A characterization of three-dimensional real hypersurfaces in non-flat complex space forms in terms of their generalized Tanaka-Webster Lie derivative

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Abstract

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1 Introduction

A *complex space form* is an n-dimensional Kähler manifold of constant holomorphic sectional curvature c. A complete and simply connected complex space form is analytically isometric to a complex projective space $\mathbb{C}P^n$ if $c > 0$, a complex Euclidean space $\mathbb{C}^n$ if $c = 0$, or a complex hyperbolic space $\mathbb{C}H^n$ if $c < 0$. Furthermore, the complex projective and complex hyperbolic spaces are called non-flat complex space forms and the symbol $M_n(c), n \geq 2$, is used to denote them when it is not necessary to distinguish them.

Let M be a *connected real hypersurface* of $M_n(c)$ without boundary. Let ∇ be the Levi-Civita connection on M and J the complex structure of $M_n(c)$. Take a locally defined unit normal vector field N on M and denote by $\xi = -JN$. This is a tangent vector field to M called the *structure vector field* on M . If it is an eigenvector of the shape operator A of M the real hypersurface is called *Hopf hypersurface* and the corresponding eigenvalue is $\alpha = g(A\xi, \xi)$. Moreover, the complex structure J induces on M an almost contact metric structure (ϕ, ξ, η, g), where ϕ is the tangential component of J and η is an one-form given by $\eta(X) = g(X, \xi)$ for any X tangent to M .

The classification of homogeneous real hypersurfaces in $\mathbb{C}P^n, n \geq 2$ was obtained by Takagi and they were divided into six type of real hypersurfaces (see [14], [15], [16]). Among them the three dimensional real hypersurfaces in $\mathbb{C}P^2$ are geodesic hyperspheres of radius r, $0 < r < \frac{\pi}{2}$, which are called real hypersurfaces of type (A) and tubes of radius r, $0 < r < \frac{\pi}{4}$, over the complex quadric, which are called real hypersurfaces of type (B). All of them are Hopf ones with constant principal curvatures (see [6]). In case of $\mathbb{C}H^n$, the study of real hypersurfaces with constant principal curvatures, was initiated by Montiel in [8] and completed by Berndt in [1]. In this case the three dimensional real hypersurfaces in $\mathbb{C}H^2$

are either a horosphere in $\mathbb{C}H^2$, or a geodesic hypersphere or a tube over a totally geodesic complex hyperbolic hyperplane $\mathbb{C}H^1$. These are known as real hypersurfaces of type (A). Furthermore, there exist tubes of radius $r > 0$ over totally real hyperbolic space $\mathbb{R}H^2$, known as real hypersurfaces of type (B). All of them are homogeneous and Hopf.

The *Jacobi operator* R_X of a Riemannian manifold $\tilde{M}$ with respect to a unit vector field X is given by $R_X = R(\cdot, X)X$, where R is the curvature tensor field on $\tilde{M}$. It is a self-adjoint endomorphism of the tangent space $T\tilde{M}$ and it is related to Jacobi vector fields, which are solutions of the second-order differential equation $\nabla_{\dot{\gamma}}(\nabla_{\dot{\gamma}}Y) + R(Y, \dot{\gamma})\dot{\gamma} = 0$ along a geodesic γ in $\tilde{M}$ (known as the Jacobi equation). In case of real hypersurfaces in $M_n(c)$ the Jacobi operator with respect to the structure vector field ξ, R_ξ, which is called the *structure Jacobi operator* on M and it plays an important role int he study of them.

