Inequalities for Fractional Integrals Related to η-Convex Functions

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Abstract

In this paper we have established generalized Hermite-Hadamard's inequalities for fractional integrals. Also some interesting integral inequalities related to η-convex functions are investigated. The obtained results have as particular cases those previously obtained for convex functions.

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1 Introduction and preliminaries

Let I be an interval in real line $\mathbb{R}$. Consider $\eta : A \times A \to B$ for appropriate $A, B \subseteq \mathbb{R}$.

Definition 1.1 ([3]). A function $f : I \to \mathbb{R}$ is called convex with respect to η (briefly η-convex), if

$$f\big(tx + (1-t)y\big) \leq f(y) + t\eta\big(f(x), f(y)\big), \tag{1.1}$$

for all $x, y \in I$ and $t \in [0, 1]$. In fact above definition geometrically says that if a function is η-convex on I, then its graph between any $x, y \in I$ is on or under the path starting from $(y, f(y))$ and ending at $\big(x, f(y) + \eta(f(x), f(y))\big)$. If $f(x)$ should be the end point of the path for every $x, y \in I$, then we have $\eta(x, y) = x - y$ and the function reduces to a convex one.

The following are two simple examples of η-convex functions.

Example 1.2. **a**. *Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} -x, & x \geq 0; \\ x, & x < 0. \end{cases}$$

and define a bifunction η as $\eta(x, y) = -x - y$, for all $x, y \in \mathbb{R}^- = (-\infty, 0]$. It is not hard to check that f is a η-convex function.

b. *Define the function $f : \mathbb{R}^+ = [0, +\infty) \to \mathbb{R}^+$ by*

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1; \\ 1, & x > 1. \end{cases}$$

and define the bifunction $\eta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *by*

$$\eta(x,y) = \begin{cases} x+y, & x \le y; \\ 2(x+y), & x > y. \end{cases}$$

Then f *is* η*-convex.*

Definition 1.3. Consider $f \in L^1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \ge 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \qquad x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \qquad x < b.$$

respectively. Here, $\Gamma(\alpha)$ is Gamma function and $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.

Lemma 1.4. *[4] Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a < b$*. If* $f' \in L[a,b]$*, then the following equality for fractional integrals holds:*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a+}(b) + J^{\alpha}_{b-}(a)] = \frac{b-a}{2} \int_0^1 [(1-t)^{\alpha} - t^{\alpha}] f'(ta + (1-t)b)dt \tag{1.2}$$

The following result is of importance [1, 2]:

Theorem 1.5. *Suppose that* $f : I \to \mathbb{R}$ *is an* η*-convex function and* η *is above bounded on* $f(I) \times f(I)$*. Then* f *satisfies a Lipschitz condition on any closed interval* $[a,b]$ *contained in the interior* I° *of* I*. Hence,* f *is absolutely continuous on* $[a,b]$ *and continuous on* I°*.*

As a consequence of Theorem 1.5, an η-convex function $f : [a,b] \to \mathbb{R}$ with respect to a bifunction η bounded from above on $f([a,b]) \times f([a,b])$, is integrable. For other results see [3].

The following theorem is a consequence of Theorem 1 of [5].

Theorem 1.6. *If* f_1 *and* f_2 *are positive increasing functions on* $[0,1]$ *.Then*

$$\int_0^1 f_1(x)dx \int_0^1 f_2(x)dx \le \int_0^1 f_1(x) f_2(x)dx.$$

Also if f_1 and f_2 are positive decreasing functions on $[0,1]$ and K is an upper bound for f_1 and f_2, then $K - f_1$ and $K - f_2$ are positive increasing functions and we have

$$\int_0^1 (K - f_1(x))dx \int_0^1 (K - f_2(x))dx \le \int_0^1 (K - f_1(x))(K - f_2(x))dx,$$

which gives again

$$\int_0^1 f_1(x)dx \int_0^1 f_2(x)dx \le \int_0^1 f_1(x) f_2(x)dx.$$

2 Main Results

In this section we give some integral inequalities related to η-convex functions specially Hermite-Hadamard type inequalities.

