Dissociation Number and Vertex 3-Path Cover Number of the Join of Graphs

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Abstract

Let $G = (V, E)$ be a graph and k be a positive integer. A subset S of V is a vertex k-path cover of G if every path of order k in G contains a vertex from S. The minimum cardinality of a vertex k-path cover of G is called the vertex k-path cover number of G, denoted by $\psi_k(G)$.

A set S of vertices of G is called a dissociation set of G if it induces a subgraph with maximum degree at most 1. The maximum cardinality of a dissociation set in G is called the dissociation number of G, denoted by $diss(G)$.

In this study, we gave the dissociation number and the vertex 3-path cover number of the join of graphs.

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1 Introduction

A vertex cover of a graph is a set of vertices such that each edge is incident to at least one vertex of the set. Brešar et al. [4] introduced a new graph

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invariant that somehow generalized this concept. The new concept is called the vertex k-path cover. The concept was motivated by a problem that sought to secure communications in wireless sensor networks. The concept vertex k-path cover will allow us to find the minimum number of protected sensors (for cost effectiveness), and how they should be placed in the network.

A couple of substantial results have been given for $\psi_3(G)$. Brešar et al. [4] gave some upperbounds for $\psi_3(G)$. Taranenko et al. [3] gave some lowerbounds and upperbounds for $\psi_3(G)$, and gave the exact value of $\psi_3(G)$ for planar grids. Jakovac et al. [7] gave the exact value of $\psi_3(G)$ for the lexicographic product of graphs.

The vertex 3-path cover number of a graph may be determined using its dissociation number. As defined in [10], a set of vertices of a graph is called a dissociation set if it induces a subgraph with maximum degree at most 1. The maximum cardinality of a dissociation set in a graph G is called the dissociation number of G and is denoted by $diss(G)$.

The dissociation number was also studied in [1], [2], [5], [6], [9] and [8].

The path $P_n = (v_1, v_2, \ldots, v_n)$ is the graph with distinct vertices v_1, v_2, $\ldots$, v_n and edges v_1v_2, v_2v_3, $\ldots$, $v_{n-1}v_n$. The *cycle* $C_n = [v_1, v_2, \ldots, v_n]$, $n \geq 3$, is the graph with vertices v_1, v_2, $\ldots$, v_n and edges v_1v_2, v_2v_3, $\ldots$, $v_{n-1}v_n$, v_nv_1. A *complete* graph of order n, denoted by K_n, is the graph in which every pair of distinct vertices are adjacent.

Let X and Y be sets. The *disjoint union* of X and Y, denoted by $X \dot\cup Y$, is found by combining the elements of X and Y, treating all elements to be distinct. Thus, $|X \dot\cup Y| = |X| + |Y|$. The *join* of two graphs G and H, denoted by $G+H$, is the graph with vertex-set $V(G+H) = V(G) \dot\cup V(H)$ and edge-set $E(G+H) = E(G) \dot\cup E(H) \dot\cup \{uv : u \in V(G), v \in V(H)\}$.

2 Preliminary Results

This section validates the dissociation number of paths and cycles, and characterized graphs with dissociation number equal to its order. Remark 2.1 follows immediately from the definition of dissociation set.

Remark 2.1 *Let G be a graph. Then S be a dissociation set in G if and only if $N_S(u) \leq 1$ for all $u \in S$.*

To see this, assume that S is a dissociation set of G, and there exists $u \in S$ such that $N_S(u) > 1$. Let $v, w \in N_S(u)$. Then $uv, uw \in E(\langle S \rangle)$. Since $v, w \in S$, $deg_{\langle S \rangle}(u) > 1$. This is a contradiction since S is a dissociation set.

Conversely, assume that $N_S(u) \leq 1$ for all $u \in S$. Let $v \in S$. Then either $N_S(v) = 0$ or $N_S(v) = 1$. If $N_S(v) = 0$, then $deg_{\langle S \rangle}(v) = 0$. On the other hand, if $N_S(v) = 1$, then $deg_{\langle S\rangle}(v) = 1$. Hence, $deg_{\langle S\rangle}(w) \leq 1$ for all $w \in S$. This shows that S is a dissociation set.

The next remark validates the dissociation number of paths.

