Topological Solutions of Noncommutative Stochastic Differential Inclusions

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Abstract

A recent approach for studying the features of an arbitrary noncommutative stochastic differential equation in a number of locally convex operator topologies, comprising the strong/$\lambda^{\star}$-topologies and the weak topologies, is extended to the more general context of a noncommutative stochastic differential inclusion (NSDI). Reformulations, corresponding to the two sets of topologies and providing equivalent forms, of the NSDI are furnished and the existence of solutions of the inclusion in the diverse topologies is established. The reformulations are amply suited for analytically and numerically characterizing the key topological features of the solutions of NDSIs.

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1 Introduction

A major preoccupation of classical stochastic analysis is the study of stochastic differential equations [15,21–23,25–29], with a view to understanding, analytically and numerically, the qualitative features of their *weak* or *strong* solutions. This, in turn, provides an insight into the fluctuations and stochastic dynamics of the systems described by the equations. By contrast, when investigating the solutions of operator-valued stochastic differential equations, we encounter a multiplicity of operator topologies, beyond the *weak* and *strong* operator topologies, which appear in analysis or applications and should also be considered.

In Ref. [6], we initiated the study of the existence and uniqueness of the solutions of a *noncommutative stochastic differential equation* (NSDE), driven by an operator-valued semimartingale, in a number of interesting locally convex operator tolopogies that are listed in Subsection 3.1 below. A major objective of the study was to provide a representation-free approach that has the potential of unifying previous publications by a number of authors [12–14, 16, 20, 24] on the subject, while also furnishing a formalism for discussing diverse interesting features of the solutions of an NSDE.

In this paper, we extend the considerations and results of Ref. [6] to the more general context of a *noncommutative stochstic differential inclusion* (NSDI) which is introduced in Subsection 3.5. This is achieved by formulating a procedure for the systematic study of the properties of the solutions of NSDIs in the locally convex operator topologies of Subsection 3.1. As in Ref. [6], we assume only that we are furnished with a notion of *noncommutative stochastic integration* in which the *stochastic integral* satisfies a *semimartingale inequality* in each of the locally convex operator topologies.

In the classical context, differential inclusions have been studied over the years by several authors [2–4, 7–11], while in the noncommutative setting, a study of a class of quantum stochastic differential inclusions was initiated by this author [17–19].

The rest of the paper is organized as follows. In Section 2, we sketch the partial *-algebraic setting [1] in which we work, especially the structure of a partial $O^\star$-algebra [1]. This enables us to consider noncommutative stochastic processes whose values are, in general, densely defined unbounded linear maps on an arbitrary Hilbert space. Section 3 contains a diversity of notions and structures that are employed in the sequel. The Section is arranged as follows. The locally convex operator topologies employed in the subsequent discussion are introduced in Subsection 3.1. These are defined on an arbitrary partial $O^\star$-algebra $\mathfrak{M}$ and used to introduce Hausdorff topologies on collections of closed subsets of $\mathfrak{M}$ in Subsection 3.2, since much of the subsequent discussion will revolve around multivalued maps. After introducing the notion of a single-valued noncommutative stochastic process in Subsection 3.3, we define noncommutative multivalued stochastic processes in Subsection 3.4. These are central to the considerations in this paper and lead to the formulation of the notion of a noncommutative stochastic differential inclusion (NSDI) in Subsection 3.5. Section 4 furnishes the notion of a Lipschitzian multifunction, as well as several examples of such a multifunction. In Section 5, we prove the existence of a solution of a Lipschitzian NSDI in the $\lambda^\star$-topology $\tau_\lambda^\star$ and the *strong* locally convex topologies $\{\tau_s, \tau_{s^\star}, \tau_{\sigma s}, \tau_{\sigma s^\star}\}$ described in Subsection 3.1. This is done under the hypotheses listed in Subsection 5.3. Similarly, in Section 6, under the hypotheses indicated in Subsection 6.3, the existence of a solution of a Lipschitzian NSDI in the *weak* locally convex operator topologies $\{\tau_w, \tau_{\sigma w}\}$ is established. As a strategem for obtaining the proofs, we first furnish two reformulations, corresponding to the two sets of topologies (strong/$\lambda^\star$ and weak) and providing equivalent forms, of the NSDI. The reformulations are themselves of independent interest and are amply suited for analytically or numerically characterizing the key topological features of the solutions of an NDSI. The results of this paper extend those of Refs. [12–14, 16, 20, 24], which use only the strong operator topology τ_s and weak operator topology τ_w.

2 Preliminaries

Throughout the paper, $\mathcal{H}$ is a Hilbert space, with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and $\mathcal{D}$ is a dense subspace of $\mathcal{H}$. To the pair $(\mathcal{D},\mathcal{H})$ we associate the linear space $L(\mathcal{D},\mathcal{H})$ of all $\mathcal{H}$-valued linear maps x whose common domain is $\mathcal{D}$ and whose adjoints x^* have domains containing $\mathcal{D}$. It follows that every operator in $L(\mathcal{D},\mathcal{H})$ is closable. We denote the closure of $x \in L(\mathcal{D},\mathcal{H})$ by $\overline{x}$. Equipped with the relation Γ given by

$$\Gamma = \{(x,y) \in L(\mathcal{D},\mathcal{H}) \times L(\mathcal{D},\mathcal{H}) : y\mathcal{D} \subseteq \mathrm{dom}(x^{+\star}) \text{ and } x^+\mathcal{D} \subseteq \mathrm{dom}(y^\star)\},$$

where $\mathrm{dom}(a)$ denotes the *domain of* a and the involution $^+$ is the map $x \mapsto x^+ = x^*\lceil_{\mathcal{D}}$, the space $L(\mathcal{D},\mathcal{H})$ acquires the structure of a *partial* *-*algebra* [1] with partial multiplication $\cdot$ defined by: $x \cdot y = x^{+*}y$ on $\mathcal{D}$. The triplet $(L(\mathcal{D},\mathcal{H}), ^+, \Gamma)$, or equivalently $(L(\mathcal{D},\mathcal{H}), ^+, \cdot)$, will be denoted simply by $L_w^+(\mathcal{D},\mathcal{H})$. We will say that a subspace $\mathfrak{M}$ of $L_w^+(\mathcal{D},\mathcal{H})$ is closed under the partial multiplication $\cdot$ if $(x,y) \in \Gamma$, with $x, y \in \mathfrak{M}$, implies $x \cdot y$ is in $\mathfrak{M}$. A $^+$-invariant subspace of $L_w^+(\mathcal{D},\mathcal{H})$ which is closed under the partial multiplication $\cdot$ will be called a *partial* $O^\star$-*algebra* on $\mathcal{D}$.

3 Fundamental notions and structures

In this Section, we introduce the main notions and structures which feature in the subsequent discussion.

3.1 Some locally convex operator tolopogies on $\mathfrak{M}$

Let $\mathfrak{M}$ be a partial O^*-algebra on $\mathcal{D}$. As in Ref. [6], we employ the following locally convex operator topologies.

(i) *strong topology* τ_s: whose family of seminorms is $\{\|\cdot\|_\xi : \xi \in \mathcal{D}\}$, where $\|x\|_\xi = \|x\xi\|$, $x \in \mathfrak{M}$, $\xi \in \mathcal{D}$;

(ii) *strong** *topology* $\tau_{s^\star}$: whose family of seminorms is $\{\|\cdot\|_{\star\xi} : \xi \in \mathcal{D}\}$, where $\|x\|_{\star\xi} = \sqrt{\|x^+\xi\|^2 + \|x\xi\|^2}$, $x \in \mathfrak{M}$, $\xi \in \mathcal{D}$;

(iii) σ-*strong topology* $\tau_{\sigma s}$: which is defined as follows: let $\mathcal{D}_\infty$ denote the set of all sequences $\boldsymbol{\xi} = \{\xi_n\}_{n=1}^\infty$ of members of $\mathcal{D}$ such that $\sum_{n=1}^\infty \|x\xi_n\|^2 < \infty$, $\forall\ x \in \mathfrak{M}$; then, the family of seminorms which generate $\tau_{\sigma s}$ is given by $\{\|\cdot\|_{\boldsymbol{\xi}} : \boldsymbol{\xi} \in \mathcal{D}_\infty\}$, where $\|x\|_{\boldsymbol{\xi}} = \sqrt{\sum_{n=1}^\infty \|x\xi_n\|^2}$, $x \in \mathfrak{M}$, $\boldsymbol{\xi} \in \mathcal{D}_\infty$;

(iv) σ-*strong** *topology* $\tau_{\sigma s^\star}$: whose family of seminorms is $\{\|\cdot\|_{\star\boldsymbol{\xi}} : \boldsymbol{\xi} \in \mathcal{D}_\infty\}$, where $\|x\|_{\star\boldsymbol{\xi}} = \sqrt{\sum_{n=1}^\infty \left(\|x^+\xi_n\|^2 + \|x\xi_n\|^2\right)}$, $x \in \mathfrak{M}$, $\boldsymbol{\xi} \in \mathcal{D}_\infty$;

(v) *weak topology* τ_w : whose family of seminorms is $\{\|\cdot\|_{\eta\xi} : \eta, \xi \in \mathcal{D}\}$, where $\|x\|_{\eta\xi} = |\langle \eta, x\xi\rangle|$, $x \in \mathfrak{M}, \eta, \xi \in \mathcal{D}$;

(vi) σ-*weak topology* $\tau_{\sigma w}$: whose family of seminorms is $\{\|\cdot\|_{\boldsymbol{\eta\xi}} : \boldsymbol{\eta}, \boldsymbol{\xi} \in \mathcal{D}_\infty\}$, where $\|x\|_{\boldsymbol{\eta\xi}} = \sum_{n=1}^\infty |\langle \eta_n, x\xi_n\rangle|$, $x \in \mathfrak{M}, \boldsymbol{\eta}, \boldsymbol{\xi} \in \mathcal{D}_\infty$;

(vii) $\lambda^\star$ *topology* $\tau_{\lambda^\star}$: this topology is defined as follows. Let $\mathfrak{M}_+$ be the positive portion of $\mathfrak{M}$ (i.e. $\mathfrak{M}_+$ is the set of members $x \in \mathfrak{M}$ such that $\langle \xi, x\xi\rangle > 0$, $\forall\ \xi \in \mathcal{D}$), and $\mathfrak{M}_e = \mathfrak{M}_+ \cup \{e\}$, where e is the identity operator in $L_w^+(\mathcal{D}, \mathcal{H})$. For $a \in \mathfrak{M}_e$, let

$$\|x\|_a = \sup_{\xi \in \mathcal{D}} \left(\frac{\sqrt{\|x^+\xi\|^2 + \|x\xi\|^2}}{\|a\xi\|} \right),\ x \in \mathfrak{M},$$

where $\alpha/0 = \infty$, for $\alpha > 0$. Set

$$\mathfrak{M}^a = \{x \in \mathfrak{M} : \|x\|_a < \infty\}.$$

Then $\{(\mathfrak{M}^a, \|\cdot\|_a) : a \in \mathfrak{M}_e\}$ is directed and covers $\mathfrak{M}$. For $a \in \mathfrak{M}_e$, let j_a be the injection of $\mathfrak{M}^a$ in $\mathfrak{M}$. Then $\tau_{\lambda^\star}$ is the *inductive topology* on $\mathfrak{M}$ determined by the collection $\{(\mathfrak{M}^a, j_a) : a \in \mathfrak{M}_e\}$. We remark that the topology $\tau_{\lambda^\star}$ reduces to the uniform topology on $B(\mathcal{H})$, the Banach space of bounded endomorphisms of $\mathcal{H}$.

The locally convex operator topologies introduced above are not metrizable, and hence are not paracompact.

Remark 3.1. We shall call the members of the set $\{\tau_s, \tau_{s^\star}, \tau_{\sigma s}, \tau_{\sigma s^\star}\}$ the *strong locally convex operator topologies* and $\{\tau_w, \tau_{\sigma w}\}$ the *weak locally convex operator topologies.*

Notation: The symbol $\tau_\Box$ denotes any member of the set $\{\tau_s, \tau_{s^\star}, \tau_{\sigma s}, \tau_{\sigma s^\star}, \tau_w, \tau_{\sigma w}, \tau_{\lambda^\star}\}$ and we write $\{\|\cdot\|_\alpha : \alpha \in \Theta(\tau_\Box)\}$ for the family of seminorms that generates $\tau_\Box$. From above, it is seen that $\Theta(\tau_s) = \Theta(\tau_{s^\star}) = \mathcal{D}$; $\Theta(\tau_{\sigma s}) = \Theta(\tau_{\sigma s^\star}) = \mathcal{D}_\infty$; $\Theta(\tau_w) = \mathcal{D} \times \mathcal{D}$; $\Theta(\tau_{\sigma w}) = \mathcal{D}_\infty \times \mathcal{D}_\infty$; and $\Theta(\tau_{\lambda^\star}) = \mathfrak{M}_e$. We denote the locally convex space $(\mathfrak{M}, \tau_\Box)$ by $\mathfrak{M}(\tau_\Box)$ and write $\mathfrak{M}^{\tau_\Box}$ for the $\tau_\Box$-completion of $\mathfrak{M}$.