Apart from the Levi-Civita connection on a non-degenerate, pseudo-Hermitian CR-manifold a canonical affine connection is defined and is called *Tanaka-Webster connection* (see [17], [19]). As a generalization of this connection, in [18] Tanno defined the *generalized Tanaka-Webster connection* for contact metric manifolds by

$$\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y.$$

Using the naturally extended affine connection of Tanno's generalized Tanaka-Webster connection, Cho defined the *k-th generalized Tanaka-Webster connection* $\hat{\nabla}^{(k)}$ on a real hypersurface M in $M_n(c)$ given by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{1.1}$$

for any vector fields X, Y tangent to M where k is a nonnull real number (see [2], [3]). Then the following relations hold

$$\hat{\nabla}^{(k)}\eta = 0, \quad \hat{\nabla}^{(k)}\xi = 0, \quad \hat{\nabla}^{(k)}g = 0, \quad \hat{\nabla}^{(k)}\phi = 0.$$

In particular, if the shape operator of a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, the generalized Tanaka-Webster connection coincides with the Tanaka-Webster connection.

Geometric conditions with respect to the k-th generalized Tanaka-Webster connection on [illegible] hypersurfaces has been studied. In [13] real hypersurfaces in $\mathbb{C}P^n, n \geq 3$, whose structure Jacobi operator satisfies relation $\mathcal{L}^{(k)}_\xi R_\xi = \mathcal{L}_\xi R_\xi$ are classified. Furthermore, the non-existence of real hypersurfaces in $\mathbb{C}P^n, n \geq 3$, whose structure Jacobi operator satisfies relation $\mathcal{L}^{(k)}_X R_\xi = \mathcal{L}_X R_\xi$, for any X orthogonal ξ is proved.

The aim of he present paper is to extend the previous results in case of three dimensional real hypersurfaces in $M_2(c)$. First, we study real hypersurfaces in $M_2(c)$ satisfies relation

$$\mathcal{L}^{(k)}_\xi R_\xi = \mathcal{L}_\xi R_\xi \tag{1.2}$$

and the following Theorem is obtained

Theorem 1.1 *Every real hypersurface in $M_2(c)$, whose structure Jacobi operator satisfies relation (1.2) is a Hopf hypersurface. Moreover, M is locally congruent*
either to a real hypersurface of type (A),
or to a Hopf hypersurface with $A\xi = 0$.

Next we study three dimensional real hypesurfaces in $M_2(c)$, whose structure Jacobi operator satisfies relation

$$\mathcal{L}_X^{(k)} R_\xi = \mathcal{L}_X R_\xi, \tag{1.3}$$

for any X orthogonal to ξ and the following Theorem is proved

Theorem 1.2 *There do not exist real hypersurface in $M_2(c)$, whose structure Jacobi operator satisfies relation (1.3).*

As an immediate consequence of the above Theorems we conclude that

Corollary 1.1 *There do not exist real hypersurfaces in $M_2(c)$ such that $\mathcal{L}_X R_\xi = \mathcal{L}_X^{(k)} R_\xi$, for any $X \in TM$.*

Finally, we remind that a tensor field T of type (1,1) is called *invariant* when the Lie derivative of it with respect to any vector fields X on M vanishes, i.e. $\mathcal{L}_X T = 0$. Moreover, it is called ξ*-invariant*, when the Lie derivative of it with respect to ξ is equal to zero, i.e. $\mathcal{L}_\xi T = 0$. Thus, as a consequence of Theorems 1.1 is concluded that

Corollary 1.2 *Every real hypersurface in $M_2(c)$ whose structure Jacobi operator is ξ-invariant with respect to the generalized Tanaka-Webster connection is a Hopf hypersurface. Moreover, M is locally congruent*
either to a real hypersurface of type (A),
or to a Hopf hypersurface with $A\xi = 0$.

2 Preliminaries

Throughout this paper all manifolds, vector fields etc are assumed to be of class C^∞ and all manifolds are assumed to be connected. Furthermore, the real hypersurfaces M are supposed to be without boundary. Thus, let M be a real hypersurface immersed in a non-flat complex space form $(M_n(c), G)$ with complex structure J of constant holomorphic sectional curvature c and N be a locally defined unit normal vector field on M and $\xi = -JN$ be the structure vector field of M . For a vector field X tangent to M relation

$$JX = \phi X + \eta(X)N$$

holds, where ϕX and $\eta(X)N$ are respectively the tangential and the normal component of JX. The Riemannian connections $\overline{\nabla}$ in $M_n(c)$ and ∇ in M are related for any vector fields X, Y on M by

