ON LINEAR OPERATORS WITH CLOSED RANGE

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ABSTRACT. Linear operators between Banach spaces such that the closedness of range of an operator implies the closedness of range of another operator are discussed.

1. Introduction

Let X and Y be normed spaces, $\mathcal{B}(X,Y)$ will denote the normed space of all continuous linear operators from X to Y. $R(T)$ and $N(T)$ will denote the range and null spaces of a linear operator T respectively. $\mathcal{CL}(X,Y)$ will denote all continuous closed range linear operators from X to Y. There are many important applications of closed range unbounded operators in the spectral study of differential operators and also in the context of perturbation theory (see e.g. [1], [2]). In this paper, we deal with continuous linear operators between Banach spaces.

A characterization of closed range bounded linear operator between two Banach spaces is given in [3]. Using this characterization, we derive results which have conclusions of having closed range of an operator when the other operator has closed range. In this section we assume that the spaces are Banach unless otherwise specified.

An operator $T \in \mathcal{B}(X)$ is called Weyl if it is Fredholm of index zero. An operator $T \in \mathcal{B}(X)$ is called Browder if it is Fredholm of finite ascent and descent. In section 2, we define norm equivalent operators, and find some properties of norm equivalent operators. Is section 3, We study Linear operators between Banach spaces such that the closedness of range of an operator implies the closedness of range of another operator. In section 4, pseudo-inverse of operators are discussed.

2. norm equivalent operators

Definition 2.1. *Let $T, S \in \mathcal{B}(X,Y)$. Two operators S and T are said to be norm equivalent and denoted by $T \sim S$ if there exist two positive real numbers k_1 and k_2 such that $k_1\|Sx\| \le \|Tx\| \le k_2\|Sx\|$, for all $x \in X$.*

Theorem 2.2. *Let $T \in \mathcal{B}(X,Y)$ and $[T] = \{S \in \mathcal{B}(X,Y) : S \sim T\}$. Then $\sim$ is an Equivalence relation, and $[T]$ is Equivalence class of T.*

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Proof. It is trivial. □

Proposition 2.3. *Let $T, S \in \mathcal{B}(X, Y)$ be norm equivalent. Then the following claims are true:*
(a) T is bounded below if and only if S is bounded below;
(b) T is closed range if and only if S is closed range;
(c) T is one-to-one if and only if S is one-to-one;
(d) αT and βS are norm equivalent, where $\alpha, \beta \in \mathbb{C} \setminus \{0\}$.

Proof. It is trivial. □

Proposition 2.4. *Let $T, S \in \mathcal{B}(X, Y)$ be norm equivalent and let $T(X) = S(X)$. Then the following claims are true:*
(a) T is Fredholm if and only if S is Fredholm;
(b) T is semi-Fredholm if and only if S is semi-Fredholm;

Proof. By hypothesis, $N(T) = N(S)$, $R(T) = R(S)$ and $\operatorname{ind} T = \operatorname{ind} S$. Therefore, the proof is trivial. □

Theorem 2.5. *Let H be a Hilbert space and $T_1, T_2 \in \mathcal{B}(H)$. Let T_1 and T_2 be norm equivalent. Let $T_1 = V_1P_1$ and $T_2 = V_2P_2$ be polar decompositions of T_1 and T_2. Then $R(P_1)$ is closed if and only if $R(P_2)$ is closed.*

Proof. By hypothesis, there exist two positive real numbers k_1 and k_2 such that $k_1\|T_1x\| \leq \|T_2x\| \leq k_2\|T_1x\|$, for all $x \in H$. Then for $x \in H$, $\|P_ix\| = \|T_ix\|$ for $i = 1, 2$. Therefore, $k_1\|P_1x\| \leq \|P_2x\| \leq k_2\|P_1x\|$. Consequently, P_1 and P_2 are norm equivalent. Hence, $R(P_1)$ is closed if and only if $R(P_2)$ is closed. □

Proposition 2.6. *Let $P \in \mathcal{B}(X, Y)$ and let $T, S \in \mathcal{B}(Y, Z)$. If T and S are norm equivalent, then $R(TP)$ is closed if and only if $R(SP)$ is closed.*

