# Viscosity approximation methods for a nonexpansive multi-valued mapping in $CAT(0)$ spaces and variational inequality

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## Abstract

The purpose of this paper is to introduce the nonexpansive multi-valued mapping in $CAT(0)$ spaces. The convexity and closednes of a fixed point set of such mapping and demiclosed principle for such mapping are also studied. By viscosity approximation methods, We prove that the proposed implicit iteration net and sequence both converges strongly to a common fixed point of nonexpansive multi-valued mappings which is also a unique solution of the variational inequality. The results presented in the paper improve and extend various results in the current literature.

**Keywords:** Nonexpansive multi-valued mapping; Viscosity approximation methods; Complete $CAT(0)$ space; Variational inequality; $\triangle - converge$; Demiclosed principle; Implicit iteration

## 1 Introduction

The concept of variational inequalities plays an important role in various kinds of problems in pure and applied sciences,viscosity approximation methods have attracted the attention of many authors. Many important results about viscosity approximation methods of nonexpansive mappings was studied in $CAT(0)$ space. In 1976, the concept of $\triangle$-convergence in general metric spaces was coined by Lim [1], Kirk et al. [11] specialized this concept to $CAT(0)$ spaces and proved that it is very similar to the weak convergence in the Banach space setting. Dhompongsa et al. [8] and Abbas et as. [4] obtained $\triangle$-convergence theorems for the

Mann and Ishikawa iterations in the $CAT(0)$ space. In 2013, Rabian [5] [6] and Xin-Dong Liu [7] proved that viscosity approximation methods for nonexpansive mappings, hierarchical optimization problems and nonexpansive semigroups in $CAT(0)$ spaces.

Let $(X,d)$ be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from $x$ to $y$) is a map $c$ from a closed interval $[0,l] \subset \mathbb{R}$ to $X$ such that $c(0) = x$,$c(l) = y$, and $d(c(t), c(t')) = |t-t'|$ for all $t, t' \in [0,l]$. In particular, $c$ is an isometry and $d(x,y) = l$. The image $\alpha$ of $c$ is called a geodesic (ormetric) segment joining $x$ and $y$. When it is unique, this geodesic segment is denoted by $[x,y]$.The space $(X,d)$ is said to be a geodesic space if every two points of $X$ are joined by a geodesic, and $X$ is said to be uniquely geodesic if there is exactly one geodesic joining $x$and $y$ for each $x, y \in X$. A subset $Y \subset X$ is said to be convex if $Y$ includes every geodesic segment joining any two of its points. A geodesic triangle $\triangle(x_1,x_2,x_3)$ in a geodesic metric space $(X,d)$ consists of three points $\triangle(x_1,x_2,x_3)$ in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle(x_1,x_2,x_3)$ in $(X,d)$ is a triangle $\overline{\triangle}(x_1,x_2,x_3) := \triangle(\overline{x}_1,\overline{x}_2,\overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$. such that $d_{\mathbb{E}^2}(\overline{x}_i,\overline{x}_j) = d(x_i,x_j)$ for all $i,j \in 1,2,3$.

A geodesic space is said to be a $CAT(0)$ space if all geodesic triangles satisfy the following comparison axiom.

$CAT(0)$:Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x,y) \leq d_{\mathbb{E}^2}(\overline{x},\overline{y}).$$

we write $(1-t)x \oplus ty$ for the unique point $z$ in the geodesic segment joining from $x$ to $y$ such that

$$d(x,z) = td(x,y), \quad d(y,z) = (1-t)d(x,y). \tag{1.1}$$

We also denote by $[x,y]$ the geodesic segment joining from $x$ to $y$, that is, $[x,y] = \{(1-t)x \oplus ty : t \in [0,1]\}$. A subset $C$ of a $CAT(0)$ space is convex if $[x,y] \subset C$ for all $x, y \in C$.

The following lemmas play an important role in our paper.

**Lemma 1.1** *Let $X$ be a $CAT(0)$ space. Then for any $x,y,z,w \in X$ and $t,s \in [0,1]$*

*(i) (see [8] ) $d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z)$;*

*(ii) (see [9] ) $d((1-t)x \oplus ty, (1-s)x \oplus sy) \leq |t-s|d(x,y)$;*

*(iii) (see [10] ) $d((1-t)x \oplus ty, (1-t)z \oplus tw) \leq (1-t)d(x,z) + td(y,w)$;*

*(iv)* *(see [11])* $d((1-t)z \oplus tx, (1-t)z \oplus ty) \le td(x,y)$;

*(v)* *(see [8] )* $d^2((1-t)x \oplus ty, z) \le (1-t)d^2(x,z) + td^2(y,z) - t(1-t)d^2(x,y)$;

If $x, y_1, y_2$ are points in a $CAT(0)$ space and if $y_0$ is the midpoint of the segment $[y_1, y_2]$, then the $CAT(0)$ inequality implies

$$d^2(y_0, x) \le \frac{1}{2}d^2(y_1, x) + \frac{1}{2}d^2(y_2, x) - \frac{1}{4}d^2(y_1, y_2) \tag{1.2}$$

This is the (CN)-inequality of Bruhat and Tits [12]. In fact ( [10], p.163), a geodesic space is a $CAT(0)$ space if and only if it satisfies the (CN)-inequality.