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N,$$

where g is the Riemannian metric induced from the metric G.
The *shape operator* A of the real hypersurface M in $M_n(c)$ with respect to N is given by

$$\overline{\nabla}_X N = -AX.$$

The real hypersurface M has an almost contact metric structure (ϕ, ξ, η, g) induced from J of $M_n(c)$, where ϕ is the *structure tensor*, which is a tensor field of type (1,1) and η is an 1-form such that

$$g(\phi X, Y) = G(JX, Y), \qquad \eta(X) = g(X, \xi) = G(JX, N).$$

Moreover, the following relations hold

$$\phi^2 X = -X + \eta(X)\xi, \qquad \eta \circ \phi = 0, \qquad \phi\xi = 0, \qquad \eta(\xi) = 1,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \phi Y) = -g(\phi X, Y).$$

The fact that J is parallel implies $\overline{\nabla} J = 0$ and this leads to

$$\nabla_X \xi = \phi A X, \qquad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{2.1}$$

The ambient space $M_n(c)$ is of constant holomorphic sectional curvature c and this results in the Gauss and Codazzi equations are respectively given by

$$R(X, Y)Z = \frac{c}{4}[g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X \tag{2.2}$$
$$-g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z] + g(AY, Z)AX - g(AX, Z)AY,$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi],$$

where R denotes the Riemannian curvature tensor on M and X, Y, Z are any vector fields on M .

The tangent space $T_P M$ at every point $P \in M$ can be decomposed as

$$T_P M = span\{\xi\} \oplus \mathbb{D},$$

where $\mathbb{D} = \ker \eta = \{X \in T_P M : \eta(X) = 0\}$ and is called (*maximal*) *holomorphic distribution* (*if* $n \geq 3$). Due to the above decomposition the vector field $A\xi$ can be written

$$A\xi = \alpha\xi + \beta U,$$

where $\beta = |\phi\nabla_\xi \xi|$ and $U = -\frac{1}{\beta}\phi\nabla_\xi\xi \in \ker(\eta)$ is a unit vector field, provided that $\beta \neq 0$.

Next, the following results concern any non-Hopf real hypersurface M in $M_2(c)$ with local orthonormal basis $\{U, \phi U, \xi\}$ at a point P of M .

Lemma 2.1 *Let M be a non-Hopf real hypersurface in $M_2(c)$. The following relations hold on M*

$$\begin{aligned}
&AU = \gamma U + \delta\phi U + \beta\xi, \qquad A\phi U = \delta U + \mu\phi U, \qquad A\xi = \alpha\xi + \beta U \\
&\nabla_U \xi = -\delta U + \gamma\phi U, \qquad \nabla_{\phi U}\xi = -\mu U + \delta\phi U, \qquad \nabla_\xi \xi = \beta\phi U, \\
&\nabla_U U = \kappa_1\phi U + \delta\xi, \qquad \nabla_{\phi U} U = \kappa_2\phi U + \mu\xi, \qquad \nabla_\xi U = \kappa_3\phi U, \\
&\nabla_U \phi U = -\kappa_1 U - \gamma\xi, \; \nabla_{\phi U}\phi U = -\kappa_2 U - \delta\xi, \; \nabla_\xi \phi U = -\kappa_3 U - \beta\xi,
\end{aligned} \tag{2.3}$$

where $\alpha, \beta, \gamma, \delta, \mu, \kappa_1, \kappa_2, \kappa_3$ are smooth functions on M and $\beta \neq 0$.