Theorem 2.1. *Suppose that $f : [a, b] \to \mathbb{R}$ is a positive η-convex function and η is bounded from above on $f(I) \times f(I)$. Then the following inequalities for fractional integrals holds.*

$$\begin{aligned}
&f\Big(\frac{a+b}{2}\Big) - \frac{M_\eta}{2} \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\Big\{J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)\Big\} \leq \\
&\min\Big\{f(b) + \frac{\eta(f(a), f(b))}{2}, f(a) + \frac{\eta(f(b), f(a))}{2}\Big\} \leq \\
&\Big\{\frac{f(a)+f(b)}{2} + \frac{\eta(f(b), f(a)) + \eta(f(a), f(b))}{4}\Big\} \leq \\
&\frac{f(a)+f(b)}{2} + \frac{M_\eta}{2},
\end{aligned}$$

where M_η is upper bound of η.

Proof. Since f is η-convex for $x, y \in [a, b]$ and $t = \frac{1}{2}$ we have

$$f\left(\frac{x+y}{2}\right) \leq f(y) + \frac{1}{2}\eta(f(x), f(y))$$

and

$$f\left(\frac{x+y}{2}\right) \leq f(x) + \frac{1}{2}\eta(f(y), f(x)).$$

Put $x := ta + (1-t)b$ and $y := (1-t)a + tb$,

$$f\left(\frac{a+b}{2}\right) \leq f((1-t)a + tb) + \frac{1}{2}\eta(f(ta + (1-t)b), f((1-t)a + tb))$$

and

$$f\left(\frac{a+b}{2}\right) \leq f(ta + (1-t)b) + \frac{1}{2}\eta(f((1-t)a + tb), f(ta + (1-t)b)).$$

Multiplying both sides of this inequalities by $2t^{\alpha-1}$ and then integrating the resulting inequality with respect to t over $[0, 1]$, we obtain

$$\begin{aligned}
&\frac{2}{\alpha}f\left(\frac{a+b}{2}\right) \leq \\
&2\int_0^1 t^{\alpha-1}f((1-t)a + tb)dt + \int_0^1 t^{\alpha-1}\eta(f(ta + (1-t)b), f((1-t)a + tb))dt \leq \\
&2\int_a^b \left(\frac{u-a}{b-a}\right)^{\alpha-1} f(u)\frac{du}{b-a} + \int_0^1 t^{\alpha-1}M_\eta dt = \\
&2\frac{\Gamma(\alpha)}{(b-a)^\alpha}J^\alpha_{b-}f(a) + \frac{M_\eta}{\alpha}.
\end{aligned}$$

So we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}J^\alpha_{b-}f(a) + \frac{M_\eta}{2},$$

and

$$
\begin{aligned}
&\frac{2}{\alpha} f\left(\frac{a+b}{2}\right) \leq \\
&2\int_0^1 t^{\alpha-1} f(ta+(1-t)b)dt + \int_0^1 t^{\alpha-1}\eta(f((1-t)a+tb), f(ta+(1-t)b)dt = \\
&2\int_a^b \left(\frac{b-u}{b-a}\right)^{\alpha-1} f(u)\frac{du}{b-a} + \int_0^1 t^{\alpha-1} M_\eta dt = \\
&2\frac{\Gamma(\alpha)}{(b-a)^\alpha} J_{a+}^{\alpha} f(b) + \frac{M_\eta}{\alpha},
\end{aligned}
$$

i.e

$$
f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} J_{a+}^{\alpha} f(b) + \frac{M_\eta}{2}.
$$

Therefore

$$
f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} \min\left\{J_{a+}^{\alpha} f(b), J_{b-}^{\alpha} f(a)\right\} + \frac{M_\eta}{2}
$$

or

$$
f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left\{J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right\} + \frac{M_\eta}{2}.
$$

The first inequality is proved. For the second inequality we first note that if f is a η-convex function, then for $t \in [0,1]$,

$$
f(ta+(1-t)b) \leq f(b) + t\eta(f(a), f(b))
$$

and

$$
f((1-t)a+tb) \leq f(b) + (1-t)\eta(f(a), f(b)).
$$

By adding this inequalities we have

$$
\begin{aligned}
f(ta+(1-t)b) + f((1-t)a+tb) &\leq f(b) + t\eta(f(a), f(b)) + f(b) + (1-t)\eta(f(a), f(b)) \\
&= 2f(b) + \eta(f(a), f(b)).
\end{aligned}
$$

