Existence of positive solutions to second-order periodic boundary value problems with impulse actions

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Abstract. In this paper we consider the existence of positive solutions for second-order periodic boundary value problems with impulse actions. By constructing a cone $K_1 \times K_2$, which is the Cartesian product of two cones in the space $C[0, 2\pi]$ and computing the fixed point index in the $K_1 \times K_2$, we establish the existence of positive solutions for the system.

Key words. Periodic boundary value problem; Second-order impulsive differential equations; Fixed point index in cones.

MR(2000) Subject Classifications: 34B15.

1. Introduction

This paper is devoted to study the existence of positive solutions for the following periodic boundary value problem with impulse effects:

$$\begin{cases} -u'' + Mu = g_1(x, u, v), & x \in I', \\ -v'' + Mv = g_2(x, v, u), & x \in I', \\ -\Delta u'|_{x=x_k} = I_{1,k}(u(x_k)), \quad \Delta u|_{x=x_k} = \overline{I}_{1,k}(u(x_k)) & k = 1, 2, \cdots, l, \\ -\Delta v'|_{x=x_k} = I_{2,k}(v(x_k)), \quad \Delta v|_{x=x_k} = \overline{I}_{2,k}(v(x_k)) & k = 1, 2, \cdots, l, \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \\ v(0) = v(2\pi), \quad v'(0) = v'(2\pi). \end{cases} \tag{1.1}$$

here $I = [0, 2\pi]$,$0 = x_0 < x_1 < x_2 < \cdots < x_l < x_{l+1} = 2\pi, M > 0$, $I' = I \setminus \{x_1, x_2, \cdots, x_l\}$ are given, $R^+ = [0, +\infty)$, $g_i \in C(I \times R^+, R^+)$, $I_{i,k}, \overline{I}_{i,k} \in C(R^+, R^+)$ with $-\frac{1}{m} I_{i,k}(u) < \overline{I}_{i,k}(u) < \frac{1}{m} I_{i,k}(u)(i = 1, 2)$, $x \in R^+, m = \sqrt{M}$, $\Delta u'|_{x=x_k} = u'(x_k^+) - u'(x_k^-)$, $\Delta u|_{x=x_k} = u(x_k^+) - u(x_k^-)$ $\Delta v'|_{x=x_k} = v'(x_k^+) - v'(x_k^-)$, $\Delta v|_{x=x_k} = v(x_k^+) - v(x_k^-)$, $u'(x_k^+), u(x_k^+), v'(x_k^+)$ *and* $v(x_k^+)$, $(u'(x_k^-), u(x_k^-), v'(x_k^-)$, $v(x_k^-))$ denote the right limit (left limit) of $u'(x), u(x), v'(x)$ *and* $v(x)$ at $x = x_k$ respectively.

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It is well known that there are abundant results about the existence of positive solutions of boundary value problems for second order ordinary differential equations. Some works can be found in $[1-3]$ and references therein.They mainly investigated the case without impulse actions.Recently, Dirichlet boundary problems of second order impulsive differential equations have been studied in $[4-6]$.Motivated by the work above, this paper attempts to study the existence of positive solutions for periodic boundary value problems. By constructing a cone $K \times K$, which is the Cartesian product of two cones in the space $C[0, 2\pi]$,and computing the fixed-point index in the $K \times K$,we establish the existence of positive solutions for the impulsive differential system (1.1).

To conclude the introduction,we introduce the following notation:

$$g_{i,0}(v) = \liminf_{u\to 0^+} \min_{x\in[0,2\pi]} \frac{g_i(x,u,v)}{u},\ I_{i,0}(k) = \liminf_{u\to 0^+} \frac{I_{i,k}(u)}{u},$$

$$g_{i,\infty}(v) = \liminf_{u\to +\infty} \min_{x\in[0,2\pi]} \frac{g_i(x,u,v)}{u},\ I_{i,\infty}(k) = \liminf_{u\to +\infty} \frac{I_{i,k}(u)}{u},$$

$$g_i^{\infty}(v) = \limsup_{u\to +\infty} \max_{x\in[0,2\pi]} \frac{g_i(x,u,v)}{u},\ I_i^{\infty}(k) = \limsup_{u\to +\infty} \frac{I_{i,k}(u)}{u},$$

$$g_i^{0}(v) = \limsup_{u\to 0^+} \max_{x\in[0,2\pi]} \frac{g_i(x,u,v)}{u},\ I_i^{0}(k) = \limsup_{u\to 0^+} \frac{I_{i,k}(u)}{u},$$

where $v \in R^+$ and $i = 1, 2$.