For $x \in \mathfrak{M}$, with closure $\overline{x}$, write $\overline{x} = v(\overline{x})\,|\overline{x}|$ for the polar decomposition of $\overline{x}$, having $v(\overline{x})$ as its partial isometry and $|\overline{x}|$ as its positive part. Let $C_b(\mathbb{R}_+)$ denote the linear space of all bounded real-valued Borel functions on $\mathbb{R}_+$. To the partial $O^\star$-algebra $\mathfrak{M}$, we associate the $W^\star$-algebra $W^\star(\mathfrak{M})$ of bounded linear operators on $\mathcal{H}$ defined by

$$W^\star(\mathfrak{M}) = \{v(\overline{x}), \varphi(|\overline{x}|) : x \in \mathfrak{M} \text{ and } \varphi \in C_b(\mathbb{R}_+)\}'',$$

where $\mathcal{C}''$ denotes the bicommutant of $\mathcal{C} \subset B(\mathcal{H})$.

3.2 Hausdorff topologies

In the sequel, we employ a number of *Hausdorff topologies*, associated with the locally convex topologies introduced above.

The symbol $\mathrm{clos}(\mathfrak{M}^{\tau_\Box})$ denotes the collection of all nonvoid *closed* subsets of $\mathfrak{M}^{\tau_\Box}$. We introduce a Hausdorff topology on $\mathrm{clos}(\mathfrak{M}^{\tau_\Box})$ as follows.

For $x \in \mathfrak{M}^{\tau_\Box}, \mathcal{M}, \mathcal{N} \in \mathrm{clos}(\mathfrak{M}^{\tau_\Box})$ and $\alpha \in \Theta(\tau_\Box)$, make the definitions:

$$\begin{aligned} d_\alpha(x, \mathcal{N}) &= \inf_{y \in \mathcal{N}} \|x - y\|_\alpha \\ \delta_\alpha(\mathcal{M}, \mathcal{N}) &= \sup_{x \in \mathcal{M}} d_\alpha(x, \mathcal{N}) \\ \rho_\alpha(\mathcal{M}, \mathcal{N}) &= \max\,(\delta_\alpha(\mathcal{M}, \mathcal{N}), \delta_\alpha(\mathcal{N}, \mathcal{M})). \end{aligned}$$

The Hausdorff topology determined on $\mathrm{clos}(\mathfrak{M}^{\tau_\Box})$ by the pseudometrics $\{\rho_\alpha(\cdot,\cdot) : \alpha \in \Theta(\tau_\Box)\}$ will be denoted by $\tau_\Box^H$.

If $\mathcal{M} \in \text{clos}(\mathfrak{M}^{\tau_\Box})$, then $\|\mathcal{M}\|_\alpha$ is defined by

$$\|\mathcal{M}\|_\alpha = \rho_\alpha(\mathcal{M}, \{0\}), \quad \alpha \in \Theta(\tau_\Box).$$

It follows that

$$\|\mathcal{M}\|_\alpha = \sup\{\|m\|_\alpha : m \in \mathcal{M}\}, \quad \alpha \in \Theta(\tau_\Box).$$

In analogy to the foregoing, for $A, B \in \text{clos}(\mathbb{C})$ and $x \in \mathbb{C}$, the complex numbers, define

$$\begin{aligned} d(x, A) &= \inf_{y \in A} |x - y|, \\ \delta(A, B) &= \sup_{x \in A} d(x, B), \\ \rho(A, B) &= \max\{\delta(A, B), \delta(B, A)\}. \end{aligned}$$

Then ρ induces a Hausdorff topology on $\text{clos}(\mathbb{C})$.

Remark 3.2. The following notions are needed in the subsequent discussion.

3.3 Noncommutative stochastic processes

Our approach to stochastic integration will be as outlined in Ref. [6], to which the reader should turn for both notation and relevant concepts, especially the definition of the $^+$-invariant linear spaces of simple processes $\text{sim}(I, \mathfrak{M}(\tau_\Box))$ on I (resp. $\text{sim}(\mathfrak{M}(\tau_\Box))$ on $\mathbb{R}$) and adapted processes $\text{Ad}(I, \mathfrak{M}(\tau_\Box))$ on I (resp. $\text{Ad}(\mathfrak{M}(\tau_\Box))$ on $\mathbb{R}$); the $^+$-invariant space $\text{Sem}(\mathfrak{M}, \tau_\Box)$ of semimartingales, each member M of which determines two subspaces $L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\Box}, dM)_L$ and $L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\Box}, dM)_R$ of $\text{Ad}(I, \mathfrak{M}(\tau_\Box))$, called the subspace of *left locally square integrable integrands* and the subspace of *right locally square integrable integrands*, respectively, with respect to M; as well as the *left stochastic integral* of f (resp. *right stochastic integral of* g) represented as:

$$(M \circ f)(t) = \int_{t_0}^t dM(s) \cdot f(s), \text{ for } f \in L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\Box}, dM)_L$$

$$(\text{resp. } (g \circ M)(t) = \int_{t_0}^t g(s) \cdot dM(s)), \text{ for } g \in L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\Box}, dM)_R.$$

These stochastic integrals are assumed to enjoy the fundamental properties listed in Ref. [6, page 968]. As we shall be employing mixed stochastic integrals, which are linear combinations of *left* and *right stochastic integrals*, we denote $L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\Box}, dM)_L \cap L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\Box}, dM)_R$ by $L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\Box}, dM)$.

As the main results in this paper may readily be reformulated for *right*, as well as *mixed*, stochastic integrals, throughout the paper, we consider only *left stochastic integrals*.

As in Ref. [6], the symbol $L^2_{\rm loc}(I, \mathfrak{M}^{\tau_\square}, ds)$ denotes the completion of the $^+$-invariant linear space $\text{sim}(I, \mathfrak{M}(\tau_\square))$ in the locally convex topology $\tau^0_\square$ whose family of seminorms is given by $\{\|\cdot\|_{\alpha,t} : \alpha \in \Theta(\tau_\square),\ t \in I\}$, where

$$\|f\|^2_{\alpha,t} = \int_{t_0}^t ds\ \|f(s)\|^2_\alpha, \quad f \in \text{sim}(I, \mathfrak{M}(\tau_\square)),\ \alpha \in \Theta(\tau_\square).$$

For $M \in \text{Sem}(\mathfrak{M}, \tau_\square)$ and $(f, g, h) \in L^2_{\rm loc}(I \times \mathfrak{M}^{\tau_\square}, dM) \times L^2_{\rm loc}(I \times \mathfrak{M}^{\tau_\square}, dM^+) \times L^2_{\rm loc}(I \times \mathfrak{M}^{\tau_\square}, ds)$, we introduce the mixed stochastic integral:

$$x(t) = x_0 + \int_{t_0}^t \big(dM(s) \cdot f(s,x) + dM^+(s) \cdot g(s,x) + h(s,x)\, ds\big), \quad t \in I.$$

This is central to the notion of stochastic integration of multivalued stochastic processes in this paper.

3.4 Noncommutative multivalued stochastic processes

Definition 1. (1) A member of $\text{clos}(\mathfrak{M}^{\tau_\square})$ will be called a *random set.*

(2) By a *multivalued stochastic process* indexed by $I \subseteq \mathbb{R}_+$, we mean a measurable multifunction on I whose values are random sets, i.e. a $\text{clos}(\mathfrak{M}^{\tau_\square})$-valued map on I such that $t \mapsto \rho_\alpha(\{x\}, \Phi(t))$ is measurable for arbitrary $x \in \mathfrak{M}^{\tau_\square},\ \alpha \in \Theta(\tau_\square)$.

(3) If Φ is a multivalued stochastic process indexed by $I \subseteq \mathbb{R}_+$, then a *selection* of Φ is a stochastic process $x : I \longrightarrow \mathfrak{M}^{\tau_\square}$ with the property that $x(t) \in \Phi(t)$ for almost all $t \in I$.

(4) A multivalued stochastic process Φ will be called

(i) *adapted* if every selection of Φ is adapted;

(ii) a member of

(a) $L^2_{\rm loc}(I, \mathfrak{M}^{\tau_\square}, dM)_{\rm mvs}$ if every selection of $t \mapsto \Phi(t),\ t \in I$, lies in $L^2_{\rm loc}(I, \mathfrak{M}^{\tau_\square}, dM)$;

(b) $L^2_{\rm loc}(I, \mathfrak{M}^{\tau_\square}, ds)_{\rm mvs}$ if every selection of $t \mapsto \Phi(t),\ t \in I$, lies in $L^2_{\rm loc}(I, \mathfrak{M}^{\tau_\square}, ds)$,

for arbitrary $\alpha \in \Theta(\tau_\square)$, and $M \in \text{Sem}(\mathfrak{M}^{\tau_\square})$.

Remark 3.3. Based on Definition 1, we will also employ the following notation.

Notation 3.1. Let $M \in \text{Sem}(\mathfrak{M}^{\tau_\square})$ and $I \subseteq \mathbb{R}_+$.

1. The set $L^2_{\text{loc}}(I \times \mathfrak{M}^{\tau_\square}, dM)_{\text{mvs}}$ (resp. $L^2_{\text{loc}}(I \times \mathfrak{M}^{\tau_\square}, ds)_{\text{mvs}}$) is the set of all maps $\Phi : I \times L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\square}, ds) \longrightarrow \text{clos}(\mathfrak{M}^{\tau_\square})$ such that $t \mapsto \Phi(t,x)$, $t \in I$, lies in $L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\square}, dM)_{\text{mvs}}$ (resp. $L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\square}, ds)_{\text{mvs}}$) for every $x \in L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\square}, ds)$.

2. If $\Phi \in L^2_{\text{loc}}(I \times \mathfrak{M}^{\tau_\square}, dM)_{\text{mvs}}$ (resp. $\Phi \in L^2_{\text{loc}}(I \times \mathfrak{M}^{\tau_\square}, ds)_{\text{mvs}}$), we use the notation

$$L_{\tau_\square}(I, \Phi, dM) = \left\{\varphi \in L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\square}, dM) : \varphi \text{ is a selection of } \Phi\right\}$$
$$(\text{resp. } L_{\tau_\square}(I, \Phi, ds) = \left\{\varphi \in L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\square}, ds) : \varphi \text{ is a selection of } \Phi\right\}).$$

3.5 Noncommutative stochastic differential inclusions

In the sequel, let dW denote any of the stochastic differentials dM, dM^+ and ds, employed as integrators above. Then we introduce multivalued stochastic expressions as follows.

If $\Phi \in L^2_{\text{loc}}(I \times \mathfrak{M}^{\tau_\square}, dW)_{\text{mvs}}$ and $(t,x) \in I \times L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\square}, ds)$, then we make the definition:

$$\int_{t_0}^{t} dW(s)\Phi(s,x) = \left\{\int_{t_0}^{t} dW(s)\,\varphi(s,x) : \varphi \in L_{\tau_\square}(I, \Phi, dW)\right\}.$$

This leads to the following notion.

Definition 2. Let $F \in L^2_{\text{loc}}(I \times \mathfrak{M}^{\tau_\square}, dM)_{\text{mvs}}$, $G \in L^2_{\text{loc}}(I \times \mathfrak{M}^{\tau_\square}, dM^+)_{\text{mvs}}$, and $H \in L^2_{\text{loc}}(I \times \mathfrak{M}^{\tau_\square}, ds)_{\text{mvs}}$. Let (t_0, x_0) be a fixed point of $I \times \mathfrak{M}(\tau_\square)$. Then a relation of the form

$$x(t) \in x_0 + \int_{t_0}^{t} \left(dM(s)F(s,x) + dM^+(s)G(s,x) + H(s,x)ds\right), \; t \in I, \quad (3.1)$$

will be called a *noncommutative stochastic integral inclusion* (NSII) with *coefficients* F, G, H, and *initial data* (t_0, x_0).

