Q-symmetry and Conditional Q-symmetries for Drinfel'd–Sokolov–Wilson (DSW) system

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Abstract

We study in this paper the Q-symmetry and conditional Q-symmetries of Drinfel'd–Sokolov–Wilson (DSW) equations. The solutions which we obtain in this paper take the form of convergent power series with easily computable components.
Keywords: Q-symmetry, conditional Q-symmetries,Lie Symmetry, (DSW) equations

1 Introduction

The investigation of exact solutions of nonlinear wave equations plays an important role in the study of nonlinear physical phenomena.

Recently, many effective methods for obtaining exact solutions of nonlinear wave equations have been proposed, such as Bäcklund transformation method [1], homogeneous balance method [2], [?], bifurcation method [4], [?], the hyperbolic tangent function expansion

method [6], [?], the Exp-function method [8], the Jacobi elliptic function expansion method [9]-[11], Hirotas bilinear method [12] and others. In this paper, we are concerned with the classical Drinfel'd–Sokolov–Wilson equation

$$\begin{aligned} &u_t + p\ v\ v_x = 0, \\ &v_t + q\ u_{xxx} + r\ u\ v_x + s\ v\ u_x = 0, \end{aligned} \tag{1}$$

where p, q, r, s are some nonzero parameters.

Recently, DSWE and the coupled DSWE, a special case of the classical DSWE, have been studied by several authors [?]-[?]. In this study, we construct the Q-symmetry and conditional Q-symmetries of Drinfel'd–Sokolov–Wilson

(DSW) equations. The solutions procedure Q-symmetry and conditional Q-symmetries, by the help of symbolic computation of Matlab or Mathematica to simplify.

Non-trivial conditional symmetries of a PDE (partial differential equation) allows us to obtain in explicit form such solutions which cannot be found by using the symmetries of the whole set of solutions of the given PDE[?].

Moreover, conditional symmetries reduce the class of PDEs to systems of ODEs (ordinary differential equations). As a rule, the reduced equations one obtains from conditional symmetries and from Q-symmetry are significantly simpler than those found by reduction using symmetries of the full set of solutions.This allows us to construct exact solutions of the reduced equations.

2 Conditional Q-symmetries

The classical symmetry properties can be extended if one studies eqs.(??) and (1) together with the invariant surface of the symmetry generator as an over-determined system of partial differential equations[?].

That is, if one studies the Lie symmetry properties of the system (??), (1) and

$$\begin{aligned} &\eta_1(x,t,u,v) - \xi_1(x,t,u,v)\ u_x - \xi_2(x,t,u,v)\ u_t = 0, \\ &\eta_2(x,t,u,v) - \xi_1(x,t,u,v)\ v_x - \xi_2(x,t,u,v)\ v_t = 0, \end{aligned} \tag{2}$$

where (??) and (2) are the invariant surfaces corresponding to the Lie symmetry group generator

$$Z = \xi_1(x,t,u,v)\frac{\partial}{\partial x} + \xi_2(x,t,u,v)\frac{\partial}{\partial t} + \eta_1(x,t,u,v)\frac{\partial}{\partial u} + \eta_2(x,t,u,v)\frac{\partial}{\partial v}, \tag{3}$$

the invariance condition, leading to conditional Q-symmetries for (??) and(1) is given by

$$Z^{(3)}F \mid_{\{F^{(j)}=0,Q^{(k)}=0\}} = 0, \tag{4}$$

where

$$\begin{aligned} F_1 &= u_t + p\ v\ v_x, \\ F_2 &= v_t + q\ u_{xxx} + r\ u\ v_x + s\ v\ u_x, \\ Q_1 &= \eta_1(x,t,u,v) - \xi_1(x,t,u,v)\ u_x - \xi_2(x,t,u,v)\ u_t, \\ Q_2 &= \eta_2(x,t,u,v) - \xi_1(x,t,u,v)\ v_x - \xi_2(x,t,u,v)\ v_t. \end{aligned} \tag{5}$$