It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a $CAT(0)$ space. Other examples include pre-Hilbert spaces, R-trees (see [10]), Euclidean buildings (see [13]), the complex Hilbert ball with a hyperbolic metric (see [14]), and many others. Complete $CAT(0)$ spaces are often called Hadamard spaces.

It is proved in [10] that a normed linear space satisfies the (CN)-inequality if and only if it satisfies the parallelogram identity, i.e., is a pre-Hilbert space; hence it is not so unusual to have an inner product-like notion in Hadamard spaces. Berg and Nikolaev [15] introduced the concept of quasilinearization as follows.

Let us formally denote a pair $(a,b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then quasilinearization is defined as a map $\langle .,. \rangle : (X \times X, X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)), \quad a,b,c,d \in X. \tag{1.3}$$

It is easily seen that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$,and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ for all $a,b,c,d,x \in X$.

We say that $X$ satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \le d(a,b)d(c,d), \quad a,b,c,d \in X. \tag{1.4}$$

It is known [15] that a geodesically connected metric space is a $CAT(0)$ space if and only if it satisfies the Cauchy-Schwarz inequality.

In 2010, Kakavandi and Amini [16] introduced the concept of a dual space for $CAT(0)$ spaces as follows. Consider the map $\Theta : \mathbb{R} \times X \times X \to C(X)$ difined by

$$\Theta(t,a,b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle,$$

where $C(X)$ is the space of all continuous real-valued functions on $X$. Then the Cauchy-Schwarz inequality implies that $\Theta(t,a,b)$ is a Lipschitz function with a Lipschitz semi-norm $L(\Theta(t,a,b)) = td(a,b)$ for all $t \in \mathbb{R}$ and $a, b \in X$, where

$$L(f) = sup\{\frac{f(x) - f(y)}{d(x,y)} : x, y \in X, x \neq y\}$$

is the Lipschitz semi-norm of the function $f : X \to \mathbb{R}$. Now, define the pseudometric $D$ on $\mathbb{R} \times X \times X$ by

$$D((t,a,b),(s,c,d)) = L(\Theta(t,a,b) - \Theta(s,c,d)).$$

**Lemma 1.2** *(see [16]) $D((t,a,b),(s,c,d)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy}\rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy}\rangle$ for all $x, y \in X$.*

For a complete $CAT(0)$ space $(X,d)$, the pseudometric space $(\mathbb{R} \times X \times X, D)$ can be considered as a subspace of the pseudometric space $(Lip(X,\mathbb{R}), L)$ of all real-valued Lipschitz functions. Also, $D$ defines an equivalence relation on $\mathbb{R} \times X \times X$, where the equivalence class of $t\overrightarrow{ab} := (t,a,b)$ is

$$[t\overrightarrow{ab}] = \{s\overrightarrow{cd} : t\langle \overrightarrow{ab}, \overrightarrow{xy}\rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy}\rangle, \forall x, y \in X\}.$$

The set $X^* := \{[t\overrightarrow{ab}] : (t,a,b) \in \mathbb{R} \times X \times X\}$ is a metric space with metric $D$, which is called the dual metric space of $(X,d)$.

In 2012, Dehghan and Rooin [17] introduced the duality mapping in $CAT(0)$ spaces and studied its relation with subdifferential, by using the concept of quasilinearization. Then they presented a characterization of metric projection in $CAT(0)$ spaces as follows.

**Lemma 1.3** *( [17],Theorem 2.4) Let $C$ be a nonempty convex subset of a complete $CAT(0)$ space $X$, $x \in X$ and $u \in C$. Then $u = P_C x$ if and only if $\langle \overrightarrow{yu}, \overrightarrow{ux}\rangle \geq 0$ for all $y \in C$*

**Definition 1.1** *Let $C$ be a nonempty subset of a complete $CAT(0)$ space $X$,and let $N(C)$ and $CB(C)$ denote the family of nonempty subsets and nonempty bounded closed subsets of C, respectively.The multi-valued mapping $T : C \to CB(C)$ is called nonexpansive iff $H(Tx,Ty) \leq d(x,y)$ for all $x, y \in C$, where $H(.,.)$ is Hausdorff metric,i.e.,$H(Tx,Ty) = \max\{\sup_{x \in Tx} d(x,Ty), \sup_{y \in Ty} d(y,Tx)\}$.*

A point $x \in C$ is called a fixed point of $T$ if $x \in Tx$. We denote by $F(T)$ the set of all fixed points of $T$.

**Remark 1.1** *The existence of fixed points for multivalued nonexpansive mappings in a $CAT(0)$ space was proved by S. Dhompongsa et al. [8].*

**Definition 1.2** *Let $C$ be a nonempty subset of a complete $CAT(0)$ space $X$, the multi-valued mapping $T : C \to CB(C)$ is called quasi- nonexpansive iff $F(T) \neq \emptyset$ and $H(Tx, p) \leq d(x, p)$ for all $x \in C, p \in F(T)$.*

A mapping $f$ of $C$ into itself is called contraction with coefficient $\alpha \in (0, 1)$ iff $d(f(x), f(y)) \leq \alpha d(x, y)$ for all $x, y \in C$. Banach's contraction principle guarantees that $f$ has a unique fixed point when $C$ is a nonempty closed convex subset of a complete metric space.