Remark 2.1 *The proof of Lemma 2.1 is included in [12].*

The Codazzi equation for $X \in \{U, \phi U\}$ and $Y = \xi$ because of Lemma 2.1 implies the following relations

$$\begin{aligned} \xi\delta &= \alpha\gamma + \beta\kappa_1 + \delta^2 + \mu\kappa_3 + \frac{c}{4} - \gamma\mu - \gamma\kappa_3 - \beta^2 & (2.4) \\ \xi\mu &= \alpha\delta + \beta\kappa_2 - 2\delta\kappa_3 & (2.5) \\ (\phi U)\alpha &= \alpha\beta + \beta\kappa_3 - 3\beta\mu & (2.6) \\ (\phi U)\beta &= \alpha\gamma + \beta\kappa_1 + 2\delta^2 + \frac{c}{2} - 2\gamma\mu + \alpha\mu & (2.7) \end{aligned}$$

and for $X = U$ and $Y = \phi U$

$$U\delta - (\phi U)\gamma = \mu\kappa_1 - \kappa_1\gamma - \beta\gamma - 2\delta\kappa_2 - 2\beta\mu \tag{2.8}$$

Furthermore, combination of the Gauss equation (2.2) with the formula of Riemannian curvature $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, taking into account relations of Lemma 2.1, implies

$$U\kappa_2 - (\phi U)\kappa_1 = 2\delta^2 - 2\gamma\mu - \kappa_1^2 - \gamma\kappa_3 - \kappa_2^2 - \mu\kappa_3 - c, \tag{2.9}$$

Relation (2.2) implies that the structure Jacobi operator R_ξ is given by

$$R_\xi(X) = \frac{c}{4}[X - \eta(X)\xi] + \alpha AX - \eta(AX)A\xi, \tag{2.10}$$

for any vector field X tangent to M , where $\alpha = \eta(A\xi) = g(A\xi, \xi)$.

Moreover, the structure Jacobi operator for $X = U$, $X = \phi U$ and $X = \xi$ due to (2.3) is given by

$$R_\xi(U) = (\frac{c}{4} + \alpha\gamma - \beta^2)U + \alpha\delta\phi U, \quad R_\xi(\phi U) = \alpha\delta U + (\frac{c}{4} + \alpha\mu)\phi U \quad \text{and} \quad R_\xi(\xi) = 0. \tag{2.11}$$

The following Theorem which in case of $\mathbb{C}P^n$ is owed to Maeda [7] and in case of $\mathbb{C}H^n$ is owed to Montiel [8] (also Corollary 2.3 in [10]).

Theorem 2.1 *Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 2$. Then*
i) α is constant.
ii) If W is a vector field which belongs to $\mathbb{D}$ such that $AW = \lambda W$, then

$$(\lambda - \frac{\alpha}{2})A\phi W = (\frac{\lambda\alpha}{2} + \frac{c}{4})\phi W.$$

iii) If the vector field W satisfies $AW = \lambda W$ and $A\phi W = \nu\phi W$ then

$$\lambda\nu = \frac{\alpha}{2}(\lambda + \nu) + \frac{c}{4}. \tag{2.12}$$

Remark 2.2 *In case of three dimensional Hopf hypersurfaces we can always consider a local orthonormal basis* $\{W, \phi W, \xi\}$ *at some point* $P \in M$ *such that* $AW = \lambda W$ *and* $A\phi W = \nu\phi W$. *Thus, relation (2.12) is satisfied. Furthermore, the structure Jacobi operator of Hopf hypersurfaces, whose shape operator is given by the previous relations for* $X = W$ *and* $X = \phi W$ *is given by*

$$R_\xi(W) = (\frac{c}{4} + \alpha\lambda)W \quad \textit{and} \quad R_\xi(\phi W) = (\frac{c}{4} + \alpha\nu)\phi W. \tag{2.13}$$

We also mention the following Theorem, which plays an important role in the study of real hypersurfaces in $M_n(c)$, which is due to Okumura in case of $\mathbb{C}P^n$ (see [11]) and to Montiel and Romero in case of $\mathbb{C}H^n$ (see [9]). It provides the classification of real hypersurfaces in $M_n(c)$, $n \geq 2$, whose shape operator A commutes with the structure tensor field ϕ.