Then multiplying both sides by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$
\int_0^1 t^{\alpha-1} f(ta+(1-t)b)dt + \int_0^1 t^{\alpha-1} f((1-t)a+tb)dt = \int_0^1 t^{\alpha-1}[2f(b) + \eta(f(a), f(b))]dt
$$

i.e

$$
\frac{\Gamma(\alpha)}{(b-a)^\alpha}[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)] \leq \frac{2f(b) + \eta(f(a), f(b))}{\alpha}
$$

or

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)]\leq f(b)+\frac{\eta(f(a),f(b))}{2}.$$

Similarly since $f(ta+(1-t)b)\leq f(a)+(1-t)\eta(f(b),f(a))$ and $f((1-t)a+tb)\leq f(a)+t\eta(f(b),f(a))$ one can show that:

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)]\leq f(a)+\frac{\eta(f(b),f(a))}{2}.$$

Therefore

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\Big\{J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)\Big\}\leq\\ &\min\Big\{f(b)+\frac{\eta(f(a),f(b))}{2},f(a)+\frac{\eta(f(b),f(a))}{2}\Big\}\\ &\leq\Big\{\frac{f(a)+f(b)}{2}+\frac{\eta(f(b),f(a))+\eta(f(a),f(b))}{4}\Big\}\\ &\leq\frac{f(a)+f(b)}{2}+\frac{M_\eta}{2}.\end{aligned}$$

This completes the proof. □

Theorem 2.2. *If $f:[a,b]\to\mathbb{R}$ is a differentiable mapping on $[a,b]$ and $|f'|$ is η-convex then the following inequality for fractional integral holds:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\Big[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)\Big]\right|\leq\\ &\frac{(1-(\frac{1}{2})^{\alpha})(b-a)}{(\alpha+1)}\min\Big\{|f'(b)|+\frac{\eta(|f'(a)|,|f'(b)|)}{2},|f'(a)|+\frac{\eta(|f'(b)|,|f'(a)|)}{2}\Big\}\leq\\ &\frac{(1-(\frac{1}{2})^{\alpha})(b-a)}{(\alpha+1)}\Big\{\frac{|f'(a)|+|f'(b)|}{2}+\frac{\eta(|f'(a)|,|f'(b)|)+\eta(|f'(b)|,|f'(a)|)}{4}\Big\}.\end{aligned}$$

Proof. We use Lemma 1.4 and η-convexity of $|f'|$ to obtain

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)]\right|=\left|\frac{b-a}{2}\int_0^1[(1-t)^{\alpha}-t^{\alpha}]f'(ta+(1-t)b)dt\right|\leq\\ &\frac{b-a}{2}\int_0^1\big|(1-t)^{\alpha}-t^{\alpha}\big|\big|f'(ta+(1-t)b)\big|dt\leq\\ &\frac{b-a}{2}\int_0^1\big|(1-t)^{\alpha}-t^{\alpha}\big|\Big[\big|f'(b)\big|+t\eta\big(\big|f'(a)\big|,\big|f'(b)\big|\big)\Big]dt=\\ &\frac{b-a}{2}\Big\{\int_0^{\frac{1}{2}}\big[(1-t)^{\alpha}-t^{\alpha}\big]\big[|f'(b)|+t\eta(|f'(a)|,|f'(b)|)\big]dt+\int_{\frac{1}{2}}^1\big[t^{\alpha}-(1-t)^{\alpha}\big]\big[|f'(b)|+t\eta(|f'(a)|,|f'(b)|)\big]dt\Big\}.\end{aligned}$$

Since we have:

$$\int_0^{\frac{1}{2}}\big[(1-t)^{\alpha}-t^{\alpha}\big]\big[|f'(b)|+t\eta(|f'(a)|,|f'(b)|)\big]dt=\frac{(1-(\frac{1}{2})^{\alpha})|f'(b)|}{(\alpha+1)}+\frac{[\eta(|f'(a)|,|f'(b)|)](1-(\alpha+2)(\frac{1}{2})^{\alpha+1})}{(\alpha+1)(\alpha+2)},$$

and

$$\int_{\frac{1}{2}}^1\big[t^{\alpha}-(1-t)^{\alpha}\big]\big[|f'(b)|+t\eta(|f'(a)|,|f'(b)|)\big]dt=\frac{(1-(\frac{1}{2})^{\alpha})|f'(b)|}{(\alpha+1)}+\frac{[\eta(|f'(a)|,|f'(b)|)]((\alpha+1)-(\alpha+2)(\frac{1}{2})^{\alpha+1})}{(\alpha+1)(\alpha+2)}.$$