We shall often abbreviate equation (3.1) as follows:

$$\begin{aligned} &dx(t) \in dM(t)F(t,x) + dM^+(t)G(t,x) + H(t,x)dt, \text{ almost all } t \in I, \\ &x(t_0) = x_0, \end{aligned} \quad (3.2)$$

and refer to this as a *noncommutative stochastic differential inclusion* (NSDI) with *coefficients* F, G, H, and *initial data* (t_0, x_0).

Definition 3. A map $x : I \longrightarrow \mathfrak{M}(\tau_\Box)$ is $\tau_\Box$-*absolutely continuous* if for arbitrary $\epsilon > 0$ and each $\alpha \in \Theta(\tau_\Box)$, there exists $\delta(\epsilon, \alpha) > 0$ such that $\sum_{k=1}^{n} \|x(t_k) - x(s_k)\|_\alpha < \epsilon$, for every pairwise disjoint family $\{(s_k, t_k)\}_{k+1}^{n}$ of open subintervals of I satisfying $|t_k - s_k| < \delta(\epsilon, \alpha)$, $k = 1, 2, \ldots$.

Notation 3.2. The set of all $\tau_\Box$-absolutely continuous members of $L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\Box}, ds)$ will be denoted by $L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\Box}, ds)_{\text{ac}}$.

Definition 4. By a $\tau_\Box$-*solution* of problem (3.2), we mean a $\tau_\Box$-*absolutely continuous* adapted stochastic process $\varphi \in L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\Box}, ds)$ such that

$$\begin{aligned} &d\varphi(t) \in dM(t)F(t, \varphi) + dM^+(t)G(t, \varphi) + H(t, \varphi)dt, \text{ almost all } t \in I, \\ &\varphi(t_0) = x_0 \in \mathfrak{M}(\tau_\Box). \end{aligned} \tag{3.3}$$

Remark 3.4. We shall prove the existence of solutions of problem (3.2) in the multiple operator topologies introduced earlier. To this end, in Sections 5 and 6, we first establish equivalent forms of problem (3.2).

4 Lipschitzian multifunctions

We shall require the following notion in the sequel.

Definition 5. Let $I \subseteq \mathbb{R}_+$ and $\mathcal{N} \subseteq L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\Box}, ds)$. A map $\Phi : I \times \mathcal{N} \longrightarrow \text{clos}(\mathfrak{M}^{\tau_\Box})$ will be called $\tau_\Box$-*Lipschitzian* if for each $\sigma \in \Theta(\tau_\Box)$, there are a continuous function $k_\sigma : I \longrightarrow \mathbb{R}_+$ and $\alpha(\sigma) \in \Theta(\tau_\Box)$ such that

$$\rho_\sigma(\Phi(t, x), \Phi(t, y)) \leq k_\sigma(t) \|x(t) - y(t)\|_{\alpha(\sigma)}$$

for all $x, y \in \mathcal{N}$ and almost all $t \in I$.

Remark. It is instructive to provide some examples of Lipschitzian multifunctions.

4.1 Examples of Lipschitzian maps

Let $L(\mathfrak{M}^{\tau_\Box})$ be the linear space of all linear maps from $\mathfrak{M}^{\tau_\Box}$ into itself and $L(\mathfrak{M}^{\tau_\Box})_{\text{con}}$ the subspace of $L(\mathfrak{M}^{\tau_\Box})$ consisting of all its continuous members. If $T \in L(\mathfrak{M}^{\tau_\Box})_{\text{con}}$, then for each $\alpha \in \Theta(\tau_\Box)$, there are $c_\alpha > 0$ and $\beta(\alpha) \in \Theta(\tau_\Box)$ such that

$$\|Tz\|_\alpha \leq c_\alpha \|z\|_{\beta(\alpha)}, \quad \text{for all } z \in \mathfrak{M}^{\tau_\Box}. \tag{$*$}$$

In the sequel,

$$|||T|||_\alpha = \inf\{c_\alpha : (*) \text{ holds}\},$$

and τ_{con} denotes the locally convex topology determined by the family $\{|||\cdot|||_\alpha : \alpha \in \Theta(\tau_\square)\}$ of semi-norms.

Proposition 4.1. *Let A be a map from I to the set of all τ_{con}-closed subsets of $L(\mathfrak{M}^{\tau_\square})_{\text{con}}$ equipped with the Hausdorff pseudometrics $\{\rho_\alpha^{\text{con}} : \alpha \in \Theta(\tau_\square)\}$ determined by τ_{con}; ω a* $\text{clos}(\mathfrak{M}^{\tau_\square})$*-valued map on I and $\Phi : I \times L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\square}, ds) \longrightarrow 2^{\mathfrak{M}^{\tau_\square}}$ the multifunction defined by $\Phi(t, x) = A(t)x(t) + \omega(t)$, $x \in L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\square}, ds)$, $t \in I$. Then, Φ is $\tau_\square$-Lipschitzian.*

Proof. First of all, it is clear that Φ has closed values. Next, for each $\sigma \in \Theta(\tau_\square)$, a straightforward calculation gives

$$\rho_\sigma(\Phi(t, x), \Phi(t, y)) \leq |||A(t)|||_\sigma^{\text{con}} \|x(t) - y(t)\|_{\alpha(\sigma)},$$

for some $\alpha(\sigma) \in \Theta(\tau)$ and almost all $t \in I$, where $|||A(t)|||_\sigma^{\text{con}} = \rho_\sigma^{\text{con}}(A(t), 0)$. □

Proposition 4.2. *There exist* $\text{clos}(I)$*-valued Lipschitzian maps on the set $I \times L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\square}, ds)$.*

Proof. Let $b : I \times \mathbb{R} \longrightarrow \text{clos}(I)$ be Lipschitzian, i.e. $\rho(b(t, \lambda_1), b(t, \lambda_2)) \leq c_b(t) |\lambda_1 - \lambda_2|$, for all $t \in I$, $\lambda_1, \lambda_2 \in \mathbb{R}$ and some continuous $c_b : I \longrightarrow \mathbb{R}_+$. Then the map $\varphi : I \times L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\square}, ds) \longrightarrow \text{clos}(I)$ defined by $\varphi(t, x) = b(t, \|x\|_{\alpha_0})$, for some $\alpha_0 \in \Theta(\tau_\square)$, is Lipschitzian, since

$$\begin{aligned}\rho(\varphi(t, x), \varphi(t, x)) &= \rho(b(t, \|x(t)\|_{\alpha_0}), b(t, \|y(t)\|_{\alpha_0}))\\ &\leq c_b(t) \left|\|x(t)\|_{\alpha_0} - \|y(t)\|_{\alpha_0}\right|\\ &\leq c_b(t) \|x(t) - y(t)\|_{\alpha_0}, \ t \in I, x, y \in L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\square}, ds).\end{aligned}$$

□

Proposition 4.3. *Let $\{a_j\}_{j=0}^\infty \subset \mathfrak{M}^{\tau_\square}$ and $\{\varphi_j\}_{j=0}^\infty$ be a sequence of Lipschitzian maps from $I \times L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\square}, ds)$ to* $\text{clos}(I)$ *such that $\rho(\varphi_j(t, x), \varphi_j(t, y)) \leq c_j(t) \|x(t) - y(t)\|_{\alpha_0}$, for some $\alpha_0 \in \Theta(\tau_\square)$, continuous maps $c_j : I \longrightarrow \mathbb{R}_+$, and each j. Suppose that the map $\Phi : I \times L^2(I, \mathfrak{M}^{\tau_\square}, ds) \longrightarrow$* $\text{clos}(\mathfrak{M}^{\tau_\square})$ *defined by*

$$\Phi(t, x) = \sum_{j=0}^{\infty} a_j \varphi_j(t, x)$$

is pointwise $\tau_\square^H$*-convergent and* $\sum_{j=0}^{\infty} \|a_j\|_\alpha c_j(t) < \infty$*, for each* $\alpha \in \Theta(\tau_\square)$ *and* $t \in I$*. Then* Φ *is* $\tau_\square$*-Lipschitzian.*

Proof. Let the hypotheses hold. Then

$$\begin{aligned}
\rho_\alpha(\Phi(t,x), \Phi(t,y)) &= \rho_\alpha(\textstyle\sum_{j=0}^{\infty} a_j\varphi_j(t,x), \sum_{j=0}^{\infty} a_j\varphi_j(t,y)) \\
&\leq \sum_{j=0}^{\infty} \|a_j\|_\alpha \rho(\varphi_j(t,x), \varphi_j(t,y)) \\
&\leq \left(\sum_{j=0}^{\infty} \|a_j\|_\alpha c_j(t)\right) \|x(t) - y(t)\|_{\alpha_0},
\end{aligned}$$

for each $\alpha \in \Theta(\tau_\square)$, $t \in I$, and $x, y \in L^2(I, \mathfrak{M}^{\tau_\square}, ds)$. □

Remark 4.1. In [6, Proposition 3.1.3], we showed that there exist I-valued Lipschitzian maps on $I \times L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\square}, ds)$. We use this fact in the next result.

Proposition 4.4. *Let* ω *be a* $\text{clos}(\mathfrak{M}(\tau_\square))$*-valued map on* I*, such that* $c_\alpha = \operatorname{ess\,sup}_{t\in I} \|\omega(t)\|_\alpha$ *is finite for each* $\alpha \in \Theta(\tau_\square)$*. Suppose* $\sigma : I \longrightarrow I$ *and* $\beta : I \times L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\square}, ds) \longrightarrow I$ *is of the form* $\beta(t,x) = \lambda(t, \|x(t)\|_{\alpha_0})$*, for some* $\alpha_0 \in \Theta(\tau_\square)$*, where* $\lambda : I \times \mathbb{R} \longrightarrow I$ *is Lipschitzian. Define* $\Phi : I \times L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\square}, ds) \longrightarrow \mathfrak{M}(\tau_\square)$ *as the* $\tau_\square$*-convergent Aumann-Lebesgue-Bochner integral [5]*

$$\Phi(t,x) = \int_{\sigma(t)}^{\beta(t,x)} \omega(s)\, ds, \quad (t,x) \in I \times L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\square}, ds).$$

Then Φ *is* $\tau_\square$*-Lipschitzian.*

Proof. Let $\alpha \in \Theta(\tau_\Box)$ be arbitrary. Then a straightforward calculation gives

$$\begin{aligned}
\rho_\alpha(\Phi(t,x),\Phi(t,y)) &= \rho_\alpha\Big(\int_{\sigma(t)}^{\beta(t,x)} \omega(s)\,ds, \int_{\sigma(t)}^{\beta(t,y)} \omega(s)\,ds\Big) \\
&\leq \int_{\beta(t,x)\wedge\beta(t,y)}^{\beta(t,x)\vee\beta(t,y)} \|\omega(s)\|_\alpha \,ds, \text{ since } \rho_\alpha \text{ is a} \\
&\qquad \text{seminorm-induced pseudometric on } \mathfrak{M}^{\tau_\Box} \\
&\leq c_\alpha \int_{\beta(t,x)\wedge\beta(t,y)}^{\beta(t,x)\vee\beta(t,y)} ds, \\
&= c_\alpha \left|\beta(t,x) - \beta(t,y)\right| \\
&= c_\alpha \left|\lambda(t, \|x(t)\|_{\alpha_0}) - \lambda(t, \|y(t)\|_{\alpha_0})\right| \\
&\leq c_\alpha k_\alpha(t) \left|\|x(t)\|_{\alpha_0} - \|y(t)\|_{\alpha_0}\right|, \text{ for some } k_\alpha : I \to \mathbb{R}_+, \\
&\leq c_\alpha k_\alpha(t) \left\|x(t) - y(t)\right\|_{\alpha_0}.
\end{aligned}$$

This concludes the proof. $\Box$

Remark 4.2. 1. Theorem 5.2 below shows how to generate $\tau_\Box^{s\lambda}$-Lipschitzian multifunctions P from some given $\tau_\Box^{s\lambda}$-Lipschitzian multifunctions F, G, H, where the topology $\tau_\Box^{s\lambda}$ is as defined in Section 5.

2. Theorem 6.2 below shows how to generate $\tau_\Box^{w}$-Lipschitzian multifunctions P from some given $\tau_\Box^{w}$-Lipschitzian multifunctions F, G, H, where the topology $\tau_\Box^{w}$ is as defined in Section 5.