Here $Z^{(3)}$ denotes the third prolongation of Z, namely

$$
\begin{aligned}
Z^{(3)} \quad = \quad & Z + \gamma_1 \frac{\partial}{\partial u_x} + \gamma_2 \frac{\partial}{\partial u_t} + \gamma_3 \frac{\partial}{\partial v_x} + \gamma_4 \frac{\partial}{\partial v_t} + \gamma_{11} \frac{\partial}{\partial u_{xx}} + \gamma_{22} \frac{\partial}{\partial u_{tt}} \\
& + \gamma_{12} \frac{\partial}{\partial u_{xt}} + \gamma_{33} \frac{\partial}{\partial v_{xx}} + \gamma_{44} \frac{\partial}{\partial v_{tt}} + \gamma_{34} \frac{\partial}{\partial v_{xt}} + \gamma_{111} \frac{\partial}{\partial u_{xxx}} \\
& + \gamma_{222} \frac{\partial}{\partial u_{ttt}} + \gamma_{333} \frac{\partial}{\partial v_{xxx}} + \gamma_{112} \frac{\partial}{\partial u_{xxt}} + \gamma_{122} \frac{\partial}{\partial u_{xtt}} + \gamma_{334} \frac{\partial}{\partial v_{xxt}} \\
& + \gamma_{344} \frac{\partial}{\partial v_{xtt}} + \gamma_{444} \frac{\partial}{\partial v_{ttt}}, \qquad (6)
\end{aligned}
$$

where

$$
\begin{aligned}
\gamma_1 &= D_x(\eta_1) - u_x \ D_x(\xi_1) - u_t \ D_x(\xi_2), \\
\gamma_2 &= D_t(\eta_1) - u_x \ D_t(\xi_1) - u_t \ D_t(\xi_2), \\
\gamma_3 &= D_x(\eta_2) - v_x \ D_x(\xi_1) - v_t \ D_x(\xi_2), \\
\gamma_4 &= D_t(\eta_2) - v_x \ D_t(\xi_1) - v_t \ D_t(\xi_2), \\
\gamma_{11} &= D_x(\gamma_1) - u_{xx} \ D_x(\xi_1) - u_{xt} \ D_x(\xi_2), \\
\gamma_{22} &= D_t(\gamma_2) - u_{xt} \ D_t(\xi_1) - u_{tt} \ D_t(\xi_2), \\
\gamma_{111} &= D_x(\gamma_{11}) - u_{xxx} \ D_x(\xi_1) - u_{xxt} \ D_x(\xi_2). \qquad (7)
\end{aligned}
$$

A generator Z which satisfies condition (4) is called a conditional Q-symmetry generator, where by the invariant surfaces (**??**) and (2). The $F^{(j)}$ and $Q^{(k)}$denote the $j-th$ and $k-th$ prolongations, respectively. D_x denotes the total derivative with respect to x and D_t with respect to t.

Now, we derive the general determining equations for the conditional Q-symmetry generators of eqs. (**??**) and (1) and consider the following special case.

We set $\xi_1 = \xi_1(x)$, $\xi_2 = \xi_2(t)$, $\eta_1 = \eta_1(u)$ and $\eta_2 = \eta_2(v)$. The invariance condition (4) leads to the following expression:

$$
\begin{aligned}
& \gamma_2 + p \ \eta_2 \ v_x + p \ v \ \gamma_3 = 0, \\
& \gamma_4 + q \ \gamma_{111} + r \ \eta_1 \ v_x + r \ u \ \gamma_3 + s \ \eta_2 \ u_x + s \ v \ \gamma_1 = 0. \qquad (8)
\end{aligned}
$$

This leads to

$$
\begin{aligned}
& p \ v_x \ \eta_2(v) + u_t \ \eta_1'(u) + p \ v \ (v_x \ \eta_2'(v) - v_x \ \xi_1'(x)) - u_t \ \xi_2'(t) = 0, \\
& r \ v_x \ \eta_1(u) + s \ u_x \ \eta_2(v) + v_t \ \eta_2'(v) + s \ v \ u_x(\ \eta_1'(u) - \ \xi_1'(x)) \\
& + r \ u \ v_x \ (\ \eta_2'(v) - \xi_1'(x)) - v_t \ \xi_2'(t) + q \ (u_{xxx} \ \eta_1'(u) - 3 \ u_{xxx} \ \xi_1'(x) \\
& + 3 \ u_x \ u_{xx} \ \eta_1''(u) - 3 \ u_{xx} \ \xi_1''(x) + u_x^3 \ \eta_1^{(3)}(u) \ - \ u_x \ \xi_1^{(3)}(x)) = 0. \qquad (9)
\end{aligned}
$$