In 2013, Ranbian Wangkeeree [19] studied the convergence theorems of the following Moudafi's viscosity iterations for a nonexpansive self mapping $T$: For a contraction $f$ on $C$ and $t(0, 1)$, let $x_t \in C$ be the unique fixed point of the contraction $x \to tf(x) \oplus (1-t)Tx$,i.e.,

$$x_t = tf(x_t) \oplus (1-t)Tx_t, \tag{1.5}$$

and $x_0 \in C$ is arbitrarily chosen and

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)Tx_n, \quad n \geq 0, \tag{1.6}$$

where $\alpha_n \in (0, 1)$.They proved that $\{x_t\}, \{x_n\}$ converges strongly to $\tilde{x} \in F(T)$ such that $\tilde{x} = P_{F(T)} f(\tilde{x})$ in the framework of a CAT(0) space,which is the unique solution of the variational inequality (VIP)

$$\langle \overrightarrow{\tilde{x} f \tilde{x}}, \overrightarrow{x \tilde{x}} \rangle \geq 0, \quad x \in F(T). \tag{1.7}$$

The purpose of this paper is to study the strong convergence about Moudafi's viscosity approximation methods for approximating a common fixed point of a nonexpansive multi-valued mapping in $CAT(0)$ spaces. We prove that the proposed implicit iteration net and sequence both converges strongly to a common fixed point of nonexpansive multi-valued mappings which is also a unique solution of the variational inequality. The convexity and closednes of a fixed point set of such mapping and demiclosed principle for such mapping are also studied. The results presented in the paper improve and extend Rabian's various results [7] [19] in the current literature and other.

## 2 Preliminaries

In order to study our results in the general setup of $CAT(0)$ spaces, we first collect some basic concepts. Let $\{x_n\}$ be a bounded sequence in $CAT(0)$ space $X$. For $p \in X$, define a continuous functional $r(., \{x_n\}) : X \to [0, +\infty)$ by

$$r(p, \{x_n\}) = \limsup_{n\to\infty} d(p, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(p, \{x_n\}) : p \in X\}.$$

The asymptotic radius $r_C(\{x_n\})$ of $\{x_n\}$ with respect to $C \subset X$ is given by

$$r_C(\{x_n\}) = \inf\{r(p, \{x_n\}) : p \in C\}.$$

The asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{p \in E : r(p, \{x_n\}) = r(\{x_n\})\}.$$

The asymptotic center $A_C(\{x_n\})$ of $\{x_n\}$ with respect to $C \subset X$ is the set

$$A_C(\{x_n\}) = \{p \in C : r(p, \{x_n\}) = r_C(\{x_n\})\}.$$

A sequence $\{x_n\}$ in $CAT(0)$ space $X$ is said to $\triangle$-converge to $p \in X$ if $p$ is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case,we call $p$ the $\triangle$-limit of $\{x_n\}$.

**Remark 2.1** *The uniqueness of an asymptotic center implies that the $CAT(0)$ space $X$ satisfies Opial's property,i.e.,for given $\{x_n\} \subset X$ such that $\{x_n\}$ $\triangle$-converge to $x$ and given $y \in X$ with $y \neq x$,*

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y)$$

The following lemmas are important in our paper.

**Lemma 2.1** *(see [8]) If $C$ is a closed convex subset of a complete $CAT(0)$ space and if $\{x_n\}$ is a bounded sequence in $C$, then the asymptotic center of $\{x_n\}$ is in $C$.*

**Lemma 2.2** *(see [8] [26]) Every bounded sequence in a complete $CAT(0)$ space always has a $\triangle$-convergent subsequence.*

**Lemma 2.3** *(see [18]) If $C$ is a closed convex subset of $X$ and $T : C \to X$ is a asymptotically nonexpansive mapping, then the conditions $\{x_n\}$ $\triangle$-convergence to $x$ and $d(x_n, Tx_n) \to 0$ imply $x \in C$ and $Tx = x$.*

Having the notion of quasilinearization, Kakavandi and Amini [20] introduced the following notion of convergence. A sequence $\{x_n\}$ in the complete $CAT(0)$ space $(X, d)$ $w$-converges to $x \in X$ if $\lim_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle = 0$,i.e.,$\lim_{n\to\infty}(d^2(x_n, x) + d^2(y, x) - d^2(x_n, y)) = 0$ for all $y \in X$.

It is obvious that convergence in the metric implies $w$-convergence, and it is easy to check that $w$-convergence implies $\triangle$-convergence , but it is showed in ( [20], Example 4.7) that the converse is not valid.

However, the following lemma shows another characterization of $\triangle$-convergence as well as, more explicitly, a relation between $w$-convergence and $\triangle$-convergence.