Remark 2.2 *Let P_n be a path of order $n \geq 2$. Then $diss(P_n) = \lceil 2n/3 \rceil$.*

To see this, let $P_n = (1, 2, \ldots, n)$ be a path of order n and let

$$S = \begin{cases} \{1, 2, 4, 5, 7, 8, \ldots, n-2, n-1\}, & \text{if } n \equiv 0(mod\, 3) \\ \{1, 2, 4, 5, 7, 8, \ldots, n-3, n-2, n\}, & \text{if } n \equiv 1(mod\, 3) \\ \{1, 2, 4, 5, 7, 8, \ldots, n-1, n\}, & \text{if } n \equiv 2(mod\, 3). \end{cases}$$

Then by Remark 2.1, S is a dissociation set. Note that $|S| = \lceil 2n/3 \rceil$. Hence, $diss(P_n) \geq \lceil 2n/3 \rceil$. Suppose that $diss(P_n) > \lceil 2n/3 \rceil$. Let S be a dissociation set with $|S| = diss(P_n)$. Then by the *Pigeonhole Principle*, S contains 3 vetices with consecutive names, say *wlog* $1, 2, 3$. This is a contradiction since S be a dissociation set and $deg_{\langle S\rangle}(2) = 2$.

This shows the Remark.

The next remark validates the dissociation number of cycles.

Remark 2.3 *Let C_n be a cycle of order $n \geq 3$. Then $diss(C_n) = \lfloor 2n/3 \rfloor$.*

Proof: Let $C_n = (1, 2, \ldots, n)$ be a cycle of order n and let

$$S = \begin{cases} \{1, 2, 4, 5, 7, 8, \ldots, n-2, n-1\}, & \text{if } n \equiv 0(mod\, 3) \\ \{1, 2, 4, 5, 7, 8, \ldots, n-3, n-2\}, & \text{if } n \equiv 1(mod\, 3) \\ \{1, 2, 4, 5, 7, 8, \ldots, n-4, n-3, n-1\}, & \text{if } n \equiv 2(mod\, 3). \end{cases}$$

Then by Remark 2.1, S is a dissociation set. Note that $|S| = \lfloor 2n/3 \rfloor$. Hence, $diss(C_n) \geq \lfloor 2n/3 \rfloor$. Suppose that $diss(C_n) \geq \lfloor 2n/3 \rfloor$. Let S be a dissociation set with $|S| = diss(C_n)$. Then by the *Pigeonhole Principle*, S contains 3 vetices with consecutive names, say *wlog* $1, 2, 3$. This is a contradiction since S be a dissociation set and $deg_{\langle S\rangle}(2) = 2$.

This shows the Remark.

Theorem 2.4 characterizes graphs with dissociation number is equal to its order.

Theorem 2.4 *Let $G = (V, E)$ be a graph of order n. Then $diss(G) = n$ if and only if $G = \overline{K}_n$ or $G = G_1 \cup G_2 \cup \cdots \cup G_m$ where G_i is a component with $|V(G_i)| \leq 2$ for all $i = 1, 2, \ldots, m$.*

Proof: If $G = \overline{K}_n$, then V is a dissociation set in G and perhaps the largest. Hence $diss(G) = n$. So we assume that $G = G$. If $|V(G_i)| \leq 2$ for all

$i = 1, 2, \ldots, m$, then by Lemma 1, $\bigcup_{i=1}^{m} V(G_i) = V(G)$ is a dissociation. Hence $diss(G) = n$.

Conversely, assume that $G \neq \overline{K}_n$ and $G \neq G_1 \cup G_2 \cup \cdots \cup G_m$ where G_i is a component with $|V(G_i)| \leq 2$ for all $i = 1, 2, \ldots, m$. Then G has a component G_j with $|V(G_j)| > 2$. Let $u \in V(G_j)$ with $deg_{G_j}(u) > 1$ and let $v, w \in N(u)$. If S is a dissociating set in G, $\{u, v, w\} \notin S$, otherwise $deg_{\langle S \rangle}(u) > 1$. Hence, $diss(G) < n$. □

3 Dissociation Number of the Join of Graphs

This section gives the dissociation number of the join of graphs. Theorem 3.1 characterizes dissociation sets in the join of graphs.

Let G be a graph and $S \subseteq V(G)$. We denote by $\langle S \rangle_G$ the induced subgraph of S in G.

Theorem 3.1 *Let G and H be graphs. A subset S of $V(G+H)$ is a dissociation set of $G+H$ if and only if S satisfies either*

1. *$S \subseteq V(G)$ and S is a dissociation set in G, or*
2. *$S \subseteq V(H)$ and S is a dissociation set in H, or,*
3. *$S = S_1 \cup S_2$, where $S_1 \subseteq V(G)$, $S_2 \subseteq V(H)$ and $|S_1| = |S_2| = 1$.*

Proof: Assume that S is a dissociation set of $G+H$ and S is neither of type (1), (2), nor (3). If S is neither of type (1), (2), nor (3), then $S_1 = S \cap V(G) \neq \varnothing$, $S_2 = S \cap V(H) \neq \varnothing$ and, $|S_1| > 1$ or $|S_2| > 1$. Without loss of generality, suppose $|S_1| > 1$. Since $|S_1| > 1$ and $S_2 \neq \varnothing$, let $u, v \in S_1$ and $w \in S_2$. Then $\langle \{u, w, v\} \rangle$ is a subgraph of S. Thus, $deg_{\langle S \rangle}(w) > 1$. This is a contradiction.