In particular, from $Q_1 = 0$ and $Q_2 = 0$ follow:

$$
\begin{aligned}
\xi_2(x,t,u,v)\ u_t &= \eta_1(x,t,u,v) - \xi_1(x,t,u,v)\ u_x, \\
\xi_2(x,t,u,v)\ v_t &= \eta_2(x,t,u,v) - \xi_1(x,t,u,v)\ v_x.
\end{aligned} \tag{10}
$$

The determining equations for the conditional Q-symmetry generator Z are obtained by equating the coefficients of the independent coordinates to zero.

By solving this system of linear partial differential equations for the infinitesimal $\xi_1(x)$, $\xi_2(t)$, $\eta_1(u)$ and $\eta_2(v)$, we get the following infinitesimal functions:

$$
\begin{aligned}
\eta_1(u) &= -\frac{2\ k_3\ u}{3}, \\
\eta_2(v) &= -\frac{2\ k_3\ v}{3}, \\
\xi_1(x) &= k_1 + \frac{k_3\ x}{3}, \\
\xi_2(t) &= k_2 + k_3\ t,
\end{aligned} \tag{11}
$$

where k_1, k_2 and k_3 are arbitrary constants.

The conditional Q-symmetry is given by:

$$
Z = (k_1 + \frac{k_3\ x}{3})\frac{\partial}{\partial x} + (k_2 + k_3\ t)\frac{\partial}{\partial t} - \frac{2\ k_3\ u}{3}\frac{\partial}{\partial u} - \frac{2\ k_3\ v}{3}\frac{\partial}{\partial v} \tag{12}
$$

The general solution of the associated invariant surface condition

$$
\begin{aligned}
(k_1 + \frac{k_3\ x}{3})\frac{\partial u}{\partial x} + (k_2 + k_3\ t)\frac{\partial u}{\partial t} &= -\frac{2\ k_3\ u}{3}, \\
(k_1 + \frac{k_3\ x}{3})\frac{\partial v}{\partial x} + (k_2 + k_3\ t)\frac{\partial v}{\partial t} &= -\frac{2\ k_3\ v}{3},
\end{aligned}
$$

are given by

$$
\begin{aligned}
u(x,t) &= \frac{\varphi_1(z)}{(3\ k_1 + k_3\ x)^2}, \\
v(x,t) &= \frac{\varphi_2(z)}{(3\ k1 + k3\ x)^2},
\end{aligned} \tag{13}
$$

where $\varphi_1(z)$ and $\varphi_2(z)$ are arbitrary functions of z and

$$
z(x,t) = \frac{k_2 + k_3\ t}{k_3\ (3\ k_1 + k_3\ x)^3}. \tag{14}
$$

Substituting (**??**) and (13) into equations (**??**) and (1), we obtain the system of nonlinear ordinary differential equations for $\varphi_1(z)$ and $\varphi_2(z)$ which take the form:

$$
\begin{aligned}
&-2k_3 p\varphi_{_2}^2(z)+\varphi_1'(z)-3pz\varphi_{_2}(z)\varphi_2'(z)=0,\\
&-24\ k_3^3\ q\ \varphi_{_1}(z)-2\ k_3\ \varphi_{_1}(z)\ \varphi_{_2}(z)\ (r+s)-3\ z\ (r\ \varphi_{_1}(z)\ \varphi_{_2}'(z)\ +\ s\ \varphi_{_2}(z)\ \varphi_{_1}'(z))\\
&+\varphi_{_2}'(z)-186\ k_3^2\ q\ z\ \varphi_1'(z)-162\ k_3\ z^2 q\ \varphi_1''-27\ z^3\ q\ \varphi_1'''(z)\ =0, \qquad (15)
\end{aligned}
$$

where $\varphi_i'=\frac{d\varphi_{_i}}{dz},\varphi_i''=\frac{d^2\varphi_{_i}}{dz^2}$ and $\varphi_i'''=\frac{d^3\varphi_{_i}}{dz^3}$; $(i=1,2)$.