Moreover,for the simplicity in the following discussion,we introduce the following hypotheses.
(H_1) :

$$[\inf_{z\in R^+} g_{1,0}(z)2\pi + \sum_{k=1}^{l} I_{1,0}(k)]\sigma > 2\pi M, \qquad \sup_{z\in R^+} g_1^{\infty}(z)2\pi + \sum_{k=1}^{l} I_1^{\infty}(k) < 2\pi\sigma M.$$

(H_2) :

$$\sup_{z\in R^+} g_2^{0}(z)2\pi + \sum_{k=1}^{l} I_2^{0}(k) < 2\pi\sigma M, \qquad [\inf_{z\in R^+} g_{2,\infty}(z)2\pi + \sum_{k=1}^{m} I_{2,\infty}(k)]\sigma > 2\pi M.$$

where $\sigma = min\{\frac{G(0)}{G(\pi)}, \frac{1}{e^{2m\pi}}\}$, $G(0) = \frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}$, $G(\pi) = \frac{2e^{m\pi}}{2m(e^{2m\pi}-1)}$,$m = \sqrt{M}$.

2. Preliminary

In this paper, we shall consider the following space
$PC(I, R) = \{u \in C(I, R); u|_{(x_k,x_{k+1})} \in C(x_k, x_{k+1}),\ u(x_k^-) = u(x_k),\ \exists\ u(x_k^+),\ k = 1, 2, \cdots, l\}$
$PC'(I, R) = \{u \in C(I, R); u|_{(x_k,x_{k+1})}, u'|_{(x_k,x_{k+1})} \in C(x_k, x_{k+1}),\ u(x_k^-) = u(x_k),\ u'(x_k^-) = u'(x_k),\ \exists\ u(x_k^+), u'(x_k^+),\ k = 1, 2, \cdots, l\}$ with the norm $\|u\|_{PC} = \sup_{x\in[0,2\pi]} |u(x)|$, $\|u\|_{PC'} = \max\{\|u\|_{PC},\ \|u'\|_{PC}\}$, Then $PC(I, R)$,$PC'(I, R)$ are Banach spaces.

Definition 2.1: A couple function $(u, v) \in PC'(I, R) \cap C^2(I', R) \times PC'(I, R) \cap C^2(I', R)$ is called a solution of system (1.1) if it satisfies system (1.1)

Lemma 2.1: The vector $(u,v)\in PC'(I,R)\cap C^2(I',R)\times PC'(I,R)\cap C^2(I',R)$ is a solution of differential system (1.1)if and only if $(u,v)\in PC'(I,R)\times PC'(I,R)$ is a solution of the following integral system

$$\begin{cases} u(x)=\int_0^{2\pi}G(x,y)g_1(y,u(y),v(y))dy+\sum\limits_{k=1}^{l}G(x,x_k)I_{1,k}(u(x_k))+\sum\limits_{k=1}^{l}\frac{\partial G(x,y)}{\partial y}|_{y=x_k}\overline{I}_{1,k}(u(x_k)),\\ v(x)=\int_0^{2\pi}G(x,y)g_2(y,v(y),u(y))dy+\sum\limits_{k=1}^{l}G(x,x_k)I_{2,k}(v(x_k))+\sum\limits_{k=1}^{l}\frac{\partial G(x,y)}{\partial y}|_{y=x_k}\overline{I}_{2,k}(v(x_k)).\end{cases} \tag{2.1}$$

where $G(x,y)$ is the Green's function to the priodic boundary value problem $-u''+Mu=0,\ u(0)=u(2\pi),u'(0)=u'(2\pi),$ and

$$G(x,y):=\frac{1}{\Gamma}\begin{cases} e^{m(x-y)}+e^{m(2\pi-x+y)}, & 0\le y\le x\le 2\pi,\\ e^{m(y-x)}+e^{m(2\pi-y+x)}, & 0\le x\le y\le 2\pi.\end{cases}$$

here $\Gamma=2m(e^{2m\pi}-1)$.