**Lemma 2.4** *(see [20], Theorem 2.6) Let $X$ be a complete $CAT(0)$ space, $\{x_n\}$ be a sequence in $X$,and $x \in X$. Then $\{x_n\}$ $\triangle$-convergence to $x$ if and only if $\limsup_{n\to\infty}\langle\overrightarrow{x_n x}, \overrightarrow{xy}\rangle \leq 0$ for all $y \in X$.*

**Lemma 2.5** *(see [19]) Let $X$ be a complete $CAT(0)$ space. Then for all $u, x, y \in X$, the following inequality holds:*

$$d^2(x, u) \leq d^2(y, u) + 2\langle\overrightarrow{xy}, \overrightarrow{xu}\rangle.$$

**Lemma 2.6** *(see [19]) Let $X$ be a complete $CAT(0)$ space. For all $u, v \in X$ and $t \in [0, 1]$, let $u_t = tu \oplus (1-t)v$. Then,for all $x, y \in X$*

(i) $\langle\overrightarrow{u_t x}, \overrightarrow{u_t y}\rangle \leq t\langle\overrightarrow{ux}, \overrightarrow{u_t y}\rangle + (1-t)\langle\overrightarrow{vx}, \overrightarrow{u_t y}\rangle$;

(ii) $\langle\overrightarrow{u_t x}, \overrightarrow{uy}\rangle \leq t\langle\overrightarrow{ux}, \overrightarrow{uy}\rangle + (1-t)\langle\overrightarrow{vx}, \overrightarrow{uy}\rangle$ *and* $\langle\overrightarrow{u_t x}, \overrightarrow{u_t y}\rangle \leq t\langle\overrightarrow{ux}, \overrightarrow{vy}\rangle + (1-t)\langle\overrightarrow{vx}, \overrightarrow{vy}\rangle$.

**Lemma 2.7** *(see [21],Lemma2.1) Let $\{a_n\}$ be sequences of nonnegative numbers such that*

$$a_{n+1} \leq (1-\delta_n)a_n + \delta_n\gamma_n, \quad \forall n \geq 1,$$

*where $\{\delta_n\}$,$\{\gamma_n\}$ satisfy following property*

(1) $\{\delta_n\} \subset (0, 1)$ *and* $\{\gamma_n\} \subset \mathbb{R}$;

(2) $\sum_{n=1}^{+\infty} \delta_n = +\infty$;

(3) $\limsup_{n\to\infty} \gamma_n \leq 0$ *or* $\sum_{n=0}^{+\infty} |\delta_n\gamma_n| < +\infty$.

*then* $\lim_{n\to\infty} a_n = 0$.

## 3 Main results

In this section,we present strong convergence theorem of Moudafi's viscosity methods for multi valued non-expansive mappings $T$ in $CAT(0)$ spaces.

For any $t \in (0, 1)$ and a contraction $f$ with coefficient $\alpha \in (0, 1)$, define the mapping $S_t : C \to CB(C)$ by

$$S_t(x) = tf(x) \oplus (1-t)u(x), \quad x \in C, u(x) \in Tx. \tag{3.1}$$

It is not hard to see that $S_t$ is a contraction on $C$. Indeed, since $d(u(x), u(y) \leq H(Tx, Ty)$ for any $x, y \in C$, we have

$$
\begin{aligned}
d(S_t(x), S_t(y)) &= d(tf(x) \oplus (1-t)u(x), tf(y) \oplus (1-t)u(y)) \\
&\leq d(tf(x) \oplus (1-t)u(x), tf(y) \oplus (1-t)u(x)) + d(tf(y) \oplus (1-t)u(x), tf(y) \oplus (1-t)u(y)) \\
&\leq td(f(x), f(y)) + (1-t)d(u(x), u(y)) \\
&\leq td(f(x), f(y)) + (1-t)H(Tx, Ty) \\
&\leq (t\alpha + (1-t))d(x, y) \\
&= (1 - t(1-\alpha))d(x, y),
\end{aligned}
$$

this implies that $S_t$ is a contraction on $C$. Then there exists a unique $q \in C$ such that

$$q = S_t(q) = tf(q) \oplus (1-t)u(q), \quad u(q) \in Tq$$

Now we prove convergence theorem for the following implicit iterative net. Let $x_t \in C$ be the unique fixed point of $S_t$. thus for any $t \in (0, 1]$,

$$x_t = S_t(x_t) = tf(x_t) \oplus (1-t)u(x_t), \quad u(x_t) \in T(x_t).$$

First,we prove following demiclosed principle for nonexpansive multi-valued mapping.

**Proposition 3.1** *If $C$ is a closed convex subset of $X$ and $T : C \to CB(C)$ is a nonexpansive multi-valued mapping, then the conditions $\{x_n\}$ $\triangle$-convergence to $p$ and $d(x_n, z_n) \to 0$ (which $z_n \in Tx_n$) imply $p \in Tp$.*

**Proof**. By lemma 2.1,2.2, since $\{x_n\}$ $\triangle$-convergence to $p$, hence $A_C\{x_n\} = p$ and $A\{x_n\} = p$.
Letting $\psi(x) := \limsup_{n\to\infty} d(x_n, x)$, from condition $d(x_n, z_n) \to 0$, we get that

$$\psi(x) = \limsup_{n\to\infty} d(z_n, x).$$

If $p* \in Tp$, then

$$\psi(p*) = \limsup_{n\to\infty} d(z_n, p*) \leq \limsup_{n\to\infty} H(Tx_n, Tp) \leq \limsup_{n\to\infty} d(x_n, p) = \psi(p).$$

From (1.2),we have that

$$d^2(x_n, \frac{1}{2}(p \oplus p*)) \leq \frac{1}{2}d^2(x_n, p) + \frac{1}{2}d^2(x_n, p*) - \frac{1}{4}d^2(p, p*).$$

Letting $n \to \infty$ and taking superior limit on the both sides, it gets that

$$\psi^2(\frac{1}{2}(p \oplus p*)) \leq \frac{1}{2}\psi^2(p) + \frac{1}{2}\psi^2(p*) - \frac{1}{4}d^2(p, p*),$$

that is,

$$d^2(p, p*) \leq 2(\psi^2(p*) - \psi^2(p)) \leq 0.$$

It implies that $p* = p$, so $Tp = \{p\}$,i.e., $p \in TP$.This completes the proof of Proposition 3.1

Next, we prove the closedness and convexity of fixed point set of nonexpansive multi-valued mapping.