We can write

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)\right]\right|\leq\frac{(1-(\frac{1}{2})^{\alpha})(b-a)|f^{'}(b)|}{(\alpha+1)}+\frac{(b-a)[\eta(|f^{'}(a)|,|f^{'}(b)|)](1-(\frac{1}{2})^{\alpha})}{2(\alpha+1)}$$

or

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)\right]\right|\leq\frac{(1-(\frac{1}{2})^{\alpha})(b-a)}{(\alpha+1)}\Big\{|f^{'}(b)|+\frac{\eta(|f^{'}(a)|,|f^{'}(b)|)}{2}\Big\}.$$

With the same argument we can show that

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)]\right|\leq\frac{(1-(\frac{1}{2})^{\alpha})(b-a)}{(\alpha+1)}\Big\{|f^{'}(a)|+\frac{\eta(|f^{'}(b)|,|f^{'}(a)|)}{2}\Big\}.$$

So

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)\right]\right|\leq\\&\frac{(1-(\frac{1}{2})^{\alpha})(b-a)}{(\alpha+1)}\min\Big\{|f^{'}(b)|+\frac{\eta(|f^{'}(a)|,|f^{'}(b)|)}{2},|f^{'}(a)|+\frac{\eta(|f^{'}(b)|,|f^{'}(a)|)}{2}\Big\}\leq\\&\frac{(1-(\frac{1}{2})^{\alpha})(b-a)}{(\alpha+1)}\Big\{\frac{|f^{'}(a)|+|f^{'}(b)|}{2}+\frac{\eta(|f^{'}(a)|,|f^{'}(b)|)+\eta(|f^{'}(b)|,|f^{'}(a)|)}{4}\Big\}.\end{aligned}$$

□

Theorem 2.3. *Suppose that $f:[a,b]\to\mathbb{R}$ is an increasing positive η-convex function , η is bounded from above on $f(I)\times f(I)$ and $\alpha>1$. Then the following inequalities for fractional integrals holds:*

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}J^{\alpha}_{a^+}f(b)+\frac{1}{2(b-a)}\int_a^b f(a+b-x)dx\leq\\&\min\Big\{f(b)+\frac{\eta(f(a),f(b))}{2},f(a)+\frac{\eta(f(b),f(a))}{2}\Big\}\leq\\&\frac{f(a)+f(b)}{2}+\frac{\eta(f(b),f(a))+\eta(f(a),f(b))}{4}\leq\\&\frac{f(a)+f(b)}{2}+\frac{M_{\eta}}{2},\end{aligned}$$

where M_{η} is upper bound of η.

Proof. For the second inequality we first note that if f is a η-convex function then for $t\in[0,1]$

$$f(ta+(1-t)b)\leq f(b)+t\eta(f(a),f(b))$$

and

$$f((1-t)a+tb)\leq f(b)+(1-t)\eta(f(a),f(b)).$$

By adding this inequalities we have

$$f(ta+(1-t)b)+f((1-t)a+tb)\leq f(b)+t\eta(f(a),f(b))+f(b)+(1-t)\eta(f(a),f(b))$$
$$=2f(b)+\eta(f(a),f(b)).$$

Then multiplying both sides by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$ along with the fact that the functions $t^{\alpha-1}$ for $\alpha>1$ and $f((1-t)a+tb)$ are increasing (with respect to t) imply that

$$\int_0^1 t^{\alpha-1}dt\int_0^1 f((1-t)a+tb)dt\leq\int_0^1 t^{\alpha-1}f((1-t)a+tb)dt.$$

Therefore

$$\int_0^1 t^{\alpha-1}f(ta+(1-t)b)dt+\int_0^1 t^{\alpha-1}f((1-t)a+tb)dt\leq\int_0^1 t^{\alpha-1}[2f(b)+\eta(f(a),f(b))]dt,$$

$$\int_0^1 t^{\alpha-1}f(ta+(1-t)b)dt+\int_0^1 t^{\alpha-1}dt\int_0^1 f((1-t)a+tb)dt\leq\int_0^1 t^{\alpha-1}[2f(b)+\eta(f(a),f(b))]dt,$$
$$\frac{\Gamma(\alpha)}{(b-a)^\alpha}J_{a+}^\alpha f(b)+\frac{1}{\alpha(b-a)}\int_a^b f(a+b-x)dx\leq\frac{2f(b)+\eta(f(a),f(b))}{\alpha},$$

i.e

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}J_{a+}^\alpha f(b)+\frac{1}{2(b-a)}\int_a^b f(a+b-x)dx\leq f(b)+\frac{\eta(f(a),f(b))}{2}.$$