Proof. By hypothesis, there exist two positive real numbers k_1 and k_2 such that $k_1\|Ty\| \leq \|Sy\| \leq k_2\|Ty\|$, for all $y \in Y$. Therefore, $k_1\|TPx\| \leq \|SPx\| \leq k_2\|TPx\|$, for all $x \in X$. Consequently, TP and SP are norm equivalent. Hence, $R(TP)$ is closed if and only if $R(SP)$ is closed. □

Corollary 2.7. *Let $T, S \in \mathcal{B}(X)$ be norm equivalent and let $TS = ST$. Then*
(a) ascent $T < \infty$ if and only if ascent $S < \infty$.
(b) T is nilpotent if and only if S is nilpotent.

Proof. It follows from Proposition 2.6 □

Proposition 2.8. *Let $T, S \in \mathcal{B}(X, Y)$ and let $P \in \mathcal{B}(Y, Z)$ be an isometry. If T and S are norm equivalent, then $R(PT)$ is closed if and only if $R(PS)$ is closed.*

Proof. By hypothesis, there exist two positive real numbers k_1 and k_2 such that $k_1\|Sx\| \leq \|Tx\| \leq k_2\|Sx\|$, $\|PTx\| = \|Tx\|$ and $\|PSx\| = \|Sx\|$ for all $x \in X$. Therefore, $k_1\|PTx\| \leq \|PTx\| \leq k_2\|PTx\|$, for all $x \in X$. Consequently, PT and PS are norm equivalent. Hence, $R(PT)$ is closed if and only if $R(PS)$ is closed. □

Proposition 2.9. *Let $T_1, S_1 \in \mathcal{B}(X_1, Y_1)$ be norm equivalent and let $T_2, S_2 \in \mathcal{B}(X_2, Y_2)$ be norm equivalent. Then $R(T_1 \oplus T_2)$ is closed if and only if $R(S_1 \oplus S_2)$ is closed.*

Proof. By hypothesis, for i=1,2, there exist two positive real numbers k_1^i and k_2^i such that $k_1^i\|T_i x_i\| \leq \|S_i x_i\| \leq k_2^i\|T_i x_i\|$, for all $x_i \in X_i$. Therefore,

$$k_1(\|T_1x_1\| + \|T_2x_2\|) \leq \|S_1x_1\| + \|S_2x_2\| \leq k_2(\|T_1x_1\| + \|T_2x_2\|),$$

where $k_1 = min\{k_1^1, k_1^2\}$ and $k_2 = max\{k_2^1, k_2^2\}$. Hence, $T_1 \oplus T_2$ and $S_1 \oplus S_2$ are norm equivalent. Therefore, $R(T_1 \oplus T_2)$ is closed if and only if $R(S_1 \oplus S_2)$ is closed. □

3. Closed Range operators

Theorem 3.1. *Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ having polar decomposition $T = VP$. THen $R(T)$ is closed if and only if $R(P)$ is closed.*

Proof. By hypothesis, P and T are norm equivalent. Therefore, according to [1, Lemma 2.2] $R(T)$ is closed if and only if $R(P)$ is closed. □

Proposition 3.2. *Let $T \in \mathcal{B}(X, Y)$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Then $R(T)$ is closed if and only if $R(\lambda T)$ is closed.*

Proof. Let $R(T)$ be closed. Then There is a constant $c > 0$ such that for given $x \in X$, there is a $y \in X$ such that $Tx = Ty$ and $\|y\| \leq c\|Tx\|$. Therefore, for $\lambda \in \mathbb{C} \setminus \{0\}$ we have $\lambda Tx = \lambda Ty$ and $\|y\| \leq \frac{c}{|\lambda|}\|\lambda Tx\|$. Consequently $R(\lambda T)$ is closed.