Conversely, let $S \subseteq V(G+H)$ and assume that S is either of type (1), type (2), or type (3). If S is of type (1), then $\langle S \rangle_G = \langle S \rangle_{G+H}$. Similarly, if S is of type (2), then $\langle S \rangle_H = \langle S \rangle_{G+H}$. So for these cases, S is a dissociation set in $G+H$. If S is of type (3), then $\langle S \rangle \cong P_2$. Hence, for this case S is a dissociation set also. □

With Theorem 3.1 the next statement is clear.

Corollary 3.2 *Let G and H be graphs. Then*

$$diss(G+H) = max\{2, diss(G), diss(H)\}.$$

4 Dissociation Number of Complete Graphs

This section validates the dissociation number of complete graphs. Remark 4.1 characterizes graphs with dissociation number equal to 1 and Theorem 4.2 characterizes graphs with dissociation number equal to 2.

Remark 4.1 *Let G be a graph. Then $diss(G) = 1$ if and only if $G = K_1$.*

Proof: Assume that $diss(G) = 1$ and $G \neq K_1$. If $G \neq K_1$, the $|V(G)| > 1$. Let $u, v \in V(G)$ and $S = \{u, v\}$. Then either $\langle S\rangle = K_2$ or $\langle S\rangle = \overline{K}_2$. This implies that $deg_{\langle S\rangle}(u) \leq 1$ and $deg_{\langle S\rangle}(v) \leq 1$. Hence, S is a dissociation set in G, that is, $diss(G) \geq 2$. This is a contradiction.

The converse is clear. □

Theorem 4.2 characterizes graphs with dissociation number equal to 2.

Theorem 4.2 *Let G be a graph of order $n \geq 2$. Then $diss(G) = 2$ if and only if*

1. *$G \cong \overline{K}_2$, or*

2. *$G \cong I + J$ with $diss(I) \leq 2$ and $diss(J) \leq 2$.*

Proof: Assume that $diss(G) = 2$. If $G = \overline{K}_2$, then we are done. So we assume that $G \ncong \overline{K}_2$. Suppose it is not true that $G \cong I + J$ with $diss(I) \leq 2$ and $diss(J) \leq 2$. Consider the following cases: (Case 1. $G \cong I+J$ with $diss(I) > 2$ or $diss(J) > 2$) If $G \cong I+J$ and without loss of generality $diss(I) > 2$, then by Corollary 3.2, $diss(G) = max\,\{diss(I), diss(J)\} > 2$. This is a contradiction. (Case 2. $G \ncong I + J$ with $diss(I) \leq 2$ and $diss(J) \leq 2$) If $G \ncong I + J$ with $diss(I) \leq 2$ and $diss(J) \leq 2$, then consider the following subcases: (Subcase 1. $diss(I) = diss(J) = 1$) If $diss(I) = 1$, then by Remark 4.1 $I = K_1$. Similarly, if $diss(J) = 1$, then by Remark 4.1 $J = K_1$. Hence, $G = \overline{K}_2$. This is a contradiction. (Subcase 2. $diss(I) = 1$ and $diss(J) = 2$) If $diss(I) = 1$, then by Remark 4.1 $I = K_1$. Since $G \ncong I + J$, either $V(I)$ is an isolated vertex. Let S be a dissociation set of J with $|S| = 2$. Then $S \cup V(I)$ is a dissociation set of G. This implies that $diss(G) > 2$. This is a contradiction. (Subcase 3. $diss(I) = 2$ and $diss(J) = 2$) Let $S_1 = \{u_1, u_2\}$ be a dissociation set of I, and $S_2 = \{v_1, v_2\}$ be a dissociation set of J. Consider the following cases for this subcase: (Case 1. $u_1v_1, u_1v_2, u_2v_1, u_2v_2 \in E(G)$) If $u_1v_1, u_1v_2, u_2v_1, u_2v_2 \in E(G)$ and $G \ncong I + J$, then there exists $v_3 \in V(J)$ distinct from v_1 and v_2. Hence, $S_1 \cup \{v_3\}$ is a dissociation set. This implies that $diss(G) > 2$. This is a contradiction. (Case 2. At least one of u_1v_1, u_1v_2, u_2v_1, u_2v_2 is not an edge.) If At least one of u_1v_1, u_1v_2, u_2v_1, u_2v_2 is not an edge, then we consider the following subcases:

(Subcase 1. There exists $v_3 \in V(J)$ distinct from v_1 and v_2.) If there exists $v_3 \in V(J)$ distinct from v_1 and v_2, then $S_1 \cup \{v_3\}$ is a dissociation set. This implies that $diss(G) > 2$. This is a contradiction. (Subcase 2. $V(I) = S_1$ and $V(J) = S_2$) If $V(I) = S_1$ and $V(J) = S_2$, then $G \cong \overline{K}_2 + P_2$. This implies that $diss(G) > 2$. This is a contradiction.