Solving a system of an ordinary differential eqs. (**??**) and (15), we have six cases of solutions for $\varphi_{_1}(z)$ and $\varphi_{_2}(z)$:

Case 1.

$$
\begin{aligned}
\varphi_{_1}(z) &= a_1 z \text{ and } \varphi_{_2}(z)=b_0,\\
\text{with } p &= \frac{a_{_1}}{2\ b_0^2},\ q=\frac{b_0\ r}{70},\ s=-r,\ k_3=1,\ a_{_1}\text{and } b_0 \text{ are arbitrary constants.}
\end{aligned}
$$

Case 2.

$$
\begin{aligned}
\varphi_{_1}(z) &= a_1 z \text{ and } \varphi_{_2}(z)=b_0,\\
\text{with } p &= \frac{a_{_1}}{2\ b_0^2},\ q=-\frac{b_0\ (2\ r\ +5\ s)}{210},\ k_3=1,\ a_{_1}\text{and } b_0 \text{ are arbitrary constants.}
\end{aligned}
$$

Case 3.

$$
\begin{aligned}
\varphi_{_1}(z) &= a_1 z\ \text{ and } \varphi_{_2}(z)=b_0,\\
\text{with } p &= -\frac{2a_{_1}}{31\ b_0^2},\ q=-\frac{8b_0\ s}{29791},\ r=-\frac{25\ s}{31},\ k_3=-\frac{31}{4},\ a_{_1} \text{ and } b_0 \text{ are arbitrary constants.}
\end{aligned}
$$

Case 4.

$$
\begin{aligned}
\varphi_{_1}(z) &= a_1 z \text{ and } \varphi_{_2}(z)=b_0,\\
\text{with } p &= -\frac{a_{_1}}{3\ b_0^2},\ q=-\frac{2b_0\ s}{279},\ r=-\frac{25\ s}{31},\ k_3=-\frac{3}{2},\ a_{_1} \text{ and } b_0 \text{ are arbitrary constants.}
\end{aligned}
$$

Case 5.

$$
\begin{aligned}
\varphi_{_1}(z) &= a_1 z \text{ and } \varphi_{_2}(z)=b_0,\\
\text{with } p &= \frac{a_{_1}}{2\ b_0^2\ k_3},\ q=-\frac{b_0\ (2\ k_3\ r+(3+2\ k_3)\ s)}{6\ k_3^2(31+4\ k_3)},\ a_{_1} \text{ and } b_0 \text{ are arbitrary constants.}
\end{aligned}
$$

Case 6.

$$\varphi_1(z) = a_1 z \text{ and } \varphi_2(z) = b_0,$$
$$\text{with } p = \frac{a_1}{2\ b_0^2},\ q = \frac{b_0\ r}{70},\ s = -r,\ k_3 = 1,\ a_1 \text{ and } b_0 \text{ are arbitrary constants.}$$

Substituting from Eqs.(??)-(??) into Eqs.(??) and (13), we obtain the solutions for the Drinfel'd–Sokolov–Wilson (DSW) system (??) and (1) in the following:

Family 1.

$$u(x,t) = \frac{a_1 z}{(3\ k_1 + x)^2}$$
$$v(x,t) = \frac{b_0}{(3\ k_1 + x)^2} \tag{16}$$

with $p = \frac{a_1}{2\ b_0^2}$, $q = \frac{b_0\ r}{70}$, $s = -r$, $k_3 = 1$, a_1 and b_0 are arbitrary constants.

Family 2.

$$u(x,t) = \frac{a_1 z}{(3\ k_1 + x)^2}$$
$$v(x,t) = \frac{b_0}{(3\ k_1 + x)^2} \tag{17}$$

with $p = \frac{a_1}{2\ b_0^2}$, $q = -\frac{b_0\ (2\ r\ +5\ s)}{210}$, $k_3 = 1$, a_1 and b_0 are arbitrary constants.