Theorem 2.2 *Let M be a real hypersurface of* $M_n(c)$, $n \geq 2$. *Then* $A\varphi = \varphi A$, *if and only if M is locally congruent to a homogeneous real hypersurface of type (A). More precisely:*
In case of $\mathbb{C}P^n$
(A_1) *a geodesic hypersphere of radius* r, *where* $0 < r < \frac{\pi}{2}$,
(A_2) *a tube of radius r over a totally geodesic* $\mathbb{C}P^k$, $(1 \leq k \leq n-2)$, *where* $0 < r < \frac{\pi}{2}$.
In case of $\mathbb{C}H^n$
(A_0) *a horosphere in* $\mathbb{C}H^n$, *i.e a Montiel tube,*
(A_1) *a geodesic hypersphere or a tube over a totally geodesic complex hyperbolic hyperplane* $\mathbb{C}H^{n-1}$,
(A_2) *a tube over a totally geodesic* $\mathbb{C}H^k$ $(1 \leq k \leq n-2)$.

Remark 2.3 *In case of three dimensional real hypersurfaces only real hypersurfaces of type* (A_1) *exist in* $\mathbb{C}P^2$ *and real hypersurfaces of type* (A_0) *and* (A_1) *exist in* $\mathbb{C}H^2$.

Finally, we mention the following Proposition (see [**?**]).

Proposition 2.1 *There do not exist real hypersurfaces in* $M_2(c)$, *whose structure Jacobi operator vanishes*

3 Proof of Theorem 1.1

Let M be a real hypersurface whose structure Jacobi operator satisfies relation (1.2). More analytically, the previous relation due to (1.1) for $X = \xi$ implies

$$k\phi R_\xi(Y) + R_\xi(\phi AY) - g(\phi A\xi, R_\xi(Y))\xi = \phi AR_\xi(Y) + \eta(Y)R_\xi(\phi A\xi) + kR_\xi(\phi Y) \tag{3.1}$$

We consider $\mathcal{N}$ the open subset of M such that

$$\mathcal{N} = \{P \in M : \beta \neq 0, \text{ in a neighborhood of } P.\}$$

On $\mathcal{N}$ the inner product of relation (3.1) for $Y = U$ with ξ due to the first of (2.11) yields

$$\alpha\delta \neq 0.$$

Suppose that $\alpha \neq 0$ then the above relation implies $\delta = 0$ and relations (2.3) and (2.11) become respectively

$$AU = \gamma U + \beta\xi, \quad A\phi U = \mu\phi U \quad \text{and} \quad A\xi = \alpha\xi + \beta U, \tag{3.2}$$
$$R_\xi(U) = (\frac{c}{4} + \alpha\gamma - \beta^2)U, \quad R_\xi(\phi U) = (\frac{c}{4} + \alpha\mu)\phi U \quad \text{and} \quad R_\xi(\xi) = 0. \tag{3.3}$$

The inner product of (3.1) for $Y = \phi U$ with ξ because of (3.2) and the second of (3.3) implies

$$R_\xi(\phi U) = 0 \Rightarrow \mu = -\frac{c}{4\alpha}.$$

Moreover, relation (3.1) for $Y = \phi U$ taking into account that $R_\xi(\phi U) = 0$ and the first of (3.3) results in

$$(\mu - k)R_\xi(U) = 0.$$

If $\mu \neq k$ then $R_\xi(U) = 0$. So the structure Jacobi operator vanishes identically, which is impossible because of Proposition 2.1 in Section 2.

Thus, $\mu = k$. Furthermore, the inner product of (3.1) for $Y = U$ with ϕU due to the first of (3.3) and $\mu = -\frac{c}{4\alpha}$ implies

$$(\gamma - k)R_\xi(U) = 0.$$

If $\gamma \neq k$ then $R_\xi(U) = 0$ and this results in the dact that the structure Jacobi operator vanishes identically, which is impossible due to Proposition 2.1.

So $\gamma = k$. Differentiation of the last relation with respect to ϕU yields $\phi U(\gamma) = 0$. Thus, relation (2.8) because of $\delta = 0$, $\mu = \gamma = k$ and $\mu = -\frac{c}{4\alpha}$ implies $c = 0$, which is a contradiction.