**Proposition 3.2** *If $C$ is a closed convex subset of $X$ and $T : C \to CB(C)$ is a nonexpansive multi-valued mapping, then F(T) is a closed and convex subset of $C$.*

**Proof**. As $T$ is continuous, so $F(T)$ is closed. In order to prove that $F(T)$ is convex, it is enough to prove that $\frac{1}{2}(p \oplus q) \in F(T)$ where $p, q \in F(T)$. Setting $w = \frac{1}{2}(p \oplus q)$ and $w* \in Tw$, using (1.2), we have

$$\begin{aligned} d^2(w*, w) &\leq \frac{1}{2}d^2(w*, p) + \frac{1}{2}d^2(w*, q) - \frac{1}{4}d^2(p, q) \\ &\leq \frac{1}{2}H^2(Tw, Tp) + \frac{1}{2}H^2(Tw, Tq) - \frac{1}{4}d^2(p, q) \\ &\leq \frac{1}{2}d^2(w, p) + \frac{1}{2}d^2(w, q) - \frac{1}{4}d^2(p, q) \\ &\leq \frac{1}{2}d^2(\frac{1}{2}(p \oplus q), p) + \frac{1}{2}d^2(\frac{1}{2}(p \oplus q), q) - \frac{1}{4}d^2(p, q) \\ &\leq \frac{1}{2}(\frac{1}{4}d^2(q, p)) + \frac{1}{2}(\frac{1}{4}d^2(p, q)) - \frac{1}{4}d^2(p, q) = 0, \end{aligned}$$

it implies that $w* = w$, so $Tw = \{w\}$,i.e., $w \in Tw$.This completes the proof of Proposition 3.2

Now, we prove strong convergence theorem of Moudafi's viscosity methods for multi-valued non- expansive mapping T in $CAT(0)$ spaces.

**Theorem 3.1** *Let $C$ be a closed convex subset of a complete $CAT(0)$ space $X$. Let $T : C \to CB(C)$ be a nonexpansive multi-valued mapping, let $f$ be a contraction on $C$ with coefficient $0 < \alpha < 1$. For each $t \in (0, 1]$, net $\{x_t\}$ be given by following implicit iterative,*

$$x_t = tf(x_t) \oplus (1 - t)u(x_t), \quad u(x_t) \in T(x_t). \tag{3.2}$$

*If $F(T) \neq \emptyset$, then $\{x_t\}$ converges strongly as $t \to 0$ to $\tilde{x} = P_{F(T)}f(\tilde{x})$ which is equivalent to the following variational inequality:*

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in F(T). \tag{3.3}$$

**Proof**. First, for any $p \in F(T)$, we have that

$$\begin{aligned} d(x_t, p) &= d(tf(x_t) \oplus (1-t)u(x_t), p) \\ &\leq td(f(x_t), p) + (1-t)d(u(x_t), p) \\ &\leq td(f(x_t), p) + (1-t)H(Tx_t, Tp) \\ &\leq t(d(f(x_t), f(p)) + d(f(p), p)) + (1-t)d(x_t, p) \\ &\leq (t\alpha + (1-t))d(x_t, p) + td(f(p), p), \end{aligned}$$

that is,

$$d(x_t, p) \leq \frac{1}{1-\alpha} d(f(p), p),$$

hence $\{x_t\}$ is bounded, so are both $\{u(x_t)\}$ and $\{f(x_t)\}$. We get that

$$\begin{aligned} \lim_{t \to 0} d(x_t, u(x_t)) &= \lim_{t \to 0} d(tf(x_t) \oplus (1-t)u(x_t), u(x_t)) \\ &\leq \lim_{t \to 0} td(f(x_t), u(x_t)) = 0 \end{aligned}$$

Now we prove that $\{x_t\}$ is relatively compact as $t \to 0$. In fact,letting $m \in \mathbb{N}$ and $x_m := x_{t_m}$ with $t_m \in (0,1]$ and $t_m \to 0$ as $m \to \infty$,

since $\{x_m\}$ is bounded, by Lemma 2.2,2.4,we may assume $\{x_m\}$ $\triangle$-converges to a point $\tilde{x} \in F(T)$. From Lemma 2.6, we have