Coversely, let $\lambda \in \mathbb{C} \setminus \{0\}$ and let $R(\lambda T)$ be closed, then $R(T) = R(\frac{1}{\lambda}\lambda T)$ is closed. □

Proposition 3.3. *Let $T \in \mathcal{B}(X, Y)$ be onto and let $S \in \mathcal{B}(Y, Z)$. If $R(ST)$ is closed, then $R(S)$ is closed.*

Proof. Let $y_1 \in Y$. Then there is $x_1 \in X$ such that $Tx_1 = y_1$. By hypothesis, there is a constant $c > 0$ such that for $x_1 \in X$, there is $x_2 \in X$ such that $STx_1 = STx_2$ and $\|x_2\| \leq c\|STx_1\|$. Therefore, $Sy_1 = Sy_2$ where $y_2 = Tx_2$ and $\|y_2\| \leq c\|T\|\|Sy_1\|$. Consequently, $R(S)$ is closed.

□

Proposition 3.4. *Let $T \in \mathcal{B}(X, Y)$ be invertible and let $S \in \mathcal{B}(Y, Z)$ be closed range. Then $R(ST)$ is closed.*

Proof. By hypothesis, there is a constant $c > 0$ such that for given $x_1 \in X$, there is a $y_2 \in Y$ such that $STx_1 = Sy_2$ and $\|y_2\| \leq c\|STx_1\|$. Also, There is $\varepsilon > 0$ such that for y_2, there is a $x_2 \in X$ such that $Tx_2 = y_2$ and $\varepsilon\|x_2\| \leq \|Tx_2\| = \|y_2\| \leq c\|STx_1\|$. Therefore, $STx_1 = STx_2$ and $\|x_2\| \leq \frac{c}{\varepsilon}\|STx_1\|$. Consequently $R(ST)$ is closed. □

Theorem 3.5. *Let $T \in \mathcal{B}(X, Y)$ be invertible and let $S \in \mathcal{B}(Y, Z)$. Then $R(ST)$ is closed if and only if $R(S)$ is closed.*

Proof. It follows from Proposition 3.3 and Proposition 3.4. □

Proposition 3.6. *Let $T \in \mathcal{B}(X,Y)$ and let $S \in \mathcal{B}(Y,Z)$ be one-to-one. If $R(ST)$ is closed, then $R(T)$ is closed.*

Proof. By hypothesis, there is a constant $c > 0$ such that for given $x_1 \in X$, there is a $x_2 \in X$ such that $STx_1 = STx_2$ and $\|x_2\| \le c\|STx_1\|$. Therefore, $Tx_1 = Tx_2$ and $\|x_2\| \le c\|S\|\|Tx_1\|$. Consequently, $R(T)$ is closed. □

Proposition 3.7. *Let $T \in \mathcal{B}(X,Y)$ be closed range and let $S \in \mathcal{B}(Y,Z)$ be bounded below. Then $R(ST)$ is closed.*

Proof. By hypothesis, there is a constant $c > 0$ such that for given $x \in X$, there is a $y \in X$ such that $Tx = Ty$ and $\|y\| \le c\|Tx\|$. Also, There is $\varepsilon > 0$ such that $\varepsilon\|Tx\| \le \|S(Tx)\|$. Therefore, $S(Tx) = S(Ty)$ and $\|y\| \le \frac{c}{\varepsilon}\|STx\|$. Consequently $R(ST)$ is closed. □

Theorem 3.8. *Let $T \in \mathcal{B}(X,Y)$ and let $S \in \mathcal{B}(Y,Z)$ be invertible. Then $R(ST)$ is closed if and only if $R(T)$ is closed.*

Proof. It follows from Proposition 3.6 and Proposition 3.7. □

Corollary 3.9. *Let $T \in \mathcal{B}(X)$ be closed range and let S be invertible. then $R(ST)$ and $R(TS)$ are closed.*

Proof. It follows from Theorem 3.5 and Theorem 3.8. □

Corollary 3.10. *Let $T \in \mathcal{B}(X)$ be bounded below and let $n \in \mathbb{N}$. Then $R(T^n)$ is closed.*