Also since we have:

$$f(ta+(1-t)b)+f((1-t)a+tb)\leq 2f(a)+\eta(f(b),f(a))$$

we can conclude that

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}J_{a+}^\alpha f(b)+\frac{1}{2(b-a)}\int_a^b f(a+b-x)dx\leq f(a)+\frac{\eta(f(b),f(a))}{2}.$$

Hence

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}J_{a+}^\alpha f(b)+\frac{1}{2(b-a)}\int_a^b f(a+b-x)dx\leq\\&\min\left\{f(b)+\frac{\eta(f(a),f(b))}{2},f(a)+\frac{\eta(f(b),f(a))}{2}\right\}\leq\\&\frac{f(a)+f(b)}{2}+\frac{\eta(f(b),f(a))+\eta(f(a),f(b))}{4}\leq\\&\frac{f(a)+f(b)}{2}+\frac{M_\eta}{2}.\end{aligned}$$

□

Corollary 2.4. *If the conditions of theorem 2.3 hold and $\eta(x,y)=x-y$ for all x,y, then*

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}J_{a+}^\alpha f(b)+\frac{1}{2(b-a)}\int_a^b f(a+b-x)dx\leq\frac{f(a)+f(b)}{2}.$$

Proof. Since

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}J^\alpha_{a+}f(b)+\frac{1}{2(b-a)}\int_a^b f(a+b-x)dx\le$$
$$\frac{f(a)+f(b)}{2}+\frac{\eta(f(b),f(a))+\eta(f(a),f(b))}{4},$$

and

$$\frac{\eta(f(b),f(a))+\eta(f(a),f(b))}{4}=0,$$

then the inequality holds.

□

The proof of the next theorem and corollary is similar to the previous one.

Theorem 2.5. *Suppose that $f:[a,b]\to\mathbb{R}$ is an increasing positive η-convex function , η is bounded from above on $f(I)\times f(I)$ and $0<\alpha<1$. Then the following inequalities for fractional integrals holds.*

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}J^\alpha_{b-}f(a)+\frac{1}{2(b-a)}\int_a^b f(x)dx\le$$
$$\min\left\{f(b)+\frac{\eta(f(a),f(b))}{2},f(a)+\frac{\eta(f(b),f(a))}{2}\right\}\le$$
$$\frac{f(a)+f(b)}{2}+\frac{\eta(f(b),f(a))+\eta(f(a),f(b))}{4}\le$$
$$\frac{f(a)+f(b)}{2}+\frac{M_\eta}{2},$$

where M_η is upper bound of η.

Corollary 2.6. *If the conditions of Theorem 2.5 hold and $\eta(x,y)=x-y$ for all x,y then*

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}J^\alpha_{b-}f(a)+\frac{1}{2(b-a)}\int_a^b f(x)dx\le\frac{f(b)+f(a)}{2}.$$

Theorem 2.7. *Let $f:I\longrightarrow(0,+\infty)$ be a η-convex function $a,b\in I$ with $a<b$. Then*

$$\frac{\Gamma(\alpha+3)}{(b-a)^\alpha}J^b_{a+}f(a)f(b)\le \tag{2.1}$$
$$(\alpha+1)(\alpha+2)f(b)\big(f(b)+\eta(f(a),f(b))\big)+\frac{\alpha(\alpha+1)}{2}(\eta(f(a),f(b)))^2+(\eta(f(a),f(b)))^2.$$

Proof. Since we have:

$$f(ta+(1-t)b)f((1-t)a+tb)\le\frac{f^2(ta+(1-t)b)+f^2((1-t)a+tb)}{2},$$

we can write

$$\int_0^1 t^{\alpha-1}f(ta+(1-t)b)f((1-t)a+tb)dt\le$$
$$\int_0^1\frac{t^{\alpha-1}f^2(ta+(1-t)b)+t^{\alpha-1}f^2((1-t)a+tb)}{2}dt=$$
$$\frac{1}{2}\Big\{\int_0^1 t^{\alpha-1}f^2(ta+(1-t)b)+\int_0^1 t^{\alpha-1}f^2((1-t)a+tb)dt\Big\}.$$