$$\begin{aligned} d^2(x_m, \tilde{x}) &= \langle \overrightarrow{x_m \tilde{x}}, \overrightarrow{x_m \tilde{x}} \rangle \\ &\leq t_m \langle \overrightarrow{f(x_m)\tilde{x}}, \overrightarrow{x_m \tilde{x}} \rangle + (1-t_m)\langle \overrightarrow{u(x_m)\tilde{x}}, \overrightarrow{x_m \tilde{x}} \rangle \\ &\leq t_m \langle \overrightarrow{f(x_m)\tilde{x}}, \overrightarrow{x_m \tilde{x}} \rangle + (1-t_m) d(u(x_m), \tilde{x}) d(x_m, \tilde{x}) \\ &\leq t_m \langle \overrightarrow{f(x_m)\tilde{x}}, \overrightarrow{x_m \tilde{x}} \rangle + (1-t_m) H(Tx_m, \tilde{x}) d(x_m, \tilde{x}) \\ &\leq t_m \langle \overrightarrow{f(x_m)\tilde{x}}, \overrightarrow{x_m \tilde{x}} \rangle + (1-t_m) d^2(x_m, \tilde{x}) \\ &\leq t_m \langle \overrightarrow{f(x_m)f(\tilde{x})}, \overrightarrow{x_m \tilde{x}} \rangle + t_m \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_m \tilde{x}} \rangle + (1-t_m) d^2(x_m, \tilde{x}) \\ &\leq t_m \alpha d^2(x_m, \tilde{x}) + t_m \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_m \tilde{x}} \rangle + (1-t_m) d^2(x_m, \tilde{x}), \end{aligned}$$

thus

$$d^2(x_m, \tilde{x}) \leq \frac{1}{1-\alpha} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_m \tilde{x}} \rangle. \tag{3.4}$$

Since $\{x_m\}$ $\triangle$-converges to a point $\tilde{x} \in F(T)$, by Lemma 2.4, we have

$$\limsup_{m \to \infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_m \tilde{x}} \rangle \leq 0. \tag{3.5}$$

From (3,4),(3.5),we get that $\lim_{m\to\infty} x_m = \tilde{x}$.

Now we show that $\tilde{x} \in F(T)$ solves the variational inequality (3.3).

By Lemma 1.1, for any $q \in F(T)$, we have

$$\begin{aligned} d^2(x_t, q) &\leq td^2(f(x_t), q) + (1-t)d^2(u(x_t), q) - t(1-t)d^2(f(x_t), u(x_t)) \\ &\leq td^2(f(x_t), q) + (1-t)H^2(Tx_t, q) - t(1-t)d^2(f(x_t), u(x_t)) \\ &\leq td^2(f(x_t), q) + (1-t)d^2(x_t, q) - t(1-t)d^2(f(x_t), u(x_t)) \\ &= td^2(f(x_t), q) + (1-t)d^2(x_t, q) - t(1-t)d^2(f(x_t), u(x_t)), \end{aligned}$$

it implies that

$$d^2(x_t, q) \leq d^2(f(x_t), q) - (1-t)d^2(f(x_t), u(x_t)).$$

so we get that

$$d^2(x_m, q) \leq d^2(f(x_m), q) - (1-t_m)d^2(f(x_m), u(x_m)).$$

Taking the limit through $m \to \infty$ and noting $d(x_t, u(x_t)) \to 0$, by Proposition 3.1, we can get that

$$d^2(\tilde{x}, q) \leq d^2(f(\tilde{x}), q) - d^2(f(\tilde{x}), \tilde{x}).$$

hence

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{q\tilde{x}} \rangle = \frac{1}{2}(d^2(f(\tilde{x}), q) + d^2(\tilde{x}, \tilde{x}) - d^2(f(\tilde{x}), \tilde{x}) - d^2(\tilde{x}, q)) \geq 0,$$

where $q \in F(T)$, it implies that $\tilde{x}$ solves the variational inequality (3.3).

Finally, we show the entire net $\{x_t\}$ converges to $\tilde{x}$.

Assume $x_{s_m} \to x^* \in F(T)$, where $s_m \to 0$ as $m \to \infty$. By same argument, we get $x^*$ solves the variational inequality (3.3),i.e.,

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{\tilde{x}x^*} \rangle \leq 0, \quad \langle \overrightarrow{x^*f(x^*)}, \overrightarrow{x^*\tilde{x}} \rangle \leq 0. \tag{3.6}$$

By (3.6), we get that

$$\begin{aligned}
0 &\geq \langle\overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{\tilde{x}x^*}\rangle - \langle\overrightarrow{x^*f(x^*)}, \overrightarrow{\tilde{x}x^*}\rangle \\
&= \langle\overrightarrow{\tilde{x}f(x^*)}, \overrightarrow{\tilde{x}x^*}\rangle + \langle\overrightarrow{f(x^*)f(\tilde{x})}, \overrightarrow{\tilde{x}x^*}\rangle - \langle\overrightarrow{x^*\tilde{x}}, \overrightarrow{\tilde{x}x^*}\rangle - \langle\overrightarrow{\tilde{x}f(x^*)}, \overrightarrow{\tilde{x}x^*}\rangle \\
&= \langle\overrightarrow{\tilde{x}x^*}, \overrightarrow{\tilde{x}x^*}\rangle - \langle\overrightarrow{f(x^*)f(\tilde{x})}, \overrightarrow{x^*\tilde{x}}\rangle \\
&\geq d^2(\tilde{x}, x^*) - d(f(x^*), f(\tilde{x}))d(x^*, \tilde{x}) \\
&\geq d^2(\tilde{x}, x^*) - \alpha d^2(\tilde{x}, x^*) = (1-\alpha)d^2(\tilde{x}, x^*),
\end{aligned}$$

it implies $d^2(\tilde{x}, x^*) = 0$, so is $\tilde{x} = x^*$.Hence the net $\{x_t\}$ converges strongly to $\tilde{x}$ which is the unique solution to the variational inequality (3.3). This completes the proof of Theorem 3.1.