One can find that

$$\frac{2e^{m\pi}}{2m(e^{2m\pi}-1)}=G(\pi)\le G(x,y)\le G(0)=\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}. \tag{2.2}$$

For every positive solution of problem (1.1),one has

$$\|u\|_{PC}=\sup_{x\in[0,2\pi]}|u(x)|$$

Without loss of generality, we assume $\lim\limits_{x\to\xi}|u(x)|=\|u\|_{PC},\xi\in[x_k,x_{k+1}],k\in\{0,1\dots,l\}$, then by(2.2)

$$\begin{aligned}
\|u\|_{PC} &\le G(0)\int_0^{2\pi}g_1(y,u(y),v(y))dy+\lim_{t\to\xi}\left\{\sum_{i=1}^{l}G(x,x_i)I_{1,i}(u(x_i))+\sum_{i=1}^{l}\frac{\partial G(x,y)}{\partial y}|_{y=x_i}\overline{I}_{1,i}(u(x_i))\right\}\\
&= G(0)\int_0^{2\pi}g_1(y,u(y),v(y))dy\\
&+\frac{1}{\Gamma}\left\{\sum_{i=1}^{k}[e^{m(\xi-x_i)}+e^{m(2\pi-\xi+x_i)}]I_{1,i}(u(x_i))+\sum_{i=k+1}^{l}[e^{m(x_i-\xi)}+e^{m(2\pi-x_i+\xi)}]I_{1,i}(u(x_i))\right\}\\
&+\frac{1}{\Gamma}\left\{\sum_{i=1}^{k}[-me^{m(\xi-x_i)}+me^{m(2\pi-\xi+x_i)}]\overline{I}_{1,i}(u(x_i))\right\}\\
&+\frac{1}{\Gamma}\left\{\sum_{i=k+1}^{l}[me^{m(x_i-\xi)}-me^{m(2\pi-x_i+\xi)}]\overline{I}_{1,i}(u(x_i))\right\}\\
&= G(0)\int_0^{2\pi}g_1(y,u(y),v(y))dy\\
&+\frac{1}{\Gamma}\left\{\sum_{i=1}^{k}[e^{m(\xi-x_i)}(I_{1,i}(u(x_i))-m\overline{I}_{1,i}(u(x_i)))+e^{m(2\pi-\xi+x_i)}(I_{1,i}(u(x_i))+m\overline{I}_{1,i}(u(x_i)))]\right\}\\
&+\frac{1}{\Gamma}\left\{\sum_{i=k+1}^{l}[e^{m(x_i-\xi)}(I_{1,i}(u(x_i))+m\overline{I}_{1,i}(u(x_i)))+e^{m(2\pi-x_i+\xi)}(I_{1,i}(u(x_i))-m\overline{I}_{1,i}(u(x_i)))]\right\}
\end{aligned}$$

It follows from $-\frac{1}{m}I_{1,i}(u) < \overline{I}_{1,i}(u) < \frac{1}{m}I_{1,i}(u)$, that$I_{1,i}(u)-m\overline{I}_{1,i}(u) > 0, I_{1,i}(u)+m\overline{I}_{1,i}(u) > 0$. So

$$\|u\|_{PC} \le G(0)\int_0^{2\pi} g_1(y,u(y),v(y))dy + \frac{2e^{2m\pi}}{\Gamma}\sum_{i=1}^{l} I_{1,i}(u(x_i)). \tag{2.3}$$

For any $x \in [0,2\pi]$, without loss of generality, we assume that $x \in [x_k, x_{k+1})$,,then

$$\begin{aligned} u(x) &\ge G(\pi)\int_0^{2\pi} g_1(y,u(y),v(y))dy + \sum_{i=1}^{l} G(x,x_i)I_{1,i}(u(x_i)) + \sum_{i=1}^{l}\frac{\partial G(x,y)}{\partial y}|_{y=x_i}\overline{I}_{1,i}(u(x_i)) \\ &= G(\pi)\int_0^{2\pi} g_1(y,u(y),v(y))dy \\ &+ \frac{1}{\Gamma}\sum_{i=1}^{k}[e^{m(x-x_i)}(I_{1,i}(u(x_i)) - m\overline{I}_{1,i}(u(x_i))) + e^{m(2\pi-x+x_i)}(I_{1,i}(u(x_i)) + m\overline{I}_{1,i}(u(x_i)))] \\ &+ \frac{1}{\Gamma}\sum_{i=k+1}^{l}[e^{m(x_i-x)}(I_{1,i}(u(x_i)) + m\overline{I}_{1,i}(u(x_i))) + e^{m(2\pi-x_i+x)}(I_{1,i}(u(x_i)) - m\overline{I}_{1,i}(u(x_i)))] \end{aligned}$$