Conversely, if $G \cong \overline{K}_2$, then clearly $diss(G) = 2$. On the other hand, if $G \cong I + J$ with $diss(I) \leq 2$ and $diss(J) \leq 2$, then by Corollary 3.2 $diss(G) = max\{2, diss(I), diss(J)\} = 2$. $\square$

The ideas from Corollary 3.2 and Remark 4.1 may be used to validate the next remark.

Remark 4.3 *Let $n \in \mathbb{N}$ with $n \geq 2$ and K_n be a complete graph of order n. Then $diss(K_n) = 2$.*

To see this we use induction. For $n = 2$, we have by Corollary 3.2 and Remark 4.1

$$diss(K_2) = dis(K_1 + K_1) = max\{2, diss(K_1)\} = 2.$$

Hence the assertion holds for $n = 2$. Let $m \geq 2$ and assume that the assertion holds for m, that is, $diss(K_m) = 2$. If $diss(K_m) = 2$, then by Corollary 3.2 and Remark 4.1 we have,

$$diss(K_{m+1}) = dis(K_1 + K_m) = max\{2, diss(K_1), diss(K_m)\} = 2.$$

Hence, the assertion holds for $m + 1$ also. Therefore, by the Principle of Mathematical Induction the assertion follows.

5 Vertex 3-Path Cover Number of the Join of Graphs

This section gives the vertex 3-path cover number of the join of graphs. Corollary 5.2 gives the vertex 3-path cover number of the join of two arbitrary graphs.

The next remark is found in [3] and [7]. We use Remark 5.1 to find the vertex 3-path cover number of the join of graphs.

Remark 5.1 *Let $G = (V, E)$ be a graph. Then $\psi_3(G) = |V| - diss(G)$.*

Corollary 5.2 *Let G and H be graphs of order n and m, respectively, with $n > 1$ or $m > 1$. Then $\psi_3(G + H) = m + n - max\{2, diss(G), diss(H)\}$.*

Proof: We have by Corollary 3.2 $diss(G+H) = max\{2, diss(G), diss(H)\}$. Therefore by Remark 5.1

$$\psi_3(K_n) = n + m - diss(G+H) = max\{2, diss(G), diss(H)\}.$$

□

The next statement, Corollary 5.3, gives the vertex 3-path cover number of the join of two paths, the join of two cycles, and the join of a path and a cycle.

Corollary 5.3 *Let P_n and P_m be paths of order n and m, respectively, and C_p and C_q be cycles of order p and q, respectively. Then*

1. $\psi_3(P_n + P_m) = m + n - max\{2, \lceil 2n/3 \rceil\}$ $(n \geq m)$
2. $\psi_3(P_n + C_q) = n + q - max\{2, \lceil 2n/3 \rceil, \lfloor 2q/3 \rfloor\}$
3. $\psi_3(C_p + C_q) = p + q - max\{2, \lfloor 2p/3 \rfloor\}$ $(p \geq q)$.

The next statement, Corollary 5.4, gives the vertex 3-path cover number of fans and wheels. We note that a fan F_n is isomorphic to $K_1 + P_n$ and a wheel W_n is isomorphic to $K_1 + C_n$.

Corollary 5.4 *Let F_n be a fan of order $n+1$ and W_m be a wheel of order $m+1$. Then*

1. $\psi_3(F_n) = 1 + n - max\{2, \lceil 2n/3 \rceil\}$
2. $\psi_3(W_m) = 1 + m - max\{2, \lfloor 2m/3 \rfloor\}$.

The next statement, Corollary 5.5, gives the vertex 3-path cover number of generalized fans and generalized wheels. We also note that a generalized fan $F_{m,n}$ is isomorphic to $\overline{K}_m + P_n$ and a generalized wheel $W_{m,n}$ is isomorphic to $\overline{K}_m + C_n$.

Corollary 5.5 *Let $F_{m,n}$ be a generalized fan of order $m+n$ and $W_{p,q}$ be a generalized wheel of order $p+q$. Then*

1. $\psi_3(F_{m,n}) = m + n - max\{2, m, \lceil 2n/3 \rceil\}$
2. $\psi_3(W_{p,q}) = p + q - max\{2, p, \lfloor 2q/3 \rfloor\}$.

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