5 Solutions in the $\lambda^\star$-topology and strong operator topologies

In this section, we discuss the existence of solutions of (3.2) in the $\lambda^\star$-topology $\tau_\lambda^\star$ and the *strong* locally convex topologies $\{\tau_s, \tau_{s^\star}, \tau_{\sigma s}, \tau_{\sigma s^\star}\}$; let $\tau_\Box^s$ denote any member of this set of topologies. Similarly, we write $\tau_\Box^{s\lambda}$ for any member of the set $\{\tau_s, \tau_{s^\star}, \tau_{\sigma s}, \tau_{\sigma s^\star}, \tau_\lambda^\star\}$.

We will first reformulate equation (3.2).

Notation 5.1. For a differentiable multifunction $\Omega : I \longrightarrow \mathcal{H}$, the symbol $\frac{d}{dt}\Omega$ denotes the set

$$\frac{d}{dt}\Omega = \left\{\frac{d\omega}{dt} : \omega \text{ is a selection of } \Omega\right\},$$

where $\dfrac{d\omega}{dt}$ is the strong derivative of ω.

5.1 The multifunction P

Define

$$P \ : \ I \times L^2_{\rm loc}(I, \mathfrak{M}^{\tau_s}, ds)_{\rm ac} \longrightarrow 2^{L^+_w(\mathcal{D},\mathcal{H})}$$

$$(t,x) \longmapsto P(t,x)$$

by

$$P(t,x)\xi = \frac{d}{dt}\left[\left(\int_{t_0}^{t} dM(s)\, F(s,x) + dM^+(s)\, G(s,x) + H(s,x)ds\right)\xi\right], \tag{5.1}$$

$\xi \in \mathcal{D}$.

Since $x \in L^2_{\rm loc}(I, \mathfrak{M}^{\tau_s}, ds)_{\rm ac}$, the map P is well defined. We shall utilize a number of properties of the map P.

5.2 Assumptions

Throughout the rest of this Section, *we work under the same set of assumptions listed as* **Assumptions 4.1**, *together with their notation, in* Ref. [6, Assumptions 4.1, page 973].

Theorem 5.1. *Assume that $F \in L^2_{\rm loc}(I \times \mathfrak{M}^{\tau_\square}, dM)_{\rm mvs}$, $G \in L^2_{\rm loc}(I \times \mathfrak{M}^{\tau_\square}, dM^+)_{\rm mvs}$, $H \in L^2_{\rm loc}(I \times \mathfrak{M}^{\tau_\square}, ds)_{\rm mvs}$ and that P is as defined in (5.1). Let $\|\cdot\|_{\sigma,M}$, denote either $\|\cdot\|_{\beta(\sigma,M)}$, for $\sigma \in \Theta(\tau^s_\square)$, or $\|\cdot\|_{\sigma,\mathcal{D}_{M\beta}}$, for $\sigma \in \mathfrak{M}_e$. Suppose that for any $\sigma \in \Theta(\tau^{s\lambda}_\square)$, there exist some $\alpha(\sigma,M) \in \Theta(\tau^{s\lambda}_\square)$ and a continuous function $k_\sigma : I \longrightarrow \mathbb{R}_+$, depending on F, G, H, such that*

$$\|F(t,x)\|^2_{\sigma,M} + \|G(t,x)\|^2_{\sigma,M} + \|H(t,x)\|^2_\sigma \le k_\sigma(t)\left(1 + \|x(t)\|^2_{\alpha(\sigma,M)}\right), \tag{5.2}$$

for all $x \in L^2_{\rm loc}(I, \mathfrak{M}^{\tau^{s\lambda}_\square}, ds)_{\rm ac}$ and almost all $t \in I$. Then there exists a function $\lambda_{\sigma,M} : I \longrightarrow \mathbb{R}_+$, depending on F, G, H, such that

$$\|P(t,x)\|^2_\sigma \le \lambda_{\sigma,M}(t)\left(1 + \|x(t)\|^2_{\alpha(\sigma,M)}\right)$$

for all $x \in L^2_{\rm loc}(I, \mathfrak{M}^{\tau^{s\lambda}_\square}, ds)_{\rm ac}$ and almost all $t \in I$.

Proof. Every selection p of the multifunction P is given by an expression of the form

$$p(t,x)\xi = \frac{d}{dt}\left[\left(\int_{t_0}^{t} dM(s)\, f(s,x) + dM^{+}(s)\, g(s,x) + h(s,x)ds\right)\xi\right], \;\; \xi \in \mathcal{D},$$

where f, g, h are selections of the multifunctions F, G, H, respectively, $x \in L^2_{\mathrm{loc}}(I, \mathfrak{M}^{\tau_\square^{s\lambda}}, ds)_{\mathrm{ac}}$, and almost all $t \in I$.

By [6, Theorem 4.2], we have

$$\|p(t,x)\|^2_{\sigma} \leq \tilde{\lambda}_{\sigma,M}(t)\left(\|f(t,x)\|^2_{\sigma,M} + \|g(t,x)\|^2_{\sigma,M} + \|h(t,x)\|^2_{\sigma}\right),$$

for each $\sigma \in \Theta(\tau_\square^{s\lambda})$, $x \in L^2_{\mathrm{loc}}(I, \mathfrak{M}^{\tau_\square^{s\lambda}}, ds)_{\mathrm{ac}}$ and almost all $t \in I$, where

$$\|\cdot\|_{\sigma,M} = \begin{cases} \|\cdot\|_{\beta(\sigma,M)}\,, & \text{for } \sigma \in \Theta(\tau_\square^{s}) \\ \|\cdot\|_{\sigma,\mathcal{D}_{M\beta}}\,, & \text{for } \sigma \in \mathfrak{M}_e \end{cases}$$

using the same notation as in Ref. [6, Assumptions 4.1]. Hence

$$\begin{aligned} &\sup\{\|p(t,x)\|^2_{\sigma} : p \text{ is a selection of } P\} \\ &\leq \tilde{\lambda}_{\sigma,M}(t)\Big(\sup\{\|f(t,x)\|^2_{\sigma,M} : f \text{ is a selection of } F\} \\ &\qquad + \sup\{\|g(t,x)\|^2_{\sigma,M} : g \text{ is a selection of } G\} \\ &\qquad + \sup\{\|h(t,x)\|^2_{\sigma} : h \text{ is a selection of } H\}\Big), \end{aligned}$$

whence

$$\|P(t,x)\|^2_{\sigma} \leq \tilde{\lambda}_{\sigma,M}(t)\left(\|F(t,x)\|^2_{\sigma,M} + \|G(t,x)\|^2_{\sigma,M} + \|H(t,x)\|^2_{\sigma}\right),$$

for each $\sigma \in \Theta(\tau_\square^{s\lambda})$, $x \in L^2_{\mathrm{loc}}(I, \mathfrak{M}^{\tau_\square^{s\lambda}}, ds)_{\mathrm{ac}}$ and almost all $t \in I$. The inequality

$$\|P(t,x)\|^2_{\sigma} \leq \lambda_{\sigma,M}(t)\left(1 + \|x(t)\|^2_{\alpha(\sigma,M)}\right),$$

$x \in L^2_{\mathrm{loc}}(I, \mathfrak{M}^{\tau_\square^{s\lambda}}, ds)_{\mathrm{ac}}$, $\sigma \in \Theta(\tau_\square^{s\lambda})$ and almost all $t \in I$, now follows from the hypotheses. This concludes the proof. □

Theorem 5.2. *Assume that* $F \in L^2_{\mathrm{loc}}(I \times \mathfrak{M}^{\tau_\square}, dM)_{\mathrm{mvs}}$, $G \in L^2_{\mathrm{loc}}(I \times \mathfrak{M}^{\tau_\square}, dM^{+})_{\mathrm{mvs}}$, $H \in L^2_{\mathrm{loc}}(I \times \mathfrak{M}^{\tau_\square}, ds)_{\mathrm{mvs}}$ *and that* P *is as defined in (5.1). Then,* P *is* $\tau_\square^{s\lambda}$*-Lipschitzian whenever* F, G, H *are* $\tau_\square^{s\lambda}$*-Lipschitzian.*

Proof. With p, f, g, h as in Theorem 5.1, we have from [6, Theorem 4.3] that

$$\|p(t,x) - p(t,y)\|_\alpha \leq \lambda_{\alpha,M}(\|f(t,x) - f(t,y)\|_{\beta(\alpha,M)} + \|g(t,x) - g(t,y)\|_{\beta(\alpha,M)} + \|h(t,x) - h(t,y)\|_\alpha),$$

whence

$$\|P(t,x) - P(t,y)\|_\alpha \leq \lambda_{\alpha,M}(\|F(t,x) - F(t,y)\|_{\beta(\alpha,M)} + \|G(t,x) - G(t,y)\|_{\beta(\alpha,M)} + \|H(t,x) - H(t,y)\|_\alpha),$$

for all $x, y \in L^2_{\mathrm{loc}}(I, \mathfrak{M}^{\tau_\square^{s\lambda}}, ds)_{\mathrm{ac}}$, where $\beta(\alpha, M) \in \Theta(\tau_\square^{s\lambda})$, $\lambda_{\alpha,M}$ is some $\mathbb{R}$-valued function on I, $\alpha \in \Theta(\tau_\square^{s\lambda})$, and almost all $t \in I$. This concludes the proof. □

Theorem 5.3. *Let $F \in L^2_{\mathrm{loc}}(I \times \mathfrak{M}^{\tau_\square}, dM)_{\mathrm{mvs}}$, $G \in L^2_{\mathrm{loc}}(I \times \mathfrak{M}^{\tau_\square}, dM^+)_{\mathrm{mvs}}$, $H \in L^2_{\mathrm{loc}}(I \times \mathfrak{M}^{\tau_\square}, ds)_{\mathrm{mvs}}$ and P be defined as in (5.1). Then, finding a $\tau_\square^{s\lambda}$-solution of the initial value stochastic differential inclusion (3.2) is equivalent to finding $x \in L^2_{\mathrm{loc}}(I, \mathfrak{M}^{\tau_\square^{s\lambda}}, ds)_{\mathrm{ac}}$ such that*

$$\begin{aligned} &\frac{d}{dt}(x(t)\xi) \in P(t,x)\xi \\ &x(t_0) = x_0 \end{aligned} \tag{5.3}$$

for arbitrary $\xi \in \mathcal{D}, x_0 \in \mathfrak{M}(\tau_\square^{s\lambda})$ and almost all $t \in I$.

Proof. This follows from the definition of P. □

Remark 5.1. 1. In view of Theorem 5.3, we may study the diverse features of the $\tau_\square^{s\lambda}$-solutions of problem 3.2, whether analytically or numerically, by equivalently studying the features of the $\tau_\square^{s\lambda}$-solutions of problem (5.3).

2. Notice that (5.3) is a differential inclusion of nonclassical type, since $P(t,x)\xi$ is, in general, not of the form $\tilde{P}(t, x(t)\xi)$, $t \in I, \xi \in \mathcal{D}, x \in L^2_{\mathrm{loc}}(I, \mathfrak{M}^{\tau_\square^{s\lambda}}, ds)_{\mathrm{ac}}$.

Notation 5.2. If z is an $\mathfrak{M}^{\tau_\square}$-valued map on I such that $\frac{d}{dt}(z(t)\xi)$ is in $\mathcal{H}$ for each $\xi \in \mathcal{D}$ and almost all $t \in I$, then $\check{z}(t)$ will denote the linear map $\xi \longmapsto \frac{d}{dt}(z(t)\xi)$, $\xi \in \mathcal{D}$. Thus $\check{z}(t) \in L^+_w(\mathcal{D}, \mathcal{H})$ and $\check{z}(t)\xi = \frac{d}{dt}(z(t)\xi)$, for all $\xi \in \mathcal{D}$ and almost all $t \in I$.

5.3 Fundamental hypotheses

In the subsequent discussion, we employ the following hypotheses and notation.