**Theorem 3.2** *Let $C$ be a nonempty bounded closed convex subset of a complete $CAT(0)$ space $X$. Let $f$ be be a contraction on $C$ with coefficient $\alpha \in (0,1)$, and let $T : C \to CB(C)$ be nonexpansive multi-valued mapping. For the arbitrary initial point $x_1 \in C$, Let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = t_n f(x_n) \oplus (1-t_n)u(x_n), \quad u(x_n) \in Tx_n, \quad n \geq 1, \tag{3.7}$$

*where sequence $\{t_n\}$ satisfies the following conditions:*

*(i)* $\{t_n\} \subset (0,1)$ *and* $\lim_{n\to\infty} t_n = 0$;

*(ii)* $\sum_{n=1}^{+\infty} t_n = +\infty$;

*(iii)* $\sum_{n=0}^{+\infty} |t_{n+1} - t_n| < +\infty$.

*If $F(T) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to some point $\tilde{x} \in F(T)$ which is equivalent to the variational inequality (3.3).*

**Proof**.

**(I)We first show that the sequence $\{x_n\}$ is bounded.**

Indeed, For $p \in F(T)$, we have

$$\begin{aligned} d(x_{n+1}, p) &= d(t_n f(x_n) \oplus (1-t_n)u(x_n), p) \\ &\leq t_n d(f(x_n), p) + (1-t_n) d(u(x_n), p) \\ &\leq t_n d(f(x_n), p) + (1-t_n) H(Tx_n, p) \\ &\leq t_n d(f(x_n), p) + (1-t_n) d(x_n, p) \\ &\leq t_n (d(f(x_n), f(p)) + d(f(p), p)) + (1-t_n) d(x_n, p) \\ &\leq t_n (\alpha d(x_n, p) + d(f(p), p)) + (1-t_n) d(x_n, p) \\ &= (t_n \alpha + (1-t_n)) d(x_n, p) + t_n d(f(p), p). \end{aligned}$$

Let $M_n = \max\{d(x_n, p), \frac{1}{1-\alpha} d(f(p), p)\}$, we have

$$d(x_{n+1}, p) \leq M_n = \max\{d(x_n, p), \frac{1}{1-\alpha} d(f(p), p)\}$$

By induction, we get that

$$d(x_{n+1}, p) \leq \max\{d(x_1, p), \frac{1}{1-\alpha} d(f(p), p)\},$$

hence $\{x_n\}$ is bounded, so are $\{u(x_n)\}$ and $\{f(x_n)\}$.

**(II) We claim that $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.**

Since

$$\begin{aligned} d(x_{n+1}, x_n) &= d(t_n f(x_n) \oplus (1-t_n)u(x_n), t_{n-1} f(x_{n-1}) \oplus (1-t_{n-1}) u(x_{n-1})) \\ &\leq d(t_n f(x_n) \oplus (1-t_n)u(x_n), t_n f(x_n) \oplus (1-t_n) u(x_{n-1}) \\ &+ d(t_n f(x_n) \oplus (1-t_n)u(x_{n-1}), t_n f(x_{n-1}) \oplus (1-t_n) u(x_{n-1})) \\ &+ d(t_n f(x_{n-1}) \oplus (1-t_n)u(x_{n-1}), t_{n-1} f(x_{n-1}) \oplus (1-t_{n-1}) u(x_{n-1})) \\ &\leq (1-t_n) d(u(x_n), u(x_{n-1})) + t_n d(f(x_n), f(x_{n-1})) + |t_n - t_{n-1}| d(f(x_{n-1}), u(x_{n-1})) \\ &\leq (1-t_n) H(Tx_n, Tx_{n-1}) + t_n d(f(x_n), f(x_{n-1})) + |t_n - t_{n-1}| d(f(x_{n-1}), u(x_{n-1})) \\ &\leq ((1-t_n) + t_n \alpha) d(x_n, x_{n-1}) + |t_n - t_{n-1}| d(u(x_{n-1}), f(x_{n-1})) \\ &\leq (1 - t_n(1-\alpha)) d(x_n, x_{n-1}) + |t_n - t_{n-1}| d(u(x_{n-1}), f(x_{n-1})). \end{aligned}$$

By condition (ii),(iii) and Lemma 2.7, we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.8}$$

Let the net $\{x_t\} \subset C$ and

$$x_t = tf(x_t) \oplus (1-t)u(x_t), \quad u(x_t) \in Tx_t, n \geq 1.$$

By theorem 3.1, we have that $\{x_t\}$ converges strongly to $\tilde{x} \in F(T)$ (as $t \to 0$), and which solves the variational inequality (3.3).