It follows from $-\frac{1}{m}I_{1,i}(u) < \overline{I}_{1,i}(u) < \frac{1}{m}I_{1,i}(u)$, that $I_{1,i}(u)-m\overline{I}_{1,i}(u) > 0, I_{1,i}(u)+m\overline{I}_{1,i}(u) > 0$. So

$$\begin{aligned} u(x) &\ge G(\pi)\int_0^{2\pi} g_1(y,u(y),v(y))dy + \frac{2}{\Gamma}\sum_{i=1}^{l} I_{1,i}(u(x_i)) \\ &\ge \frac{G(\pi)}{G(0)} \bullet G(0)\int_0^{2\pi} g_1(y,u(y),v(y))dy + \frac{1}{e^{2m\pi}}\frac{2e^{2m\pi}}{\Gamma}\sum_{i=1}^{l} I_{1,i}(u(x_i)) \\ &\ge min\{\frac{G(\pi)}{G(0)}, \frac{1}{e^{2m\pi}}\}\|u\|_{PC} := \sigma\|u\|_{PC}. \end{aligned} \tag{2.4}$$

Similarly, $v(x) \ge \sigma\|v\|_{PC}$.

In applications below, we take $E = C(I,R)$ and define

$$K = \{u \in C(I,R) : u(x) \ge \sigma\|u\|_{PC}, x \in [0,2\pi]\}.$$

It is easy to see that K is a closed convex cone in E. For $r > 0$, let $K_r = \{u \in K : ||u|| < r\}$ and $\partial K_r = \{u \in K : ||u|| = r\}$. For any $(u,v) \in K \times K$,define mappings $\Phi_v : K \to C(I,R^+)$, $\Psi_u : K \to C(I,R^+)$,and $T : K \times K \to C(I,R^+) \times C(I,R^+)$ as follows

$$\begin{aligned} &\Phi_v(u)(x) = \int_0^{2\pi} G(x,y)g_1(y,u(y),v(y))dy + \sum_{k=1}^{l} G(x,x_k)I_{1,k}(u(x_k)) + \sum_{k=1}^{l}\tfrac{\partial G(x,y)}{\partial y}|_{y=x_k}\overline{I}_{1,k}(u(x_k)), \\ &\Psi_u(v)(x) = \int_0^{2\pi} G(x,y)g_2(y,v(y),u(y))dy + \sum_{k=1}^{l} G(x,x_k)I_{2,k}(v(x_k)) + \sum_{k=1}^{l}\tfrac{\partial G(x,y)}{\partial y}|_{y=x_k}\overline{I}_{2,k}(v(x_k)), \\ &T(u,v)(x) = (\Phi_v(u)(x), \Psi_u(v)(x)), \qquad x \in [0,2\pi]. \end{aligned} \tag{2.5}$$

Lemma 2.2: $T : K \times K \to K \times K$ is completely continuous.Moreover, $T(K \times K) \subset K \times K$.

Proof It is easy to see that $T : K \times K \to K \times K$ is completely continuous. Thus we only need to show $T(K \times K) \subset K \times K$, For any $(u,v) \in K \times K$, we prove $T(u,v) \in K \times K$,i.e. $\Phi_v(u) \in K$ and $\Psi_u(v) \in K$. By using inequalities (2.3) and (2.4), we have that

$$\|\Phi_v(u)\| \le G(0) \int_0^{2\pi} g_1(y, u(y), v(y))dy + \frac{2e^{2m\pi}}{\Gamma} \sum_{i=1}^{l} I_{1,i}(u(x_i))$$

$$\Phi_v(u)(x) \ge min\{\frac{G(\pi)}{G(0)}, \frac{1}{e^{2m\pi}}\}\|\Phi_v(u)\|_{PC} := \sigma\|\Phi_v(u)\|_{PC},\ x \in [0, 2\pi]$$

Similarly, $\Psi_u(v)(x) \ge \sigma\|\Psi_u(v)\|_{PC}$. Thus, $\Phi_v(u)(x) \in K$ and $\Psi_u(v)(x) \in K$. Consequently, $T(K \times K) \subset K \times K$

Lemma 2.3: Let $\Phi : K \to K$ be a completely continuous mapping with $\mu\Phi u \ne u$ for every $u \in \partial K_r$ and $0 < \mu \le 1$. Then $i(\Phi, K_r, K) = 1$.

