DCOMPOSITION OF A VECTOR MEASURE

ABALO **DOUHADJI**[1*] AND YAOVI **AWUSSI**[2]

[1] Departement of mathematics, University of Lom, PObox 1515, Lom,Togo.
douhadjiabalo@gmail.com
[2] Departement of mathematics, University of Lome, Togo.
Jawussi@yahoo.fr
[*] Correspondence:douhadjiabalo@gmail.com

Abstract. in this paper we show the vector form of the decomposition of a measure . We consider a bounded vector measure m on $K(G;E)$ and we prove that it decomposes into two measures, one of which is absolutely continuous with respect to the Haar measure and the other foreign to the Haar measure.

Keywords: vector measure, Haar measure, Singular measures, absolute continuity.

1. Introduction

In this article we prove and extend in vector measure case a theorem due to Radon-Nikodym in the single case of positive measure. First we prove a theorem which allow us to get a vector measure at each we have a complex measure and then we get the decomposition of the vector measure.

2. Preliminary notes

Definition 2.1 (**A vector measure**)**.** *Let G be a locally compact group and*
$K(G;E)$ the space of continuous functions with compact support on G in E
We call a vector measure on G with respect to the Banach spaces E and F;a linear map:

$$\begin{aligned} m: \quad K(G;E) &\to F \\ f &\mapsto m(f) \end{aligned}$$

such as $\forall K$ compact of $G \quad \exists a_K > 0, \|m(f)\|_F \leq a_K \|f\|_\infty$.

where $\|.\|_F$ designates the norm on the Banach space F and
$\|f\|_\infty = sup\{\|f(t)\|_E, t \in K\}$,the norm on $K(G;E)$

Definition 2.2 (**Absolute continuity**)**.** *Let E be a set and $\mathfrak{B}$ a sigma algebra of subsets of E.*
m a bounded complex or vector measure and μ a positive measure on $\mathfrak{B}$

We say that m is absolutely continue with respect to μ if

$$\forall A \in \mathfrak{B} \quad such\, as\, \mu(A) = 0 \quad then \quad m(A) = 0 \tag{1}$$

On note: $m \ll \mu$

Definition 2.3 (**Singular measures**). *:*
Two measures ν *and* μ *are foreign (or singular)if there exists a partition* (E_1, E_2) *of* E *such as* $|\mu|(E_1) = 0$ *and* $|\nu|(E_2) = 0$.
we note : $\mu \perp \nu$

Theorem 2.4 (**Lebesgue Radon Nikodym**). *:*
Let μ *be a positive measure; any real or complex measure* ν *can be uniquely written in the form :* $\nu = \nu_a + \nu_s$ *where* ν_a *is positive and absolutely continuous with respect to* μ *and* ν_s *singular; positive and foreign to* μ

Theorem 2.5. *:*
E *and* F *two Banach spaces,* G *a locally compact group*
Let $\nu \in K(G, E)$ *be a complex measure* σ*-finite,* $w \in F$
The mapping

$$\begin{aligned} m : K(G,E) &\to F \\ f &\mapsto w\nu(f) \end{aligned}$$

is a vector measure.

Proof.
Show before that $w\nu(f) \in F$.
$\nu(f) \in \mathbb{C}$ is a scalar and $w \in F$ a vector so $w\nu(f) \in F$.
then
m is linear because ν is.
ν being a complex Radon measure we have $\forall\, K\, compact\, of\, G \quad \exists a_k$ such as $|\nu(f)| \leqslant a_k \|f\|_\infty$ we have:

$$\begin{aligned} \|m(f)\|_F &= \|w\nu(f)\|_F \\ &\leqslant \|w\|_F |\nu(f)| \\ &\leqslant \|w\|_F \times a_k \|f\|_\infty \\ &\leqslant M_k \|f\|_\infty \quad with \quad M_K = \|w\| a_k \end{aligned}$$

hence m is continuous and therefore m is a vector measure.