**(III) We show that**

$$\limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle \leq 0.$$

Indeed, it follows from Lemma 2.6 that

$$\begin{aligned}
d^2(x_t, x_n) &= \langle \overrightarrow{x_t x_n}, \overrightarrow{x_t x_n} \rangle \\
&\leq t\langle \overrightarrow{f(x_t)x_n}, \overrightarrow{x_t x_n} \rangle + (1-t)\langle \overrightarrow{u(x_t)x_n}, \overrightarrow{x_t x_n} \rangle \\
&\leq t\langle \overrightarrow{f(x_t)f(\tilde{x})}, \overrightarrow{x_t x_n} \rangle + t\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_t x_n} \rangle + t\langle \overrightarrow{\tilde{x} x_t}, \overrightarrow{x_t x_n} \rangle + t\langle \overrightarrow{x_t x_n}, \overrightarrow{x_t x_n} \rangle \\
&+ (1-t)\langle \overrightarrow{u(x_t)u(x_n)}, \overrightarrow{x_t x_n} \rangle + (1-t)\langle \overrightarrow{u(x_n)x_n}, \overrightarrow{x_t x_n} \rangle \\
&\leq t\alpha d(x_t, \tilde{x})d(x_t, x_n) + t\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_t x_n} \rangle + td(x_t, \tilde{x})d(x_t, x_n) + td^2(x_t, x_n) \\
&+ (1-t)d^2(x_t, x_n) + (1-t)d(u(x_n), x_n)d(x_t, x_n).
\end{aligned}$$

Let $M := \sup d(\{x_t, x_n\})$, we get that

$$\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n x_t} \rangle \leq (1+\alpha)Md(x_t, \tilde{x}) + M\frac{d(u(x_n), x_n)}{t}.$$

We have

$$\begin{aligned}
&\limsup_{t\to 0}\limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n x_t} \rangle \\
&\leq \limsup_{t\to 0}\limsup_{n\to\infty}((1+\alpha)Md(x_t, \tilde{x}) + M\frac{H(Tx_n, x_n)}{t}) = 0
\end{aligned}$$

Since $\lim_{t\to 0} x_t = \tilde{x}$ and by the continuity of $d(.,.)$. For any fixed $n$, we have that

$$\begin{aligned}
&\lim_{t\to 0} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n x_t} \rangle \\
&= \frac{1}{2}\lim_{t\to 0}(d^2(f(\tilde{x}), x_t) + d^2(\tilde{x}, x_n) - d^2(f(\tilde{x}), x_n) - d^2(\tilde{x}, x_t)) \\
&= \frac{1}{2}(d^2(f(\tilde{x}), \tilde{x}) + d^2(\tilde{x}, x_n) - d^2(f(\tilde{x}), x_n) - d^2(\tilde{x}, \tilde{x})) \\
&= \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle
\end{aligned}$$

which implies that, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle < \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n x_t} \rangle + \varepsilon, \quad t \in (0, \delta). \tag{3.9}$$

Hence, by the upper limit as $n \to \infty$ first and then $t \to 0$, we get that

$$\limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle \leq \varepsilon,$$

which implies that

$$\limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle \leq 0.$$

**(VI) Finally,we prove that $x_n$ converges strongly to $\tilde{x}$.**

Let $y_n = t_n\tilde{x} \oplus (1-t_n)u(x_n)$ ,by Lemma 2.5,2.6, we have that

$$\begin{aligned} d^2(x_{n+1}, \tilde{x}) &\leq d^2(y_n, \tilde{x}) + 2\langle \overrightarrow{x_{n+1}y_n}, \overrightarrow{x_{n+1}\tilde{x}} \rangle \\ &\leq (1-t_n)^2 d^2(x_n, \tilde{x}) + 2(t_n^2 \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle \\ &+ t_n(1-t_n)\langle \overrightarrow{f(x_n)u(x_n)}, \overrightarrow{x_{n+1}\tilde{x}} \rangle + t_n(1-t_n)\langle \overrightarrow{u(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle) \\ &= (1-t_n)^2 d^2(x_n, \tilde{x}) + 2(t_n^2 \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle + t_n(1-t_n)\langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle) \\ &= (1-t_n)^2 d^2(x_n, \tilde{x}) + 2t_n \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle \\ &= (1-t_n)^2 d^2(x_n, \tilde{x}) + 2t_n(\langle \overrightarrow{f(x_n)f(\tilde{x})}, \overrightarrow{x_{n+1}\tilde{x}} \rangle + \langle \overrightarrow{f(\tilde{x}\tilde{x})}, \overrightarrow{x_{n+1}\tilde{x}} \rangle) \\ &\leq (1-t_n)^2 d^2(x_n, \tilde{x}) + 2t_n(\alpha d(x_n, \tilde{x}) d(x_{n+1}, \tilde{x}) + \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle) \\ &\leq (1-t_n)^2 d^2(x_n, \tilde{x}) + 2t_n \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle) + \alpha t_n (d^2(x_n, \tilde{x}) + d^2(x_{n+1}, \tilde{x})), \end{aligned}$$

which implies that

$$d^2(x_{n+1}, \tilde{x}) \leq \frac{1-(2-\alpha)t_n}{1-t_n\alpha} d^2(x_n, \tilde{x}) + \frac{2t_n}{1-t_n\alpha} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle + \alpha_n^2 L,$$

where $L = \sup_{n\geq 1}\{d^2(x_n, \tilde{x})\}$.
Letting $\delta_n = \frac{2(1-\alpha)t_n}{1-t_n\alpha}$ and $\gamma_n = \frac{1}{1-\alpha}\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}} \rangle + \frac{(1-\alpha t_n)t_n}{2(1-\alpha)} L$.
we have that

$$d^2(x_{n+1}, \tilde{x}) \leq (1-\delta_n) d^2(x_n, \tilde{x}) + \delta_n \gamma_n.$$

By Lemma 2.7,we get that $\lim_{n\to\infty} x_n = \tilde{x}$ and which solves the variational inequality (3.3). This completes the proof of Theorem 3.2.

## Competing interests

The authors declare that they have no competing interests.

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