Lemma 2.4: Let $\Phi : K \to K$ be a completely continuous mapping. Suppose that the following two conditions are satisfied:

(i) $\inf_{u \in \partial K_r} ||\Phi u|| > 0$; (ii) $\mu\Phi u \ne u$ for every $u \in \partial K_r$ and $\mu \ge 1$.

Then, $i(\Phi, K_r, K) = 0$.

Lemma 2.5: Let E be a Banach space and $K_i \subset K$ $(i = 1, 2)$ be a closed set in E. For $r_i > 0\,(i = 1, 2)$, denote $K_{r_i} = \{u \in K_i : ||u|| < r_i\}$ and $\partial K_{r_i} = \{u \in K_i : ||u|| = r_i\}$. Suppose $\Phi_i : K_i \to K_i$ is completely continuous. If $u_i \ne \Phi_i u_i$ for any $u_i \in \partial K_{r_i}$, then

$$i(\Phi, K_{r_1} \times K_{r_2}, K_1 \times K_2) = i(\Phi_1, K_{r_1}, K_1) \times i(\Phi_2, K_{r_2}, K_2)$$

where $\Phi(u,v) =: (\Phi_1 u, \Phi_2 v)$ for any $(u,v) \in K_1 \times K_2$.

3. Main Results

Theorem 3.1: Assume that $(H_1) - (H_2)$ are satisfied. Then problem (1.1) has at least one positive solution (u,v).

To prove Theorem3.1,we first give the following lemmas.

Lemma 3.1: If (H_1) is satisfied, then $i(\Phi_v, K_{R_1} \setminus \overline{K}_{r_1}, K) = 1$.

Proof Since (H_1) holds, then there exists $0 < \varepsilon < 1$ such that

$$(1-\varepsilon)[\inf_{z \in R^+} g_{1,0}(z)2\pi + \sum_{k=1}^{l} I_{1,0}(k)]\sigma > 2\pi M,$$

$$2\pi\sigma M > \sum_{k=1}^{l}(I_1^{\infty}(k) + \varepsilon) + 2\pi(\sup_{z \in R^+} g_1^{\infty}(z) + \varepsilon). \tag{3.1}$$

By the definitions of $g_{1,0},\ I_{1,0}$, one can find $r_0 > 0$ such that for any $x \in [0, 2\pi],\ 0 < u < r_0,\ v \in R^+$

$$g_1(x, u, v) \ge g_{1,0}(v)(1-\varepsilon)u,\ I_{1,k}(u) \ge I_{1,0}(k)(1-\varepsilon)u.$$

Let $r_1 \in (0, r_0)$, then for $u \in \partial K_{r_1}$, we have

$$u(x) \geq \sigma \|u\| = \sigma r_1 > 0. \quad \forall x \in [0, 2\pi]$$

Thus

$$\begin{aligned}
\Phi_v u(x) &= \int_0^{2\pi} G(x,y) g_1(y,u(y),v(y))dy + \sum_{k=1}^{l} G(x,x_k) I_{1,k}(u(x_k)) + \sum_{k=1}^{l} \frac{\partial G(x,y)}{\partial y}|_{y=x_k} \overline{I}_{1,k}(u(x_k)) \\
&\geq G(\pi) \int_0^{2\pi} g_1(y,u(y),v(y))dy + \frac{2}{\Gamma} \sum_{k=1}^{l} I_{1,k}(u(x_k)) \\
&\geq G(\pi)(1-\varepsilon) \int_0^{2\pi} g_{1,0}(v(y)) u(y) dy + \frac{2}{\Gamma}(1-\varepsilon) \sum_{k=1}^{l} I_{1,0}(k) u(x_k) \\
&\geq (1-\varepsilon)\sigma r_1 \left(\inf_{z \in R^+} g_{1,0}(z) G(\pi) 2\pi + \frac{2}{\Gamma} \sum_{k=1}^{l} I_{1,0}(k) \right)
\end{aligned}$$

from which we see that $\inf_{u \in \partial K_{r_1}} ||\Phi_v u||_{PC} > 0$, namely, hypothesis (i) of Lemma 2.4 holds. Next we show that $\mu \Phi_v u \neq u$ for any $u \in \partial K_{r_1}$, $v \in K$ and $\mu \geq 1$.