(H$_1$) The map y is a member of $L^2_{\mathrm{loc}}(I, \mathfrak{M}^{\tau_\square}, ds)_{\mathrm{ac}}$ possessing the property that for each $\alpha \in \Theta(\tau_\square)$ and almost all $t \in I$, there is a positive number p_α such that

$$d_\alpha\left(\breve{y}(t), P(t, y)\right) \leq p_\alpha(t).$$

(H$_2$) $\gamma > 0$ is an arbitrary but fixed number and $Q_{y,\gamma}$ is the set

$$Q_{y,\gamma} \equiv \{(t, x) \in I \times L^2_{\mathrm{loc}}(I, \mathfrak{M}^{\tau_\square}, ds) : \|x(t) - y(t)\|_\alpha \leq \gamma, \ \forall\, \alpha \in \Theta(\tau_\square)\}.$$

(H$_3$) Each of the maps $F \in L^2_{\mathrm{loc}}(I \times \mathfrak{M}^{\tau_\square}, dM)_{\mathrm{mvs}}$, $G \in L^2_{\mathrm{loc}}(I \times \mathfrak{M}^{\tau_\square}, dM^+)_{\mathrm{mvs}}$, and $H \in L^2_{\mathrm{loc}}(I \times \mathfrak{M}^{\tau_\square}, ds)_{\mathrm{mvs}}$ is Lipschitzian from $Q_{y,\gamma}$ to $(\mathrm{clos}(\mathfrak{M}^{\tau_\square}), \tau^H_\square)$.

(H$_4$) (a) For each $\alpha \in \Theta(\tau_\square)$, $\quad \delta_\alpha \equiv \|x_0 - y(t_0)\|_\alpha$.

(b) We assume that $\quad \delta_\alpha \leq \gamma, \ \forall\, \alpha \in \Theta(\tau_\square)$.

(H$_5$) Define

(a) for arbitrary $\alpha \in \Theta(\tau_\square)$, the map Ξ_α by

$$\Xi_\alpha(t) \equiv \delta_\alpha \exp\left[\int_{t_0}^t ds\, k_\alpha(s)\right] + \int_{t_0}^t ds\, p_\alpha(s) \exp\left[\int_s^t dr\, k_\alpha(r)\right], \ t \in I,$$

with k_α as in Theorem 5.1, and

(b) the subset J of I by

$$J \equiv \{t \in I : \Xi_\alpha(t) \leq \gamma, \ \forall\, \alpha \in \Theta(\tau_\square)\}.$$

Remark 5.2. We shall need the following result in the sequel.

Proposition 5.4. *Let $\{\varphi_j\}_{j=1}^\infty$ be a sequence of members of $L^2_{\mathrm{loc}}(I, \mathfrak{M}^{\tau_\square}, ds)_{\mathrm{ac}}$ which satisfy*

(i) $(t, \varphi_j(t)) \in Q_{y,\gamma}$, $j \geq 1$, *for almost all* $t \in J$;

(ii) *there exists a sequence* $\{v_j\}_{j=1}^\infty \subset L^1_{\mathrm{loc}}(I, \mathfrak{M}^{\tau_\square}, ds)$ *such that*

(a) $\varphi_j(t) = x_0 + \displaystyle\int_{t_0}^t ds\, v_{j-1}(s), \ j \geq 1;$

(b) $\|\breve{\varphi}_j(t) - \breve{\varphi}_{j-1}(t)\|_\alpha$

$$\le k_\alpha(t)\left\{\frac{\delta_\alpha(m_\alpha(t))^{j-2}}{(j-2)!} + \int_{t_0}^t ds\, \frac{[m_\alpha(t)-m_\alpha(s)]^{j-2}}{(j-2)!}p_\alpha(s)\right\}$$

$\equiv b_{\alpha,j-2}(t)$ *for almost all* $t \in J$*, where* $m_\alpha(t) = \int_{t_0}^t ds\, k_\alpha(s)$,

$\alpha \in \Theta(\tau_\Box)$.

Then

$\|\varphi_j(t) - \varphi_{j-1}(t)\|_\alpha \le b_{\alpha,j-1}(t), \quad j \ge 2,$ *all* $\alpha \in \Theta(\tau_\Box)$*, and almost all* $t \in J$.

Proof. Assume (i) and (ii), and let $\alpha \in \Theta(\tau_\Box)$. Then

$$\begin{aligned}
\|\varphi_j(t) - \varphi_{j-1}(t)\|_\alpha &= \left\|\int_{t_0}^t ds\,(v_{j-1}(s) - v_{j-2}(s))\right\|_\alpha \\
&= \left\|\int_{t_0}^t ds\,(\breve{\varphi}_j(s) - \breve{\varphi}_{j-1}(s))\right\|_\alpha, \text{ by (ii)(a)} \\
&\le \int_{t_0}^t ds\, \|\breve{\varphi}_j(s) - \breve{\varphi}_{j-1}(s)\|_\alpha, \\
&\le \int_{t_0}^t ds\, b_{\alpha,j-2}(s), \text{ by (ii)(b)} \\
&= b_{\alpha,j-1}(t), \quad j \ge 2, \text{ all } \alpha \in \Theta(\tau_\Box), \text{ and almost all } t \in J.
\end{aligned}$$

This concludes the proof. □

Remark 5.3. In the sequel, we discuss the existence of the solutions of problem (5.3) in the strong/$\lambda^\star$ locally convex operator topologies $\{\tau_s, \tau_{s^\star}, \tau_{\sigma s}, \tau_{\sigma s^\star}, \lambda^\star\}$. Our main result is the following.

Theorem 5.5. *Suppose that the hypotheses* (H_1)-(H_5) *hold and* F, G, *and* H *are continuous from* $I \times L^2_{\mathrm{loc}}(I, \mathfrak{M}^{\tau_\Box}, ds)$ *to* $(\mathrm{clos}(\mathfrak{M}^{\tau_\Box}), \tau_\Box^H)$. *Then there exists a solution* φ *of problem (5.3) such that*

$$\|\varphi(t) - y(t)\|_\alpha \le \Xi_\alpha(t), \quad t \in J$$

and

$$\|\breve{\varphi}(t) - \breve{y}(t)\|_\alpha \le k_\alpha(t)\Xi_\alpha(t) + p_\alpha(t),$$

for all $\alpha \in \Theta(\tau_\Box)$ *and almost all* $t \in J$.

Proof. We first give the proof in the *strong topology* τ_s. In this case, $F \in L^2_{\text{loc}}(I \times \mathfrak{M}^{\tau_s}, dM)_{\text{mvs}}$, $G \in L^2_{\text{loc}}(I \times \mathfrak{M}^{\tau_s}, dM^+)_{\text{mvs}}$ and $H \in L^2_{\text{loc}}(I \times \mathfrak{M}^{\tau_s}, ds)_{\text{mvs}}$.

Let y be as in (H_1) and $\xi \in \mathcal{D}$. Since $\dfrac{d}{dt}(y(t)\xi)$ may not be in $P(t,y)\xi$, almost all $t \in I$, y is not necessarily a τ_s-solution of problem (5.3).

Define φ_0 as y. Then φ_0 is adapted. By [4, Corollary 8.2.13], there is a measurable selection $t \mapsto v_0(t)(\xi)$ in $t \mapsto P(t,\varphi_0)\xi$, $t \in I$, $\xi \in \mathcal{D}$ such that

$$\|v_0(t) - \breve{\varphi}_0(t)\|_\xi = d_\xi(\breve{\varphi}_0(t), P(t,\varphi_0)), \tag{5.4}$$

almost all $t \in I$. By (H_1), the right-hand side is majorized by $p_\xi(t)$. As the map $\xi \mapsto v_0(t)(\xi)$ is linear for almost all $t \in J$, there is a $v_0(t) \in \mathfrak{M}^{\tau_s}$ such that $v_0(t)(\xi) = v_0(t)\xi$, for arbitrary $\xi \in \mathcal{D}$ and almost all $t \in J$. As $t \mapsto v_0(t)\xi$ is locally Lebesgue-Bochner integrable, it determines a map φ_1 through the definition

$$\varphi_1(t)\xi = x_0\xi + \int_{t_0}^t ds\, v_0(t)\xi, \quad \text{almost all } t \in J,\ \xi \in \mathcal{D},$$

and since $v_0(t)$ is in $\mathfrak{M}^{\tau_s}$ for almost all $t \in J$, it follows that $\varphi_1(t)$ is affiliated to $\mathfrak{M}_t$, i.e. φ_1 is adapted. Moreover, for $t \in J$,

$$\begin{aligned}
\|\varphi_1(t) - \varphi_0(t)\|_\xi &\leq \|x_0 - \varphi(t_0)\|_\xi + \int_{t_0}^t ds\ \|v_0(s) - \breve{\varphi}_0(s)\|_\xi, \\
&\qquad \text{since } \breve{\varphi}_0 \text{ is the zero map,} \\
&= \delta_\xi + \int_{t_0}^t ds\, d_\xi(\breve{\varphi}_0(t), P(t,\varphi_0)), \text{ by (H)}_1 \text{ and equation (5.4)} \\
&\leq \delta_\xi + \int_{t_0}^t ds\, p_\xi(s), \text{ by (H)}_4
\end{aligned}$$

by (H_1) and (H_4).

We claim that there exists, indeed, a sequence $\{\varphi_j\}_{j\geq 0}$ of τ_s-absolutely continuous maps from I to $\mathfrak{M}^{\tau_s}$ satisfying (i) and (ii) of Proposition 5.4, and hence its conclusion. To prove this, assume that $\{\varphi_j\}_{0\leq j\leq n}$ has already been defined and satisfies (i) and (ii) of Proposition 5.4. By [4, Corollary 8.2.13], there exists $v_n(\cdot)(\xi) \in P(\cdot,\varphi_n)\xi$ such that

$$\|v_n(t) - \breve{\varphi}_n(t)\|_\xi = d_\xi(\breve{\varphi}_n(t), P(t,\varphi_n)), \quad \text{a.e on } J,$$

where $v_n(t)$ is the linear map $\xi \mapsto v_n(t)(\xi),\ \xi \in \mathcal{D}$, almost all $t \in J$. Now define φ_{n+1} by

$$\varphi_{n+1}(t)\xi = x_o\xi + \int_{t_0}^{t} ds\, v_n(t)\xi, \quad \text{almost all } t \in J. \tag{5.5}$$

Then

$$\begin{aligned}
\|\breve{\varphi}_{n+1}(t) - \breve{\varphi}_n(t)\|_\xi &= \|v_n(t) - v_{n-1}(t)\|_\xi \\
&\le \rho_\xi(P(t,\varphi_n), P(t,\varphi_{n-1})) \\
&\le k_\xi(t)\,\|\varphi_n(t) - \varphi_{n-1}(t)\|_{\alpha(\xi)}\,, \quad \text{for some } \alpha(\xi) \in \mathcal{D}, \\
&\qquad\qquad \text{since } P \text{ is Lipschitzian,} \\
&\le k_\xi(t) b_{\alpha(\xi),n-1},\ \xi \in \mathcal{D}, \text{ and almost all } t \in J.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\|\varphi_{n+1}(t) - \varphi_0(t)\|_\xi &\le \|\varphi_1(t) - \varphi_0(t)\|_\xi + \cdots + \|\varphi_{n+1}(t) - \varphi_n(t)\|_\xi \\
&\le \sum_{k=0}^{n} b_{\xi,k}(t) \le \Xi_\xi(t) \le \gamma.
\end{aligned}$$

It follows from Proposition 5.4 that $\{\varphi_n\}_{n\ge 0}$ is τ_s-Cauchy and converges uniformly to $\varphi(t)$ in $\mathfrak{M}^{\tau_s}$. Also from Proposition 5.4, $\{v_n\}_{n\ge 0}$ is τ_s-Cauchy, whence $\{v_n\}_{n\ge 0}$ converges pointwise, for almost all $t \in J$, to a map $v \in L^1_{\text{loc}}(I, \mathfrak{M}^{\tau_s}, ds)$. From (5.5), we get

$$\varphi(t)\xi = x_o\xi + \int_{t_0}^{t} ds\, v(t)\xi, \quad \forall\ \xi \in \mathcal{D}, \text{ and almost all } t \in J.$$

As P is continuous on $I \times L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_s}, ds)_{\text{ac}}$ and has closed values, its graph is closed. Hence, since $v_i(t) \in P(t,\varphi_i)$, for almost all t in J, it follows that $v(t) \in P(t,\varphi)$, for almost all $t \in J$, whence

$$\frac{d}{dt}(\varphi(t)\xi) \in P(t,\varphi)\xi, \quad \forall\ \xi \in \mathcal{D}, \text{ and almost all } t \in J,$$

showing that φ is a τ_s-solution of problem (5.3). This concludes the proof in the case of the strong topology τ_s.

Strong topology* $\tau_{\star s}$

In this case, $F \in L^2_{\text{loc}}(I \times \mathfrak{M}^{\tau_{\star s}}, dM)_{\text{mvs}}$, $G \in L^2_{\text{loc}}(I \times \mathfrak{M}^{\tau_{\star s}}, dM^+)_{\text{mvs}}$ and $H \in L^2_{\text{loc}}(I \times \mathfrak{M}^{\tau_{\star s}}, ds)_{\text{mvs}}$. Let ξ be an arbitrary member of $\Theta(\tau_{\star s}) = \mathcal{D}$.