Proof. By hypothesis, There is $\varepsilon > 0$ such that $\varepsilon\|x\| \le \|Tx\|$ for all $x \in X$. Therefore, $\varepsilon^n\|x\| \le \|T^n x\|$ for all $x \in X$. Consequently, T^n is bounded below, then $R(T^n)$ is closed. □

Theorem 3.11. *$\mathcal{CL}(X)$ is open in $\mathcal{B}(X)$.*

Proof. Let $T \in \mathcal{CL}(X)$ and let $A = \{ST : S \in \mathcal{B}(X) \quad \text{is invertible}\}$. Then A is open and $T \in A$. Moreover $A \subseteq \mathcal{CL}(X)$ by Corollary 3.9. □

Corollary 3.12. *Let $T, T_n \in \mathcal{B}(X)$. Let $\lim T_n = T$. If $T \in \mathcal{CL}(X)$ then, there is $N \in \mathbb{N}$ such that $T_n \in \mathcal{CL}(X)$ for all $n > N$.*

Proof. It follows from Theorem 3.11. □

Proposition 3.13. *Let $T_i \in \mathcal{B}(X_i, Y_i) \qquad i = 1, 2$. If $R(T_1)$ and $R(T_2)$ are closed, then*

(a) $R(T_1 \oplus T_2)$ is closed;
(b) $R(T_1 \otimes T_2)$ is closed.

Proof. (a) By hypothesis, there is a constant $c_i > 0$ such that for given $x_{1i} \in X_i$, there is a $x_{2i} \in X_i$ such that $T_i x_{1i} = T_i x_{2i}$ and $\|x_{2i}\| \le c_i\|T_i x_{1i}\|$. Therefore, $\|(x_{21}, x_{22})\| \le \max\{c_1, c_2\}\|(T_1 \oplus T_2)(x_{11}, x_{12})\|$. Consequently, $R(T_1 \oplus T_2)$ is closed.

(b)By hypothesis, there is a constant $c_i > 0$ such that for given $x_{1i} \in X_i$, there is a $x_{2i} \in X_i$ such that $T_i x_{1i} = T_i x_{2i}$ and $\|x_{2i}\| \leq c_i \|T_i x_{1i}\|$. Therefore, $\|(x_{21} \otimes x_{22})\| \leq c_1 c_2 \|(T_1 \otimes T_2)(x_{11} \otimes x_{12})\|$. Consequently, $R(T_1 \otimes T_2)$ is closed. □

We know that if T_1 is a linear bounded operator on a Hilbert space H_1 and T_2 is a linear bounded operator on a Hilbert space H_2 there exists a unique linear bounded operator T on $H_1 \otimes H_2$ such that

$$T(x_1 \otimes x_2) = T_1 x_1 \otimes T_2 x_2$$

for all x_1 in H_1 and x_2 in H_2. This operator is called a tensor product of operators T_1 and T_2 and denoted by $T_1 \otimes T_2$.

Corollary 3.14. *Let $T_1 \in \mathcal{B}(H_1)$ and let $T_2 \in \mathcal{B}(H_2)$. If $R(T_1)$ and $R(T_2)$ are closed, then $R(T_1 \otimes T_2)$ is closed.*

Proof. Let $R(T_1)$ and $R(T_2)$ be closed. Then $R(T_1 \otimes T_2)$ is closed by proposition 3.13. □

Theorem 3.15. *Let $T, S \in \mathcal{B}(X, Y)$ and let $T \neq 0$. If $k_1|f(Tx)| \leq |f(Sx)| \leq k_2|f(Tx)|$, for all $x \in X$, for all $f \in Y^*$, and for some $k_1 > 0$ and $k_2 > 0$, then*

(a) S is a Ferholm operator if and only if T a Ferholm operator;
(b) S is a Weyl operator if and only if T a Weyl operator;
(c) S is a Browder operator if and only if T a Browder operator;
(d) S is a compact operator if and only if T a compact operator;
(e) S is an invertible operator if and only if T an invertible operator.