If this is not true, then there exist $u_0 \in \partial K_{r_1}$ and $\mu_0 \geq 1$ such that $\mu_0 \Phi_v u_0 = u_0$. Note that $u_0(x)$ satisfies

$$\begin{cases}
-u_0''(x) + M u_0(x) = \mu_0 g_1(x, u_0(x), v(x)), \qquad x \in I', \\
-\Delta u_0'|_{x=x_k} = \mu_0 I_{1,k}(u_0(x_k)), \quad k = 1, 2, \cdots, l, \\
\Delta u_0|_{x=x_k} = \mu_0 \overline{I}_{1,k}(u_0(x_k)), \quad k = 1, 2, \cdots, l, \\
u_0(0) = u_0(2\pi), \\
u_0'(0) = u_0'(2\pi).
\end{cases} \tag{3.2}$$

Integrate from 0 to2π, using integration by parts in the left side, notice that

$$\begin{aligned}
\int_0^{2\pi} [-u_0''(x) + M u_0(x)] dx &= \sum_{k=1}^{l} \Delta u_0'(x_k) + M \int_0^{2\pi} u_0(x) dx \\
&= -\mu_0 \sum_{k=1}^{l} I_{1,k}(u_0(x_k)) + M \int_0^{2\pi} u_0(x) dx
\end{aligned}$$

So we obtain

$$\begin{aligned}
M \int_0^{2\pi} x_0(t) dt &= \mu_0 \sum_{k=1}^{l} I_{1,k}(u_0(x_k)) + \mu_0 \int_0^{2\pi} g_1(y, u_0(y), v(y)) dy \\
&\geq (1-\varepsilon) \sum_{k=1}^{l} [(I_{1,0}(k) + \inf_{z \in R^+} g_{1,0}(z) 2\pi] \sigma r_1
\end{aligned}$$

$$2\pi M r_1 \geq (1-\varepsilon)[\sum_{k=1}^{l} (I_{1,0}(k) + \inf_{z \in R^+} g_{1,0}(z) 2\pi] \sigma r_1,$$

which contradicts with (3.1) . Hence,from Lemma 2.4 we have

$$i(\Phi, K_{r_1}, K) = 0. \qquad \forall v \in K \tag{3.3}$$

On the other hand, from (H_1), there exists $H > r_1$ such that for any $x \in [0, 2\pi],\ u \geq H,\ v \in R^+$

$$g_1(x, u, v) \leq (g_1^\infty(v) + \varepsilon)u,\ I_{1,k}(u) \leq (I_1^\infty(k) + \varepsilon)u, \tag{3.4}$$

Choose $R_1 > R_0 := \max\{\frac{H}{\sigma}, r_1\}$ and let $u \in \partial K_{R_1}, v \in K$. Since $u(x) \geq \sigma||u||_{PC} = \sigma R_1 > H$ for $x \in [0, 2\pi],\ \ v \in K$. Now we show that $\mu\Phi_v u \neq u$ for any $u \in \partial K_{R_1},\ v \in K$ and $0 < \mu \leq 1$. In fact, if there exist $u_0 \in \partial K_{R_1}$ and $0 < \mu_0 \leq 1$ such that $\mu_0 \Phi_v u_0 = u_0$, then $u_0(x)$ satisfies equation (3.2). Integrating from $0\ to\ 2\pi$,we obtain

$$\begin{aligned} M\int_0^{2\pi} u_0(x)dx &= \mu_0[\sum_{k=1}^{l} I_{1,k}(u_0(x_k)) + \int_0^{2\pi} g_1(x, u_0(x), v(x))dx] \\ &\leq \sum_{k=1}^{l}(I_1^\infty(k) + \varepsilon)u_0(x_k) + \int_0^{2\pi} u_0(x)dx(\sup_{z\in R^+} g_1^\infty(z) + \varepsilon) \\ &\leq R_1[\sum_{k=1}^{l}(I_1^\infty(k) + \varepsilon) + 2\pi(\sup_{z\in R^+} g_1^\infty(z) + \varepsilon)] \end{aligned}$$

i.e.,

$$2\pi\sigma M R_1 \leq R_1[\sum_{k=1}^{l}(I_1^\infty(k) + \varepsilon) + 2\pi(\sup_{z\in R^+} g_1^\infty(z) + \varepsilon)]$$

which is a contradiction with (3.1) .

