces and complex submanifolds in complex projective space, *Trans. A.M.S.* **296** (1986), 137-149.

[7] Y. Maeda, On real hypersurfaces of a complex projective space, *J. Math. Soc. Japan* **28** (1976), 529-540.

[8] S. Montiel, Real hypersurfaces of a complex hyperbolic space, *J. Math. Soc. Japan* **35** (1985), 515-535.

[9] S. Montiel and A. Romero, On some real hypersurfaces of a complex hyperbolic space. *Geom. Dedicata* **20** (1986), 245-261.

[10] R. Niebergall and P.J. Ryan, Real hypersurfaces in complex space forms, in Tight and Taut Submanifolds, MSRI Publications, Vol. 32, 1997, 233-305.

[11] M. Okumura, On some real hypersurfaces of a complex projective space. *Trans. Amer. Math. Soc.* **212** (1975), 355-364.

[12] K. Panagiotidou and Ph. J. Xenos, Real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ whose structure Jacobi operator is Lie $\mathbb{D}$-parallel. Note Mat. **32** (2012), 89-99.

[13] J. D. Pérez, Lie derivatives and structure Jacobi operator on real hypersurfaces in complex projective spaces, submitted.

[14] R. Takagi, On homogeneous real hypersurfaces in a complex projective space, *Osaka J. Math.* **10** (1973), 495-506.

[15] R. Takagi, Real hypersurfaces in complex projective space with constant principal curvatures, *J. Math. Soc. Japan* **27** (1975), 43-53.

[16] R. Takagi, Real hypersurfaces in complex projective space with constant principal curvatures II, *J. Math. Soc. Japan* **27** (1975), 507-516.

[17] N. Tanaka, On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections, *Japan. J. Math.* **2** (1976), 131-190.

[18] S. Tanno, Variational problems on contact Riemennian manifolds, *Trans. A.M.S.* **314** (1989), 349-379.

[19] S.M. Webster, Pseudohermitian structures on a real hypersurface, *J. Diff. Geom.* **13** (1978), 25-41.

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