Proof. By hypothesis, $S = \alpha T$ for some $\alpha \neq 0$. Therefore, $N(T) = N(S)$, $R(T) = R(S)$, $indT = indS$, $ascentT = ascentS$ and $descentT = descentS$. Hence, the proof is trivial. □

4. PSEUDO-INVERSE OF OPERATOR

Let $T \in \mathcal{B}(X, Y)$. A pseudo-inverse of T is a $S \in \mathcal{B}(Y, X)$ such that $TST = T$.

Proposition 4.1. *Let $T_i \in \mathcal{B}(X_i, Y_i)$ and let $S_i \in \mathcal{B}(Y_i, X_i)$ be a pseudo-inverse of T_i $(i = 1, 2)$. Then*

$S_1 \oplus S_2$ is a pseudo-inverse of $T_1 \oplus T_2$;

Proof. Let $x_1 \in X_1$ and $x_2 \in X_2$. Then,

$$\begin{aligned}(T_1 \oplus T_2)(S_1 \oplus S_2)(T_1 \oplus T_2)(x_1, x_2) =& (T_1 \oplus T_2)(S_1 \oplus S_2)(T_1 x_1, T_2 x_2)\\ =& (T_1 \oplus T_2)(S_1 T_1 x_1, S_2 T_2 x_2)\\ =& (T_1 S_1 T_1 x_1, T_2 S_2 T_2 x_2)\\ =& (T_1 x_1, T_2 x_2)\\ =& (T_1 \oplus T_2)(x_1, x_2)\end{aligned}$$

Therefore,$S_1 \oplus S_2$ is a pseudo-inverse of $T_1 \oplus T_2$.

□

Proposition 4.2. *Let $T \in \mathcal{B}(X,Y)$ and let $S \in \mathcal{B}(Y,X)$ be a pseudo-inverse of T. Then*

(a) If T is Fredholm operator, then S is Fredholm operator;

(b) If S is compact operator, then T is compact operator.

Proof. (a) Let T be Fredholm operator. Then TST is a Fredholm operator. Therefore, TS is a Fredholm operator. Hence, S is Fredholm operator.

(b) Let S be a compact operator. Therefore, TST is a compact operator. Hence, T is compact operator. □

Theorem 4.3. *Let $T \in \mathcal{B}(X)$ and let $S \in \mathcal{B}(X)$ be a pseudo-inverse of T. Then*

(a) If T is Freholm operator, then S is Fredholm operator;

(b) If T is Weyl operator, then S is Weyl operator;

(c) If T is invertible operator, then S is invertible operator and $S = T^{-1}$.

Proof. (a) It follows from Theorem 4.3 (a).

(b) Let T be a Weyl operator. Then S is Fredholm operator. Also

$$0 = indT = indT + indS + indT = 0 + indS + 0.$$

Therefore, $indS = 0$. Consequently, S is Weyl operator.

(c) Let T be invertible operator. Then $TS = I$ and $ST = I$. Therefore, S is invertible operator and $S = T^{-1}$. □

Proposition 4.4. *Let $T \in \mathcal{B}(H)$ and let $S \in \mathcal{B}(H)$ be a pseudo-inverse of T. Then*

(a) S^* is a pseudo-inverse of T^*;

(b)If T is unitary operator, then S is unitary operator;

Proof. (a) Let $TST = T$. Then $T^*S^*T^* = (TST)^* = T^*$. Therefore, S^* is a pseudo-inverse of T^*.

(b) Let T be unitary operator, then T is invertible and $S = T^*$. Therefore, $SS^* = S^*S = I$. Consequently, S is unitary operator; □

Proposition 4.5. *Let $T \in \mathcal{B}(X,Y)$. let $P_T = \{S \in \mathcal{B}(Y,X) : TST = T\}$. Then, P_T is convex;*

Proof. (a) Let $S_1, S_2 \in P_T$ and let $0 < \lambda < 1$. Then, $T(\lambda S_1 + (1-\lambda)S_2)T = \lambda TS_1T + (1-\lambda)TS_2T = T$. Therefore, $\lambda S_1 + (1-\lambda)S_2 \in P_T$. Consequently, P_T is convex. □

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