Let $R_1 = \max\{r_1, \frac{H}{\sigma}\}$, then for any $u \in \partial K_{R_1}, v \in K$ and $0 < \mu \leq 1$, we have $\mu\Phi_v u \neq u$. Thus

$$i(\Phi, K_{R_1}\ , K) = 1. \tag{3.5}$$

In view of (3.3) and (3.5), we obtain

$$i(\Phi, K_{R_1} \setminus \overline{K}_{r_1}, K) = 1.$$

Lemma 3.2: If (H_2) is satisfied, then $i(\Psi_u, K_{R_2} \setminus \overline{K}_{r_2}\ , K) = -1$.

Proof Since (H_2) holds, there exists $0 < \varepsilon < 1$ such that

$$2\pi\sigma M > \sum_{k=1}^{l}(I_2^0(k) + \varepsilon) + 2\pi(\sup_{z\in R^+} g_2^0(z) + \varepsilon),$$

$$(1 - \varepsilon)[\inf_{z\in R^+} g_{2,\infty}(z)2\pi + \sum_{k=1}^{l} I_{2,\infty}(k)]\sigma > 2\pi M. \tag{3.6}$$

One can find $r_0 > 0$ such that for any $x \in [0, 2\pi],\ 0 \leq v \leq r_0,\ u \in R^+$

$$g_2(x, v, u) \leq (g_2^0(u) + \varepsilon)v, \quad I_{2,k}(v) \leq (I_2^0(k) + \varepsilon)v, \tag{3.7}$$

Let $r_2 \in (0, r_0)$. Now we prove that $\mu \Psi_u v \neq v$ for any $v \in \partial K_{r_2},\ u \in K$ and $0 < \mu \leq 1$. If this is not true, then there exist $v_0 \in \partial K_{r_2}$ and $0 < \mu_0 \leq 1$ such that $\mu_0 \Psi_u v_0 = v_0$. Note that $v_0(x)$ satisfies

$$\begin{cases} -v_0''(x) + Mv_0(x) = \mu_0 g_2(x, v_0(x), u(x)), \qquad x \in I', \\ -\Delta v_0'|_{x=x_k} = \mu_0 I_{2,k}(v_0(x_k)), \quad k = 1, 2, \cdots, l, \\ \Delta v_0|_{x=x_k} = \mu_0 \overline{I}_{2,k}(v_0(x_k)), \quad k = 1, 2, \cdots, l, \\ v_0(0) = v_0(2\pi), \\ v_0'(0) = v_0'(2\pi). \end{cases} \tag{3.8}$$

Integrating from 0 *to* 2π,we obtain

$$\begin{aligned} M \int_0^{2\pi} v_0(x)dx &= \sum_{k=1}^{l} \mu_0 I_{2,k}(v_0(x_k)) + \mu_0 \int_0^{2\pi} g_2(x, v_0(x), u(x))dx \\ \leq & \sum_{k=1}^{l} (I_2^0(k) + \varepsilon) v_0(x_k) + \int_0^{2\pi} v_0(x)dx (\sup_{z \in R^+} g_2^0(z) + \varepsilon) \\ \leq & \ r_2 [\sum_{k=1}^{l} (I_2^0(k) + \varepsilon) + 2\pi (\sup_{z \in R^+} g_2^0(z) + \varepsilon)]. \end{aligned}$$

so

$$2\pi \sigma M r_2 \leq r_2 [\sum_{k=1}^{l} (I_2^0(k) + \varepsilon) + 2\pi (\sup_{z \in R^+} g_2^0(z) + \varepsilon)].$$

which is a contradiction with (3.6). By Lemma 2.3, we have

$$i(\Psi_u, K_{r_2}, K) = 1. \tag{3.9}$$

On the other hand, from (H_2), there exists $H > r_2$ such that for any $x \in [0, 2\pi],\ v \geq H,\ u \in R^+$

$$g_2(x, v, u) \geq g_{2,\infty}(u)(1 - \varepsilon)v,\ I_{2,k}(v) \geq I_{2,\infty}(k)(1 - \varepsilon)v, \tag{3.10}$$