Then, arguing as above, there exists a sequence $\{\varphi_n\}_{n\geq 0}$ of τ_s-absolutely continuous maps from I to $\mathfrak{M}^{\tau_s}$ satisfying the inequalities

$$\begin{aligned} \|\varphi_1(t) - \varphi_0(t)\|_{\star\xi} &\leq \delta_{\star\xi} + \int_{t_0}^{t} ds\, p_\xi(s), \text{ with } \delta_{\star\xi} = \|x_0 - \varphi(t_0)\|_{\star\xi} \\ &\qquad \text{and } p_\xi \text{ as in } (\mathrm{H})_1, \end{aligned}$$

$$\begin{aligned} \|\varphi_{n+1}(t) - \varphi_0(t)\|_{\star\xi} &\leq \sum_{k=0}^{n} b_{\xi,k}(t) \leq \Xi_\xi(t) \leq \gamma \\ \|\breve{\varphi}_{n+1}(t) - \breve{\varphi}_n(t)\|_{\star\xi} &\leq k_\xi(t) b_{\alpha(\xi),n-1}, \end{aligned}$$

for some $\alpha(\xi)$ depending on $\xi \in \Theta(\tau_{\star s}) = \mathcal{D}$, and almost all $t \in J$, where we have invoked the Lipschitzian character of P. That the $\tau_{\star s}$-limit $\varphi(t)$ of $\{\varphi_n(t)\}_{n\geq 0}$ is a $\tau_{\star s}$-solution of problem (5.3), i.e. that φ satisfies

$$\frac{d}{dt}(\varphi(t)\xi) \in P(t,\varphi)\xi, \quad \forall\, \xi \in \mathcal{D}, \text{ and almost all } t \in J,$$

is shown in the same way as in the case of the *strong topology* above.

σ-Strong topology $\tau_{\sigma s}$ In this case, $F \in L^2_{\mathrm{loc}}(I \times \mathfrak{M}^{\tau_{\sigma s}}, dM)_{\mathrm{mvs}}$, $G \in L^2_{\mathrm{loc}}(I \times \mathfrak{M}^{\tau_{\sigma s}}, dM^+)_{\mathrm{mvs}}$ and $H \in L^2_{\mathrm{loc}}(I \times \mathfrak{M}^{\tau_{\sigma s}}, ds)_{\mathrm{mvs}}$. Let $\boldsymbol{\xi} = \{\xi_n\}_{n=1}^{\infty}$ be an arbitrary member of $\Theta(\tau_{\sigma s}) = \mathcal{D}_\infty$. As above, there exists a sequence $\{\varphi_n\}_{n\geq 0}$ of $\tau_{\sigma s}$-absolutely continuous maps from I to $\mathfrak{M}^{\tau_{\sigma s}}$ which satisfy the inequalities

$$\begin{aligned} \|\varphi_1(t) - \varphi_0(t)\|_{\boldsymbol{\xi}} &\leq \delta_{\boldsymbol{\xi}} + \int_{t_0}^{t} ds\, p_{\boldsymbol{\xi}}(s), \text{ with } \delta_{\boldsymbol{\xi}} = \|x_0 - \varphi(t_0)\|_{\boldsymbol{\xi}} \\ &\qquad \text{and } p_{\boldsymbol{\xi}} \text{ as in } (\mathrm{H})_1, \end{aligned}$$

$$\begin{aligned} \|\varphi_{n+1}(t) - \varphi_0(t)\|_{\boldsymbol{\xi}} &\leq \sum_{k=0}^{n} b_{\boldsymbol{\xi},k}(t) \leq \Xi_{\boldsymbol{\xi}}(t) \leq \gamma \\ \|\breve{\varphi}_{n+1}(t) - \breve{\varphi}_n(t)\|_{\boldsymbol{\xi}} &\leq k_{\boldsymbol{\xi}}(t) b_{\alpha(\boldsymbol{\xi}),n-1}, \end{aligned}$$

for some $\alpha(\boldsymbol{\xi})$ depending on $\boldsymbol{\xi} \in \Theta(\tau_{\sigma s}) = \mathcal{D}_\infty$, and almost all $t \in J$, where we have used the Lipschitzian character of P. We are then able to conclude, in the same way as in the case of the *strong topology* above, that the $\tau_{\sigma s}$-limit $\varphi(t)$ of $\{\varphi_n(t)\}_{n\geq 0}$ is a $\tau_{\sigma s}$-solution of problem (5.3), i.e. that φ satisfies

$$\frac{d}{dt}(\varphi(t)\xi) \in P(t,\varphi)\xi, \quad \forall\, \xi \in \mathcal{D}, \text{ and almost all } t \in J.$$

σ-Strong topology $\tau_{\sigma s^\star}$*

In this case, $F \in L^2_{\rm loc}(I \times \mathfrak{M}^{\tau_{\sigma s^\star}}, dM)_{\rm mvs}$, $G \in L^2_{\rm loc}(I \times \mathfrak{M}^{\tau_{\sigma s^\star}}, dM^+)_{\rm mvs}$ and $H \in L^2_{\rm loc}(I \times \mathfrak{M}^{\tau_{\sigma s^\star}}, ds)_{\rm mvs}$. Let $\boldsymbol{\xi} = \{\xi_n\}_{n=1}^\infty$ be an arbitrary member of $\Theta(\tau_{\sigma s^\star}) = \mathcal{D}_\infty$. Then there is a sequence $\{\varphi_n\}_{n\geq 0}$ of $\tau_{\sigma s^\star}$-absolutely continuous maps from I to $\mathfrak{M}^{\tau_{\sigma s^\star}}$ which satisfy the inequalities

$$\begin{aligned}
\|\varphi_1(t) - \varphi_0(t)\|_{\star\boldsymbol{\xi}} &\leq \delta_{\star\boldsymbol{\xi}} + \int_{t_0}^t ds\, p_{\boldsymbol{\xi}}(s), \text{ with } \delta_{\star\boldsymbol{\xi}} = \|x_0 - \varphi(t_0)\|_{\star\boldsymbol{\xi}} \\
&\qquad \text{and } p_{\boldsymbol{\xi}} \text{ as in } (\mathrm{H})_1, \\
\|\varphi_{n+1}(t) - \varphi_0(t)\|_{\star\boldsymbol{\xi}} &\leq \sum_{k=0}^n b_{\boldsymbol{\xi},k}(t) \leq \Xi_{\boldsymbol{\xi}}(t) \leq \gamma \\
\|\breve{\varphi}_{n+1}(t) - \breve{\varphi}_n(t)\|_{\star\boldsymbol{\xi}} &\leq k_{\boldsymbol{\xi}}(t) b_{\alpha(\boldsymbol{\xi}),n-1},
\end{aligned}$$

for some $\alpha(\boldsymbol{\xi})$ depending on $\boldsymbol{\xi} \in \mathcal{D}_\infty$, and almost all $t \in J$, where we have utilized the Lipschitzian character of P. Hence as in the case of the *strong topology* above, the $\tau_{\sigma s^\star}$-limit $\varphi(t)$ of $\{\varphi_n(t)\}_{n\geq 0}$ is a $\tau_{\sigma s^\star}$-solution of problem (5.3), i.e. φ satisfies

$$\frac{d}{dt}(\varphi(t)\xi) \in P(t,\varphi)\xi, \quad \forall\, \xi \in \mathcal{D}, \text{ and almost all } t \in J.$$

$\lambda^\star$-topology $\tau_{\lambda^\star}$

In this case, $F \in L^2_{\rm loc}(I \times \mathfrak{M}^{\tau_{\lambda^\star}}, dM)_{\rm mvs}$, $G \in L^2_{\rm loc}(I \times \mathfrak{M}^{\tau_{\lambda^\star}}, dM)_{\rm mvs}$ and $H \in L^2_{\rm loc}(I \times \mathfrak{M}^{\tau_{\lambda^\star}}, ds)_{\rm mvs}$. Let a be an arbitrary member of $\mathfrak{M}_e$. Then there is a sequence $\{\varphi_n\}_{n\geq 0}$ of $\tau_{\lambda^\star}$-absolutely continuous maps from I to $\mathfrak{M}^{\tau_{\lambda^\star}}$ which satisfy the inequalities

$$\begin{aligned}
\|\varphi_1(t) - \varphi_0(t)\|_a &\leq \delta_a + \int_{t_0}^t ds\, p_a(s), \text{ with } \delta_a = \|x_0 - \varphi(t_0)\|_a \\
&\qquad \text{and } p_a \text{ as in } (\mathrm{H})_1, \\
\|\varphi_{n+1}(t) - \varphi_0(t)\|_a &\leq \sum_{k=0}^n b_{a,k}(t) \leq \Xi_a(t) \leq \gamma \\
\|\breve{\varphi}_{n+1}(t) - \breve{\varphi}_n(t)\|_a &\leq k_a(t) b_{\alpha(a),n-1},
\end{aligned}$$

for some $\alpha(a)$ depending on $a \in \Theta(\tau_{\lambda^\star}) = \mathfrak{M}_e$, and almost all $t \in J$, since P is Lipschizian. Hence as in the case of the *strong topology* above, the $\tau_{\lambda^\star}$-limit $\varphi(t)$ of $\{\varphi_n(t)\}_{n\geq 0}$ is a $\tau_{\lambda^\star}$-solution of problem (5.3), i.e. φ satisfies

$$\frac{d}{dt}(\varphi(t)\xi) \in P(t,\varphi)\xi, \quad \forall\, \xi \in \mathcal{D}, \text{ and almost all } t \in J.$$

This concludes the proof of the theorem. □

6 Solutions in the weak topologies

In this section, we discuss the existence of the solutions of equation 3.2 in the *weak* locally convex operator topologies $\{\tau_w, \tau_{\sigma w}\}$; let $\tau_\square^w$ denote any member of this set of topologies.

We first reformulate equation 3.2 as follows.

6.1 The multifunction P

Define

$$P \; : \; I \times L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_w}, ds)_{\text{ac}} \longrightarrow 2^{\text{sesq}(\mathcal{D})}$$

$$(t, x) \longmapsto P(t, x)$$

by

$$P(t,x)(\eta,\xi) = \frac{d}{dt}\left[\langle \eta, \left(\int_{t_0}^{t} dM(s)\, F(s,x) + dM^+(s)\, G(s,x) + H(s,x)ds \right) \xi \rangle \right], \tag{6.1}$$

$\eta, \xi \in \mathcal{D}$, where $\text{sesq}(\mathcal{D})$ denotes the set of all sesquilinear forms on $\mathcal{D} \times \mathcal{D}$.

Since $x \in L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_w}, ds)_{\text{ac}}$, the map P is well defined. We shall utilize a number of properties of the map P.

6.2 Assumptions

Throughout the rest of this Section, *we work under the same set of assumptions listed as* **Assumptions 5.1**, *together with their notation, in* Ref. [6, Assumptions 5.1, page 985].

Theorem 6.1. *Assume that* $F \in L^2_{\text{loc}}(I \times \mathfrak{M}^{\tau_\square^w}, dM)_{\text{mvs}}$*,* $G \in L^2_{\text{loc}}(I \times \mathfrak{M}^{\tau_\square^w}, dM^+)_{\text{mvs}}$*,* $H \in L^2_{\text{loc}}(I \times \mathfrak{M}^{\tau_\square^w}, ds)_{\text{mvs}}$ *and that* P *is as defined in (6.1). Suppose that for any* $\sigma \in \Theta(\tau_\square^w)$*, there exist some* $\alpha(\sigma, M) \in \Theta(\tau_\square^w)$ *and a continuous function* $k_\sigma : I \longrightarrow \mathbb{R}_+$*, depending on* F*,* G*,* H*, such that*

$$\|F(t,x)\|^2_{\sigma,M} + \|G(t,x)\|^2_{\sigma,M} + \|H(t,x)\|^2_{\sigma} \leq k_\sigma(t)\left(1 + \|x(t)\|^2_{\alpha(\sigma,M)}\right), \tag{6.2}$$

for all $x \in L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\square^w}, ds)_{\text{ac}}$ *and almost all* $t \in I$*. Then there exists a function* $\lambda_{\sigma,M} : I \longrightarrow \mathbb{R}_+$*, depending on* F*,* G*,* H*, such that*

$$\|P(t,x)\|^2_{\sigma} \leq \lambda_{\sigma,M}(t)\left(1 + \|x(t)\|^2_{\alpha(\sigma,M)}\right)$$

for all $x \in L^2_{\text{loc}}(I, \mathfrak{M}^{\tau_\square^w}, ds)_{\text{ac}}$ *and almost all* $t \in I$*.*

Proof. Every selection p of the multifunction P is given by an expression of the form

$$p(t,x)(\eta,\xi) = \frac{d}{dt}\left[\langle \eta, \left(\int_{t_0}^{t} dM(s)\, f(s,x) + dM^{+}(s)\, g(s,x) + h(s,x)ds\right) \xi\rangle\right],$$

$\eta, \xi \in \mathcal{D}$, where f, g, h are selections of the multifunctions F, G, H, respectively, $x \in L^2_{\text{loc}}(I, \mathfrak{M}^{\tau^w_\square}, ds)_{\text{ac}}$, and almost all $t \in I$.