Therefore, on M we have $\alpha = 0$ and relation (2.11) becomes

$$R_\xi(U) = (\frac{c}{4} - \beta^2)U, \quad R_\xi(\phi U) = \frac{c}{4}\phi U \quad \text{and} \quad R_\xi(\xi) = 0. \tag{3.4}$$

The inner product of relation (3.1) for $Y = \phi U$ with ξ because of the second relation of (3.4) gives $c = 0$, which is a contradiction.

Thus, $\mathcal{N}$ is empty and the following Proposition is proved

Proposition 3.1 *Every real hypersurface in $M_2(c)$ whose structure Jacobi operator satisfies relation (1.2) is a Hopf hypersurface.*

Due to the above Proposition, relations in Theorem 2.1 and remark 2.2 hold. Relation (3.1) for $Y = W$ and $Y = \phi W$ because of (2.13) implies respectively

$$k\alpha(\lambda + \nu) = \lambda\alpha(\lambda - \nu) = 0 \quad \text{and} \quad k\alpha(\lambda + \nu) = -\nu\alpha(\lambda - \nu).$$

Combination of the last two relations results in

$$\alpha(\lambda + \nu)(\lambda - \nu) = 0.$$

If $\lambda + \nu = 0$ then relation (2.12) yields $\lambda^2 = -\frac{c}{4}$. This case occurs when the ambient space is $\mathbb{C}H^2$. Furthermore, the last relation leads to the conclusion that λ and ν are constant. In this case M is locally congruent to a real hypersurface of type (B). Substitution of the eigenvalues of such real hypersurfaces in relation $\lambda + \nu = 0$ leads to a contradiction.

Therefore, on M relation $\alpha(\lambda - \nu) = 0$. Thus, locally either $\alpha = 0$ or $\lambda = \nu$. If $\alpha = 0$ in case of $\mathbb{C}P^2$ we have two cases:
1) if $\lambda \neq \nu$ then M is locally congruent to a non-homegeneous real hypersurface considered as a tube of radius $r = \frac{\pi}{4}$ over a holomorphic curve,
2) if $\lambda = \nu$ then M is locally congruent to a geodesic hypersphere of radius $r = \frac{\pi}{4}$.

In case of $\mathbb{C}H^2$ if $\alpha = 0$, M is a Hopf hypersurface with $A\xi = 0$ (for the construction of such real hypersurfaces see [5]).

If $\alpha \neq 0$ then $\lambda = \nu$ and this implies

$$(A\varphi - \varphi A)X = 0$$

for any X tangent to M. So due to Theorem 2.2 M is locally congruent to a real hypersurface of type (A) and this completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

More analytically relation (1.3) because of (1.1) is written

$$g(\phi AX, R_\xi(Y))\xi + g(A\phi X, R_\xi(Y))\xi + \eta(Y)R_\xi(\phi AX) - k\eta(Y)R_\xi(\phi X) = 0, \tag{4.1}$$

where $X \in \mathbb{D}$ and $Y \in TM$.

First we prove the following Proposition

Proposition 4.1 *There do not exist Hopf hypersurfaces in $M_2(c)$ whose structure Jacobi operator satisfies relation (1.3).*

Proof: Let M be a Hopf hypersurface. Then we have $A\xi = \alpha\xi$, where α is constant. Relation (4.1) for $X = W$ and $Y = \xi$ because of (2.13) implies

$$(\lambda - k)R_\xi(\phi W) = 0.$$

Suppose that $\lambda \neq k$ then $R_\xi(\phi W) = 0$. The latter due to (2.13) yields $\frac{c}{4} + \alpha\nu = 0$. Relation (4.1) for $X = \phi W$ and $Y = W$ results in

$$(\nu + \lambda)(\frac{c}{4} + \alpha\lambda) = 0.$$

If $\nu + \lambda \neq 0$ then $\frac{c}{4} + \alpha\lambda = 0$ and this leads to $R_\xi(W) = 0$. From the last relation we conclude that the structure Jacobi operator vanishes identically. It is known because of Proposition that such real hypersurfaces do not exist.