Choose $R_2 > R_0 := \max\{\frac{H}{\sigma}, r_2\}$ and let $v \in \partial K_{R_2}, u \in K$. Since $v(x) \geq \sigma ||v||_{PC} = \sigma R_2 > H$ for $x \in [0, 2\pi],\ \ u \in K$, from (3.10) we see that

$$g_2(x, v(x), u(x)) \geq g_{2,\infty}(u(x))(1 - \varepsilon)v(x) \geq \sigma g_{2,\infty}(u(x))(1 - \varepsilon)R_2,$$

$$I_{2,k}(v(x_k) \geq \sigma I_{2,\infty}(k)(1 - \varepsilon)R_2,$$

Essentially the same reasoning as above yields $\inf_{v \in \partial K_{R_2}} ||\Psi_u v||_{PC} > 0$. Next we show that if R_2 is large enough, then $\mu \Psi_u v \neq v$ for any $v \in \partial K_{R_2},\ u \in K$ and $\mu \geq 1$. In fact, if there exist $v_0 \in \partial K_{R_2}$ and $\mu_0 \geq 1$ such that $\mu_0 \Psi_u v_0 = v_0$, then $v_0(x)$ satisfies equation (3.8) . Integrate from 0 *to* 2π, using integration by parts in the left side to obtain

$$\begin{aligned} M \int_0^{2\pi} v_0(x)dx &= \sum_{k=1}^{l} \mu_0 I_{2,k}(v_0(x_k)) + \mu_0 \int_0^{2\pi} g_2(x, v_0(x), u(x))dx \\ &\geq (1 - \varepsilon)[\sum_{k=1}^{l} I_{2,\infty}(k) + \inf_{z \in R^+} g_{2,\infty}(z) 2\pi] \sigma R_2. \end{aligned}$$

So we obtain

$$2\pi M R_2 \geq (1-\varepsilon)[\sum_{k=1}^{l} I_{2,\infty}(k) + \inf_{z\in R^+} g_{2,\infty}(z)2\pi]\sigma R_2$$

which contradicts with (3.6),too.

Hence hypothesis (ii) of Lemma 2.4 is satisfied and

$$i(\Psi_u, K_{R_2}\ , K) = 0. \tag{3.11}$$

In view of (3.9) and (3.11), we obtain

$$i(\Psi_u, K_{R_2} \setminus \overline{K}_{r_2}, K) = -1$$

Proof of Theorem 3.1. Since $(H_1)-(H_2)$ are satisfied ,from Lemma2.2 we get $\Phi_v : K \to K$, $\Psi_u : K \to K$ and $T : K \times K \to K \times K$ are completely continuous. From Lemma3.1,3.2 and 2.5 we have

$$i(T, K_{R_1} \setminus \overline{K}_{r_1} \times K_{R_2} \setminus \overline{K}_{r_2}, K \times K) = i(\Phi_v, K_{R_1} \setminus \overline{K}_{r_1}, K) \times i(\Psi_u, K_{R_2} \setminus \overline{K}_{r_2}, K) = -1$$

Thus, system (1.1)has at least one positive solution (u,v).

Corollary 3.1:The conclusion of Theorem 3.1 is valid if (H_1) and (H_2) are replaced by

$$(H_1^*)\ \inf_{z\in R^+} g_{1,0}(z) = \infty \ or \ \sum_{k=1}^{l} I_{1,0}(k) = \infty;$$

$$\sup_{z\in R^+} g_1^{\infty}(z) = 0 \ and \ I_1^{\infty}(k) = 0, \quad k = 1, 2, ...l.$$

$$(H_2^*)\ \sup_{z\in R^+} g_2^{0}(z) = 0 \ and \ I_2^{0}(k) = 0, \quad k = 1, 2, ...l;$$

$$\inf_{z\in R^+} g_{2,\infty}(z) = \infty \ or \ \sum_{k=1}^{l} I_{2,\infty}(k) = \infty.$$

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