By [6, Theorem 4.2], we have

$$\|p(t,x)\|^2_\sigma \le \tilde{\lambda}_{\sigma,M}(t)\left(\|f(t,x)\|^2_{\sigma,M} + \|g(t,x)\|^2_{\sigma,M} + \|h(t,x)\|^2_\sigma\right),$$

for each $\sigma \in \Theta(\tau^w_\square)$, $x \in L^2_{\text{loc}}(I, \mathfrak{M}^{\tau^w_\square}, ds)_{\text{ac}}$ and almost all $t \in I$. Hence

$$\begin{aligned}
&\sup\{\|p(t,x)\|^2_\sigma : p \text{ is a selection of } P\} \\
&\le \tilde{\lambda}_{\sigma,M}(t)\Big(\sup\{\|f(t,x)\|^2_{\sigma,M} : f \text{ is a selection of } F\} \\
&\qquad + \sup\{\|g(t,x)\|^2_{\sigma,M} : g \text{ is a selection of } G\} \\
&\qquad + \sup\{\|h(t,x)\|^2_{\sigma} : h \text{ is a selection of } H\}\Big),
\end{aligned}$$

whence

$$\|P(t,x)\|^2_\sigma \le \tilde{\lambda}_{\sigma,M}(t)\left(\|F(t,x)\|^2_{\sigma,M} + \|G(t,x)\|^2_{\sigma,M} + \|H(t,x)\|^2_\sigma\right),$$

for each $\sigma \in \Theta(\tau^w_\square)$, $x \in L^2_{\text{loc}}(I, \mathfrak{M}^{\tau^w_\square}, ds)_{\text{ac}}$ and almost all $t \in I$. The inequality

$$\|P(t,x)\|^2_\sigma \le \tilde{\lambda}_{\sigma,M}(t)\left(1 + \|x(t)\|^2_{\alpha(\sigma,M)}\right),$$

$x \in L^2_{\text{loc}}(I, \mathfrak{M}^{\tau^w_\square}, ds)_{\text{ac}}$, $\sigma \in \Theta(\tau^w_\square)$ and almost all $t \in I$, now follows from the hypotheses. This concludes the proof. □

Theorem 6.2. *Assume that* $F \in L^2_{\text{loc}}(I \times \mathfrak{M}^{\tau^w_\square}, dM)_{\text{mvs}}$, $G \in L^2_{\text{loc}}(I \times \mathfrak{M}^{\tau^w_\square}, dM^+)_{\text{mvs}}$, $H \in L^2_{\text{loc}}(I \times \mathfrak{M}^{\tau^w_\square}, ds)_{\text{mvs}}$ *and that* P *is as defined in (6.1). Then,* P *is* $\tau^w_\square$*-Lipschitzian whenever* F, G, H *are* $\tau^w_\square$*-Lipschitzian.*

Proof. With p, f, g, h as in Theorem 5.1, we have from [6, Theorem 4.3] that

$$\begin{aligned}
\|p(t,x) - p(t,y)\|_\alpha \quad \le \quad & \tilde{\lambda}_{\alpha,M}(t)(\|f(t,x) - f(t,y)\|_{\beta(\alpha,M)} + \|g(t,x) - g(t,y)\|_{\beta(\alpha,M)} \\
& + \|h(t,x) - h(t,y)\|_\alpha),
\end{aligned}$$

whence

$$\begin{aligned}\|P(t,x)-P(t,y)\|_\alpha &\leq \tilde{\lambda}_{\alpha,M}(t)(\|F(t,x)-F(t,y)\|_{\beta(\alpha,M)} \\ &\qquad + \|G(t,x)-G(t,y)\|_{\beta(\alpha,M)} + \|H(t,x)-H(t,y)\|_\alpha),\end{aligned}$$

for all $x, y \in L^2_{\text{loc}}(I, \mathfrak{M}^{\tau^w_\square}, ds)_{\text{ac}}$, where $\beta(\alpha, M) \in \Theta(\tau^w_\square)$, $\tilde{\lambda}_{\alpha,M}$ is some $\mathbb{R}$-valued function on I, $\alpha \in \Theta(\tau^w_\square)$, and almost all $t \in I$. This concludes the proof. □

Theorem 6.3. *$F \in L^2_{\text{loc}}(I \times \mathfrak{M}^{\tau^w_\square}, dM)_{\text{mvs}}$, $G \in L^2_{\text{loc}}(I \times \mathfrak{M}^{\tau^w_\square}, dM^+)_{\text{mvs}}$, $H \in L^2_{\text{loc}}(I \times \mathfrak{M}^{\tau^w_\square}, ds)_{\text{mvs}}$ and P be defined as in (6.1). Then, finding a $\tau^w_\square$-solution of the initial value stochastic differential inclusion 3.2 is equivalent to finding $x \in L^2_{\text{loc}}(I, \mathfrak{M}^{\tau^w_\square}, ds)_{\text{ac}}$ such that*

$$\begin{aligned}\frac{d}{dt}\langle \eta, x(t)\xi\rangle &\in P(t,x)(\eta,\xi) \\ x(t_0) &= x_0\end{aligned} \tag{6.3}$$

for arbitrary $\eta, \xi \in \mathcal{D}, x_0 \in \mathfrak{M}(\tau^w_\square)$ and almost all $t \in I$.

Proof. This follows from the definition of P. □

Remark 6.1. 1. In view of Theorem 6.3, we may study the diverse features of the $\tau^w_\square$-solutions of problem (3.2), whether analytically or numerically, by equivalently studying the features of the $\tau^w_\square$-solutions of problem (6.3).

2. Notice that (6.3) is a differential inclusion of nonclassical type, since $P(t,x)(\eta,\xi)$ is, in general, not of the form $\tilde{P}(t, \langle \eta, x(t)\xi\rangle)$, $t \in I$, $\eta, \xi \in \mathcal{D}, x \in L^2_{\text{loc}}(I, \mathfrak{M}^{\tau^w_\square}, ds)_{\text{ac}}$.

3. In Ref. [17, page 2009, equations 4.1a & 4.1b], we considered the *quantum stochastic differential inclusion*:

$$\begin{aligned}dX(t) &\in E(X(t),t)d\Lambda(t) + F(X(t),t)dA(t) + G(X(t),t)dA^+(t) \\ &\qquad + H(X(t),t)dt, \\ X(t_0) &= x_0, \quad \text{for almost all } t \in [0,T],\end{aligned} \tag{6.4}$$

where Λ, A, and A^+ are the gauge, creation and annihilation operators of quantum field theory, while E, F, G and H are stochastic processes such that the maps $t \mapsto E(X(t),t)$, $t \mapsto F(X(t),t)$, $t \mapsto G(X(t),t)$, and $t \mapsto H(X(t),t)$, $t \in [0,T]$, are adapted processes whenever X is adapted. By transforming problem (6.4) into the equivalent form of problem (6.3) (*vide* [17, page 2014, Theorem 6.2]), we obtained results about the existence of solutions of problem (6.4) in the *weak* operator topology τ_w.

Notation 6.1. 1. If z is an $\mathfrak{M}^{\tau_\square^w}$-valued map on I such that $(\eta,\xi) \longmapsto \frac{d}{dt}\langle \eta, z(t)\xi\rangle$ is in sesq$(\mathcal{D})$ for $\eta,\xi \in \mathcal{D}\times\mathcal{D}$ and almost all $t \in I$, then $\check{z}(t)$ will denote the sesquilinear map $(\eta,\xi) \longmapsto \frac{d}{dt}\langle \eta, z(t)\xi\rangle$, $(\eta,\xi) \in \mathcal{D}\times\mathcal{D}$. Thus $\check{z}(t) \in$ sesq$(\mathcal{D})$ and $\check{z}(t)(\eta,\xi) = \frac{d}{dt}\langle \eta, z(t)\xi\rangle$, for $(\eta,\xi) \in \mathcal{D}\times\mathcal{D}$ and almost all $t \in I$.

2. If $q \in$ sesq$(\mathcal{D})$, we denote $|q(\eta,\xi)|$ by $\|q\|_{\eta\xi}$.

6.3 Fundamental hypotheses

We employ the hypotheses and notation listed in $(\mathbf{H}_1)$-$(\mathbf{H}_5)$, with the topology $\tau_\square$ interpreted as either τ_w or $\tau_{\sigma w}$ and $\alpha \in \Theta(\tau_w) = \mathcal{D}$ or $\alpha \in \Theta(\tau_{\sigma w}) = \mathcal{D}_\infty \times \mathcal{D}_\infty$.

Theorem 6.4. *Suppose that the hypotheses* (H_1)-(H_5) *hold and F, G, and H are continuous from $I \times L^2_{\mathrm{loc}}(I, \mathfrak{M}^{\tau_\square^w}, ds)$ to* $(\mathrm{clos}(\mathfrak{M}^{\tau_\square^w}), \tau_\square^H)$. *Then there exists a solution φ of problem (6.3) such that*

$$\|\varphi(t) - y(t)\|_\alpha \leq \Xi_\alpha(t), \quad t \in J$$

and

$$\|\check{\varphi}(t) - \check{y}(t)\|_\alpha \leq k_\alpha(t)\Xi_\alpha(t) + p_\alpha(t),$$

for almost all $t \in J$ and all $\alpha \in \Theta(\tau_\square)$.

Proof. We first give the proof in the *weak topology* τ_s. In this case, $F \in L^2_{\mathrm{loc}}(I \times \mathfrak{M}^{\tau_w}, dM)_{\mathrm{mvs}}$, $G \in L^2_{\mathrm{loc}}(I \times \mathfrak{M}^{\tau_w}, dM^+)_{\mathrm{mvs}}$ and $H \in L^2_{\mathrm{loc}}(I \times \mathfrak{M}^{\tau_w}, ds)_{\mathrm{mvs}}$.

Let y be as in (H_1) and $\eta,\xi \in \Theta(\tau_w) = \mathcal{D}$. Since $\frac{d}{dt}\langle \eta, y(t)\xi\rangle$ may not be in $P(t,y)(\eta,\xi)$, almost all $t \in I$, y is not necessarily a τ_w-solution of problem (6.3).

Define φ_0 as y. Then φ_0 is adapted. By [3, Theorem 1.14.2], there is a measurable selection $t \mapsto v_0(t)(\eta,\xi)$ in $t \mapsto P(t,\varphi_0)(\eta,\xi)$, $t \in I$, $\eta,\xi \in \mathcal{D}$ such that

$$\|v_0(t) - \check{\varphi}_0(t)\|_{\eta\xi} = d_{\eta\xi}(\check{\varphi}_0(t), P(t,\varphi_0)),$$

almost all $t \in I$. By (H_1), the right-hand side is majorized by $p_{\eta\xi}(t)$. As the map $(\eta,\xi) \mapsto v_0(t)(\eta,\xi)$ is linear for almost all $t \in J$, there is a $v_0(t) \in \mathfrak{M}^{\tau_w}$ such that $v_0(t)(\eta,\xi) = \langle \eta, v_0(t)\xi\rangle$, for arbitrary $\eta,\xi \in \mathcal{D}$ and almost all $t \in J$.