So $\nu+\lambda=0$ and relation (2.12) because of the latter implies $\lambda^2=-\frac{c}{4}$. This case occurs when the ambient space is the complex hyperbolic space $\mathbb{C}H^2$. Furthermore, we conclude that the real hypersurface has three constant eigenvalues since $\frac{c}{4}+\alpha\nu=0$ and $\lambda^2=-\frac{c}{4}$. So it is locally congruent to a real hypersurface of type (B). Substitution of the eigenvalues in relation $\lambda+\nu=0$ leads to a contradiction.

Therefore, on M we have $\lambda=k$. Relation (4.1) for $X=W$ and $Y=\phi W$ because of (2.13) yields

$$(\lambda+\nu)(\frac{c}{4}+\alpha\nu)=0.$$

If $\lambda+\nu\neq 0$ then the last relation gives $\frac{c}{4}+\alpha\nu=0$. Thus, M has three dinstict eigenvalues constant and this implies that M is locally congruent to a real hypersurface of type (B). Substitution of the eigenvalues of sych real hypersurface in $\frac{c}{4}+\alpha\nu=0$ leads to a contradiction.

So on M we have $\lambda+\nu=0$ and relation (2.12) because of the latter implies $\lambda^2=-\frac{c}{4}$. This case occurs when the ambient space is the complex hyperbolic space $\mathbb{C}H^2$. Moreover, we conclude that the real hypersurface has three constant eigenvalues and so is locally congruent to a real hypersurface of type (B). Substitution of the eigenvalues in relation $\lambda^2=-\frac{c}{4}$ leads to a contradiction and this completes the proof of the Proposition. $\square$

Next we examine non-Hopf hypersurfaces, whose structure Jacobi satisfies relation (4.1). Since M is a non-Hopf hypersurface we have that $\beta\neq 0$ and relation (2.3) holds. Relation (4.1) for $X=U$ and $Y=\phi U$ implies

$$(\gamma+\mu)g(R_\xi(\phi U),\phi U)=0.$$

Suppose that $\gamma+\mu\neq 0$ then the above relation yields $g(R_\xi(\phi U),\phi U)=0$. Moreover, relation (4.1) for $X=\phi U$ and $Y=U$ gives $g(R_\xi(U),U)=0$. Furthermore, relation (4.1) for $X=U$ and $Y=\xi$ because of $g(R_\xi(\phi U,\phi U)=0$ and (2.11) results in $\alpha\delta=0$. Thus, we conclude that $R_\xi(U)=R_\xi(\phi U)=0$ and the structure Jacobi operator vanishes identically, which is impossible because of Proposition 2.1.

Therefore, on M we have $\gamma+\mu=0$ and the inner product of relation (4.1) for $X=U$ and $Y=\xi$ with U due to (2.11) gives

$$\delta(\beta^2-\alpha k-\frac{c}{4})=0.$$

If $\delta\neq 0$ then the above relation yields $\beta^2=\frac{c}{4}+\alpha k$. Moreover, the inner product of relation (4.1) for $X=\phi U$ and $Y=\xi$ with U because of (2.11) implies $\frac{c}{4}+\alpha k=0$. Substitutition of the latter in $\beta^2=\frac{c}{4}+\alpha k$ results in $\beta=0$, which is impossible.