Now let $u := ta + (1-t)b$ then $(1-t)a + tb = a + b - u$ and $dt = -\frac{du}{b-a}$. Therefore

$$\int_a^b (\frac{b-u}{b-a})^{\alpha-1} f(u)f(a+b-u)du \le \frac{1}{2}\Big\{ \int_0^1 t^{\alpha-1} f^2(ta+(1-t)b) + \int_0^1 t^{\alpha-1} f^2((1-t)a+tb)dt \Big\}.$$

On the other hand, since f is η-convex

$$\begin{cases} f^2(ta+(1-t)b) \le \big\{f(b) + t\eta(f(a), f(b))\big\}^2 = f^2(b) + 2tf(b)\eta(f(a), f(b)) + t^2(\eta(f(a), f(b)))^2 \\ f^2((1-t)a+tb) \le \big\{f(b) + (1-t)\eta(f(a), f(b))\big\}^2 = f^2(b) + 2(1-t)f(b)\eta(f(a), f(b)) + (1-t)^2(\eta(f(a), f(b)))^2. \end{cases}$$

So we get

$$\begin{aligned} &\int_a^b \Big(\frac{b-u}{b-a}\Big)^{\alpha-1} f(u)f(a+b-u)du \le \\ &\frac{1}{2}\Big\{ \int_0^1 2f^2(b)t^{\alpha-1}dt + \int_0^1 2f(b)\eta(f(a), f(b))t^{\alpha}dt + \int_0^1 (\eta(f(a), f(b)))^2 t^{\alpha+1}dt + \\ &\int_0^1 2(1-t)f(b)\eta(f(a), f(b))t^{\alpha-1}dt + \int_0^1 (1-t)^2(\eta(f(a), f(b)))^2 t^{\alpha-1}dt \Big\} = \\ &\frac{1}{2}\Big\{ \frac{2}{\alpha}f^2(b) + \frac{2}{\alpha}f(b)\eta(f(a), f(b)) + \frac{1}{\alpha+2}(\eta(f(a), f(b)))^2 + \frac{2}{\alpha(\alpha+1)(\alpha+2)}(\eta(f(a), f(b)))^2 \Big\} = \\ &\frac{1}{\alpha}f^2(b) + \frac{1}{\alpha}f(b)\eta(f(a), f(b)) + \frac{1}{2(\alpha+2)}(\eta(f(a), f(b)))^2 + \frac{1}{\alpha(\alpha+1)(\alpha+2)}(\eta(f(a), f(b)))^2. \end{aligned}$$

Also

$$\int_a^b \Big(\frac{b-u}{b-a}\Big)^{\alpha-1} f(u)f(a+b-u)du = \frac{\Gamma(\alpha)}{(b-a)^{\alpha}} J_{a^+}^b f(a)f(b).$$

It follows that

$$\frac{\Gamma(\alpha)}{(b-a)^{\alpha}} J_{a^+}^b f(a)f(b) \le \frac{1}{\alpha}f^2(b) + \frac{1}{\alpha}f(b)\eta(f(a), f(b)) + \frac{1}{2(\alpha+2)}(\eta(f(a), f(b)))^2 + \frac{1}{\alpha(\alpha+1)(\alpha+2)}(\eta(f(a), f(b)))^2,$$

or

$$\frac{\Gamma(\alpha+3)}{(b-a)^{\alpha}} J_{a^+}^b f(a)f(b) \le (\alpha+1)(\alpha+2)f^2(b) + (\alpha+1)(\alpha+2)f(b)\eta(f(a), f(b)) + \frac{\alpha(\alpha+1)}{2}(\eta(f(a), f(b)))^2 + (\eta(f(a), f(b)))^2,$$

or

$$\frac{\Gamma(\alpha+3)}{(b-a)^{\alpha}} J_{a^+}^b f(a)f(b) \le (\alpha+1)(\alpha+2)f(b)\Big(f(b) + \eta(f(a), f(b))\Big) + \frac{\alpha(\alpha+1)}{2}(\eta(f(a), f(b)))^2 + (\eta(f(a), f(b)))^2.$$

□

Corollary 2.8. *In Theorem 2.7, if $\eta(x, y) = x - y$ for all x, y then*

$$\frac{\Gamma(\alpha+3)}{(b-a)^{\alpha}} J_{a^+}^b f(a)f(b) \le (\alpha+1)(\alpha+2)f(b)f(a) + \frac{\alpha(\alpha+1)}{2}(f(a) - f(b))^2 + (f(a) - f(b))^2.$$

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