Since $t \mapsto v_0(t)(\eta, \xi)$ is locally absolutely integrable, it determines a map φ_1 through the definition

$$\langle \eta, \varphi_1(t)\xi \rangle = \langle \eta, x_0\xi \rangle + \int_{t_0}^{t} ds \, \langle \eta, v_0(t)\xi \rangle, \quad \text{almost all } t \in J, \ \eta, \xi \in \mathcal{D}.$$

Since $v_0(t)$ is in $\mathfrak{M}^{\tau_w}$ for almost all $t \in J$, it follows that $\varphi_1(t)$ is affiliated to $\mathfrak{M}_t$, i.e. φ_1 is adapted. Moreover, for $t \in J$,

$$\begin{aligned} \|\varphi_1(t) - \varphi_0(t)\|_{\eta\xi} &\leq \|x_0 - \varphi_0(t_0)\|_{\eta\xi} + \int_{t_0}^{t} ds \, \|v_0(s) - \breve{\varphi}_0(s)\|_{\eta\xi} \\ &\leq \delta_{\eta\xi} + \int_{t_0}^{t} ds \, p_{\eta\xi}(s), \end{aligned}$$

by (H_1) and (H_4).

We claim that there exists, indeed, a sequence $\{\varphi_j\}_{j\geq 0}$ of τ_w-absolutely continuous maps from I to $\mathfrak{M}^{\tau_w}$ satisfying (i) and (ii) of Proposition 5.4, and hence its conclusion. To prove this, assume that $\{\varphi_j\}_{0\leq j\leq n}$ has already been defined and satisfies (i) and (ii) of Proposition 5.4. By [3, Theorem 1.14.2], there exists $v_n(\cdot)(\eta, \xi) \in P(\cdot, \varphi_n)(\eta, \xi)$ such that

$$\|v_n(t) - \breve{\varphi}_n(t)\|_{\eta\xi} = d_{\eta\xi}(\breve{\varphi}_n(t), P(t, \varphi_n)), \quad \text{almost all } t \text{ on } J,$$

where $v_n(t)$ is the linear map given by $v_n(t)(\eta, \xi) = \langle \eta, v_n(t)\xi \rangle, \ \eta, \xi \in \mathcal{D}$, almost all $t \in J$. Now define φ_{n+1} by

$$\langle \eta, \varphi_{n+1}(t)\xi \rangle = \langle \eta, x_o\xi \rangle + \int_{t_0}^{t} ds \, \langle \eta, v_n(t)\xi \rangle, \quad \text{almost all } t \in J. \tag{6.5}$$

Then

$$\begin{aligned} \|\breve{\varphi}_{n+1}(t) - \breve{\varphi}_n(t)\|_{\eta\xi} &= \|v_n(t) - v_{n-1}(t)\|_{\eta\xi} \\ &\leq \rho_{\eta\xi}(P(t, \varphi_n), P(t, \varphi_{n-1})) \\ &\leq k_{\eta\xi}(t) \, \|\varphi_n(t) - \varphi_{n-1}(t)\|_{\alpha(\eta,\xi)}, \quad \text{for some } \alpha(\eta, \xi) \in \mathcal{D} \times \mathcal{D}, \\ &\qquad \text{since } P \text{ is Lipschitzian} \\ &\leq k_{\eta\xi}(t) b_{\alpha(\eta,\xi),n-1}, \ \eta, \xi \in \mathcal{D}, \text{ and almost all } t \in J, \ n \geq 1. \end{aligned}$$

Moreover,

$$\begin{aligned} \|\varphi_{n+1}(t) - \varphi_0(t)\|_{\eta\xi} &\leq \|\varphi_1(t) - \varphi_0(t)\|_{\eta\xi} + \cdots + \|\varphi_{n+1}(t) - \varphi_n(t)\|_{\eta\xi} \\ &\leq \sum_{k=0}^{n} b_{\eta\xi,k}(t) \leq \Xi_{\eta\xi}(t) \leq \gamma. \end{aligned}$$

It follows from Proposition 5.4 that $\{\varphi_n\}_{n\geq 0}$ is τ_w-Cauchy and converges uniformly to $\varphi(t)$ in $\mathfrak{M}^{\tau_w}$. Also from Proposition 5.4, $\{v_n\}_{n\geq 0}$ is τ_w-Cauchy, whence $\{v_n\}_{n\geq 0}$ converges pointwise, for almost all $t\in J$, to a map $v\in L^1_{\text{loc}}(I,\mathfrak{M}^{\tau_w},ds)$. From (6.5), we get

$$\langle \eta, \varphi(t)\xi\rangle = \langle \eta, x_o\xi\rangle + \int_{t_0}^t ds\, \langle \eta, v(t)\xi\rangle, \quad \forall\, \eta,\xi\in\mathcal{D}, \text{ and almost all } t\in J.$$

As P is continuous on $I\times L^2_{\text{loc}}(I,\mathfrak{M}^{\tau_w},ds)_{\text{ac}}$ and has closed values, its graph is closed. Hence, as $\langle \eta, v_i(t)\xi\rangle \in P(t,\varphi_i)(\eta,\xi)$ for arbitrary $\eta,\xi\in\mathcal{D}$ and almost all $t\in J$, it follows that $\langle \eta, v(t)\xi\rangle\in P(t,\varphi)(\eta,\xi)$, for $\eta,\xi\in\mathcal{D}$ and almost all $t\in J$, whence

$$\frac{d}{dt}\langle \eta, \varphi(t)\xi\rangle \in P(t,\varphi)(\eta,\xi), \quad \forall\, \eta,\xi\in\mathcal{D}, \text{ and almost all } t\in J,$$

showing that φ is a τ_w-solution of problem (6.3).

σ-weak topology $\tau_{\sigma w}$

In this case, $F\in L^2_{\text{loc}}(I\times\mathfrak{M}^{\tau_{\sigma w}},dM)_{\text{mvs}}$, $G\in L^2_{\text{loc}}(I\times\mathfrak{M}^{\tau_{\sigma w}},dM^+)_{\text{mvs}}$ and $H\in L^2_{\text{loc}}(I\times\mathfrak{M}^{\tau_{\sigma w}},ds)_{\text{mvs}}$. Let $\boldsymbol{\eta}=\{\eta_n\}_{n=1}^\infty$, $\boldsymbol{\xi}=\{\xi_n\}_{n=1}^\infty$ be arbitrary members of $\mathcal{D}_\infty$. As above, there exists a sequence $\{\varphi_n\}_{n\geq 0}$ of $\tau_{\sigma w}$-absolutely continuous maps from I to $\mathfrak{M}^{\tau_{\sigma w}}$ which satisfy the inequalities

$$\begin{aligned}\|\varphi_1(t)-\varphi_0(t)\|_{\boldsymbol{\eta\xi}} &\leq \delta_{\boldsymbol{\eta\xi}} + \int_{t_0}^t ds\, p_{\boldsymbol{\eta\xi}}(s), \text{ with } \delta_{\boldsymbol{\eta\xi}} = \|x_0-\varphi(t_0)\|_{\boldsymbol{\eta\xi}}\\ &\qquad\qquad \text{and } p_{\boldsymbol{\eta\xi}} \text{ as in } (\mathrm{H})_1,\end{aligned}$$

$$\begin{aligned}\|\varphi_{n+1}(t)-\varphi_0(t)\|_{\boldsymbol{\eta\xi}} &\leq \sum_{k=0}^n b_{\boldsymbol{\eta\xi},k}(t)\leq \Xi_{\boldsymbol{\eta\xi}}(t)\leq\gamma\\ \|\breve{\varphi}_{n+1}(t)-\breve{\varphi}_n(t)\|_{\boldsymbol{\eta\xi}} &\leq k_{\boldsymbol{\eta\xi}}(t) b_{\alpha(\boldsymbol{\eta},\boldsymbol{\xi}),n-1},\end{aligned}$$

for some $\alpha(\boldsymbol{\eta},\boldsymbol{\xi})\in\Theta(\tau_{\sigma w})=\mathcal{D}_\infty\times\mathcal{D}_\infty$ depending on $(\boldsymbol{\eta},\boldsymbol{\xi})\in\Theta(\tau_{\sigma w})=\mathcal{D}_\infty\times\mathcal{D}_\infty$, and almost all $t\in J$, where we have invoked the Lipschitzian character of P. We are then able to conclude, in the same way as in the case of the *weak topology* above, that the $\tau_{\sigma w}$-limit $\varphi(t)$ of $\{\varphi_n(t)\}_{n\geq 0}$ is a $\tau_{\sigma w}$-solution of problem (6.3), i.e. that φ satisfies

$$\frac{d}{dt}\langle \eta, \varphi(t)\xi\rangle \in P(t,\varphi)(\eta,\xi), \ \forall\, \eta,\xi\in\mathcal{D} \text{ and almost all } t\in J.$$

This concludes the proof. □

References

[1] J.-P. Antoine, A. Inoue and C. Trapani, *Partial *-Algebras and their Operator Realizations*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2002.

[2] R. P. Agarwal and D. O'Regan (editors), *Set-valued mappings with applications in nonlinear analysis*, Taylor and Francis, London, New York, 2002.

[3] J.-P. Aubin and A. Cellina, *Differential inclusions*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984.

[4] J. P. Aubin and H Frankowska, *Set-Valued Analysis*, Birkhauser, Boston, 1990.

[5] R. J. Aumann, Integrals of set valued functions, Journal of Mathematical Analysis and Applications, **12(3)** (1965), 1-12.

[6] G. O. S. Ekhaguere, *Topological Solutions of Noncommutative Stochastic Differential Equations*, Stochastic Analysis and Applications **25** (2007), 961-993.

[7] A. F. Filippov, *Differential equations with discontinuous righthand side*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1988.

[8] M. D. P. Monteiro Marques, *Differential inclusions in nonsmooth mechanical problems: shocks and dry friction*, Birkhauser Verlag, Basel, Boston, Berlin, 1993.

[9] D. Repovs and P. V. Semenov, *Continuous selections of multivalued mappings*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1998.

[10] G. V. Smirnov, *Introduction to the theory of differential inclusions*, American Mathematical Society, Providence, Rhode Island, 2001.

[11] A. Tolstonogov, *Differential inclusions in a Banach space*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.

[12] L. Accardi, A. Fagnola and J. Quegebeur, *A representation free quantum stochastic calculus*, Journal of Functional Analysis **104** (1992), 149–197.

[13] S. Attal and J. M. Lindsay, *Quantum stochastic calculus with maximal operator domains*, Annals of Probability **32** (2004), 488–529.

[14] C. Barnett, R. F. Streater and I. F. Wilde, Quasifree quantum stochastic integrals for the CAR and CCR, Journal of Functional Analysis **52** (1983), 19–47.

[15] K. Bichteler, *Stochastic integration with jumps*, Cambridge University Press, Cambridge, 2002.

[16] G. O. S. Ekhaguere, *On noncommutative stochastic integration II*, Journal of the Nigerian Mathematical Society **6** (1987), 13–32.

[17] G. O. S. Ekhaguere, *Lipschitzian quantum stochastic differential inclusions*, International Journal of Theoretical Physics **31** (1992), 2003-2027.

[18] G. O. S. Ekhaguere, *Quantum stochastic differential inclusions of hypermaximal monotone type*, International Journal of Theoretical Physics **34** (1995), 323–353.

[19] G. O. S. Ekhaguere, *Quantum stochastic evolutions*, International Journal of Theoretical Physics **35** (1996), 1909–1946.

[20] R. Hudson and K. R. Parthasarathy, *Quantum Ito's formula and stochastic evolutions*, Communications in Mathematical Physics **93** (1984), 301–323.

[21] K. Ito, *Stochastic integral*, Proceedings of the Imperial Academy of Tokyo **20** (1944), 519–524.

[22] P. E. Kloeden and E. Platen, *The numerical solution of stochastic differential equations*, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1992.

[23] P. E. Kloeden, E. Platen and H. Schurz, *The numerical solution of stochastic differential equations through computer experiments*, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1994.

[24] K. R. Parthasarathy, *An introduction to quantum stochastic calculus*, Birkhauser Verlag, Bessel, 1992.

[25] P. Protter, Stochastic integration and differential equations: a new approach, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, Hong Kong, 1992.

[26] D. Revux and M. Yor, *Continuous martingales and Brownian motion*, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, Hong Kong, 1991.

[27] L. C. G. Rogers and D. Williams, *Diffusions, markov processes and martingales, Vol 2: Ito Calculus*, John Wiley, Chichester, New York, Brisbane, Toronto, Singapore, 1987.

[28] D. W. Stroock and S. R. S. Varadhan, *Multidimensional diffusion processes*, Springer-Verlag, Berlin, Heidelberg, New York, 1979.

[29] N. Ikeda and S. Watanabe, Stochastic differential equations and diffusion processes, North-Holland Publishing Company, Amsterdam, Oxford, New York, 1989.