So on M we have $\delta=0$ and since $\gamma=-\mu$ relation (2.11) becomes

$$R_\xi(U)=(\frac{c}{4}-\alpha\mu-\beta^2)U,\ \ R_\xi(\phi U)=(\frac{c}{4}+\alpha\mu)\phi U\ \text{ and }\ R_\xi(\xi)=0. \tag{4.2}$$

The inner product of relation (4.1) for $X = U$ and $Y = \xi$ with ϕU because of (4.2) yields

$$(k + \mu)(\frac{c}{4} + \alpha\mu) = 0.$$

If $k + \mu \neq 0$ then $\frac{c}{4} + \alpha\mu = 0$, which implies that $\alpha\mu \neq 0$. The inner product of relation (4.1) for $X = \phi U$ and $y = \xi$ bearing in mind all the previous relations yields

$$(k - \mu)(\frac{c}{2} - \beta^2) = 0.$$

If $\beta^2 \neq \frac{c}{2}$ then the above relation yields $k = \mu$ and $\gamma = -k$. Since $k = \mu$ we obtain $\xi\mu = 0$ and relation (2.6) implies $\kappa_2 = 0$. Furthermore, differantiation of the latter with respect to ϕU gives

$$(\phi U)\mu = (\phi U)\gamma = 0.$$

Furthermore, differenatiation of $\frac{c}{4} + \alpha\mu = 0$ with respect to ϕU because of the above relation and relation (2.6) gives $\kappa_3 = 3\mu - \alpha$. Since $(\phi U)\gamma = 0$ relation (2.8) implies $\kappa_1 = \frac{\beta}{2}$. So relation (2.4) bearing in mind all the previous relations gives $\frac{\beta^2}{2} = c + 7\mu^2$. Differentiatin of the last relation with respect to ϕU yields $(\phi U)\beta = 0$ and relation (2.7) implies $\frac{\beta^2}{2} + \frac{c}{2} + 2\mu^2 = 0$. Moreover, since $\kappa_1 = \frac{\beta}{2}$ we conclude that $(\phi U)\kappa_1 = 0$ and relation (2.9) due to $\gamma = -\mu$, $\kappa_1 = \frac{\beta^2}{2}$, $\kappa_3 = 3\mu - \alpha$ and $\kappa_2 = 0$ results in $\frac{\beta^2}{2} = 4\mu^2 - 2c$. Combination of the last one with $\frac{\beta^2}{2} = c + 7\mu^2$ implies $c = 3\mu^2$. Substitution of the latter in $\frac{\beta^2}{2} + 2\mu^2 + \frac{c}{2} = 0$ leads to a contradiction.

Therefore, we have $\beta^2 = \frac{c}{2}$. Differentiation of the latter with respect to ϕU because of relation (2.7) implies $\beta\kappa_1 + \frac{c}{2} + 2\mu^2 = 0$. Relation (2.4) due to the last one implies $2\mu\kappa_3 = \mu^2 + \frac{c}{2}$. Moreover, differentiating $\gamma = -\mu$ with respect to ϕU due to (2.8) implies $(\phi U)\mu = 2\mu\kappa_1 + \beta\mu$. Furthermore, relation $\frac{c}{4} + \alpha\mu = 0$ holds and differentiation of the latter with respect to (ϕU) and bearing in mind all the previous relations results in $2\mu^2 = c$. The last one gives $(\phi U)\mu = 0$. So we have that since $(\phi U)(\frac{c}{4} + \alpha\mu) = 0$ due to $\mu \neq 0$ and $(\phi U)\mu = 0$ we obtain $(\phi U)\alpha = 0$, which because of relation (2.6) yields $\kappa_3 = 3\mu - \alpha$. Substitution of the latter and $2\mu^2 = c$ in $2\mu\kappa_3 = \mu^2 + \frac{c}{2}$ implies $c = 0$, which is a contradiction.

Thus, on M we have $\mu + k = 0$. Summarizing on M the following relations hold

$$\delta = 0, \ \ \gamma + \mu = 0 \ \text{ and } \ \mu + k = 0.$$

The inner product of relation (4.1) for $X = U$ and $Y = \xi$ with ϕU and for $X = \phi U$ and $Y = \xi$ with U because of (4.2) and $k \neq 0$ implies

$$R_\xi(U) = R_\xi(\phi U) = 0.$$

The latter results in the fact that the structure Jacobi operator vanishes identically and because of Proposition 2.1 we conclude that such real hypersurfaces do not exist and this completes the proof of Theorem 1.2.

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