□

3. Main result

Theorem 3.1. *:*
Any vector measure m *on* $K(G, E)$ *decomposes uniquely into*

$$m = m_a + m_s \quad with \quad m_a \ll \mu\, and\, m_s \perp \mu$$

Proof.
The uniqueness
Suppose there exists m'_a et m'_s such as $m = m'_a + m'_s = m_a + m_s$ with $m_a \ll \mu\, and\, m_s \perp \mu$

on one hand and $m'_a \ll \mu \, and \, m'_s \perp \mu$ on the second .
which equals $m'_a - m_a = m_s - m'_s$
Like $m_s \perp \mu$ then there exists $G_1 \subset G$ such as $\mu(G_1) = 0$ and $m_s(\bar{G_1}) = 0$
Like $m'_s \perp \mu$ then there exists $G_2 \subset G$ such as $\mu(G_2) = 0$ and $m'_s(\bar{G_2}) = 0$
$(m_s - m'_s)(\bar{G_1} \cap \bar{G_2}) = 0$
$\mu(G_1 \cup G_2) = \mu(G_1) + \mu(G_2) = 0$
Since $m_a \ll \mu$ and $m'_a \ll \mu$ then $m_a(G_1 \cup G_2) = 0$ and $m'_a(G_1 \cup G_2) = 0$

$$\begin{aligned} (m'_a - m_a)(G_1 \cup G_2) &= m'_a(G_1 \cup G_2) - m_a(G_1 \cup G_2) \\ &= 0 \quad \text{hence} \\ m'_a - m_a &= 0 \Rightarrow m'_a = m_a \end{aligned}$$

We got $m'_a - m_a \ll \mu$ so $m_s - m'_s = 0$ on $G \Rightarrow m_s = m'_s$. hence the uniqueness
<u>Existence</u>
Let ν be a complex measure on $K(G; E)$, according to the theorem 1.1;
there exists $\nu_a \ll \mu \quad and \, \nu_s \perp \mu$ such as $\nu = \nu_a + \nu_s$.

$$\forall f \in K(G, E) \quad \forall w \neq 0 \in F, w\nu(f) = w\nu_a(f) + w\nu_s(f)$$

$$\begin{aligned} \nu_a \ll \mu \Leftrightarrow \mu(A) = 0 &\Rightarrow \nu(A) = 0 \quad \forall A \subset G. \\ &\Rightarrow w\nu(A) = 0 \quad \forall w \neq 0 \\ &so \quad w\nu_a \ll \mu \end{aligned}$$

$$\begin{aligned} \nu_s \perp \mu \Leftrightarrow \mu(A) = 0 \qquad &and \quad \nu_s(\bar{A}) = 0 \quad \forall A \subset G \\ &and \quad w\nu_s(\bar{A}) = 0 \quad \forall w \neq 0 \\ &so \qquad w\nu_s \perp \mu \end{aligned}$$

according to the theorem 1.2; $w\nu_a(f)$ and $w\nu_s(f)$ are the vectors measures belonging to F Banach space from which we put

$$m_a(f) = w\nu_a(f)$$
$$m_s(f) = w\nu_s(f)$$

thus giving $m(f) = m_a(f) + m_s(f)$ and consequently $m = m_a + m_s$ □

References

[1] Y.AWUSSI: *On the absolute continuity of vector measure*, theoretical mathematics and aplications, vol.3,no.4.2013,41-45
[2] Y.MENSAN: *Facts about the Fourier-Stieljes transform of vector measures on compact groups*, international journal of analysis and applications, 2(1),19-25 (2013)
[3] A.DOUHADJI et Y.AWUSSI: *Absolute continuity of vector measure on compact Lie group* submitted
[4] Y.MENSAN: $S_p(\sigma, A)$ *des transforms de Fourier-Stieltjes des mesures vectorielles dfinies sur un groupe compact*, thse de doctorat, 2008, p: 13-14
[5] J.GENET: *Measure and intgration*, course and exercises; vuibert library 1976
[6] W.RUDIN: *Real and complex analysis*, course; Masson 1980
[7] N.DINCULEANU:*Integration on locally spaces*, Noorhoff International Publishing, Leyden 1974.