Family 3.

$$u(x,t) = \frac{a_1 z}{(3\ k_1 + x)^2}$$
$$v(x,t) = \frac{b_0}{(3\ k_1 + x)^2} \tag{18}$$

with $p = -\frac{2a_1}{31\ b_0^2}$, $q = -\frac{8b_0\ s}{29791}$, $r = -\frac{25\ s}{31}$, $k_3 = -\frac{31}{4}$, $a_1$and b_0 are arbitrary constants.

Family 4.

$$u(x,t) = \frac{a_1 z}{(3\ k_1 + k_3\ x)^2}$$
$$v(x,t) = \frac{b_0}{(3\ k_1 + k_3\ x)^2} \tag{19}$$

with $p = -\frac{a_1}{3\ b_0^2}$, $q = -\frac{2b_0\ s}{279}$, $r = -\frac{25\ s}{31}$, $k_3 = -\frac{3}{2}$, $a_1$and b_0 are arbitrary constants.

Family 5.

$$u(x,t) = \frac{a_1 z}{(3\ k_1 + k_3\ x)^2}$$
$$v(x,t) = \frac{b_0}{(3\ k_1 + k_3\ x)^2} \tag{20}$$

with $p = \frac{a_1}{2\ b_0^2\ k_3}$, $q = -\frac{b_0\ (2\ k_3\ r+(3+2\ k_3)\ s)}{6\ k_3^2(31+4\ k_3)}$, a_1 and b_0 are arbitrary constants.

Family 6.

$$\begin{aligned} u(x,t) &= \frac{a_1 z}{(3\ k_1 + k_3\ x)^2} \\ v(x,t) &= \frac{b_0}{(3\ k_1 + k_3\ x)^2} \end{aligned} \tag{21}$$

with $p = \frac{a_1}{2\ b_0^2}$, $q = \frac{b_0\ r}{70}$, $s = -r$, $k_3 = 1$, a_1 and b_0 are arbitrary constants and $z = \frac{k_2+k_3\ t}{k_3\ (3\ k_1+k_3\ x)^3}$.

3 Q-symmetry generators

Before we consider conditional symmetries of (**??**) and (1), let us briefly describe the classical Lie approach and introduce our notation [**?**]. We are concerned with a partial differential equation of order r with $m+1$ independent variables $(x_0, x_1, ..., x_m)$ and two field variable u and v, i.e. an

equation of the form

$$\begin{aligned} F_1(x_0, x_1, ..., x_m, u, \frac{\partial u}{\partial x_0}, ..., \frac{\partial^r u}{\partial x_{j_1} ... \partial x_{j_r}}) = 0, \\ F_2(x_0, x_1, ..., x_m, v, \frac{\partial v}{\partial x_0}, ..., \frac{\partial^r v}{\partial x_{j_1} ... \partial x_{j_r}}) = 0, \end{aligned} \tag{22}$$

where $0 \le j_1 \le j_2 \le ... \le j_r \le m, j = 0, ..., m$. A Lie transformation group that leave (**??**) and (22) invariant is generated by a Lie symmetry generator

Z which is defined by

$$Z = \sum_{j=0}^{m} \xi_j (x_0, x_1, ..., x_m, u, v) \frac{\partial}{\partial x_j} + \eta_1 (x_0, x_1, ..., x_m, u, v) \frac{\partial}{\partial u} + \eta_2 (x_0, x_1, ..., x_m, u, v) \frac{\partial}{\partial v}. \tag{23}$$

Z_{w_1} and Z_{w_2} are the associated vertical form of (23), defined by

$$\begin{aligned} Z_{w_1} &= (\eta_1 - \sum_{j=0}^{m} \xi_j\ u_j) \frac{\partial}{\partial u}, \\ Z_{w_2} &= (\eta_2 - \sum_{j=0}^{m} \xi_j\ v_j) \frac{\partial}{\partial v}, \end{aligned} \tag{24}$$

where $Z_{w_1} \mid_{\theta_1} = Z \mid_{\theta_1}$ and $Z_{w_2} \mid_{\theta_2} = Z \mid_{\theta_2}$. Here θ_1 and θ_2 are a differential

1-form, called the contact form, which defined by

$$\theta_1 = du - \sum_{j=0}^{m} u_j \ dx_j,$$
$$\theta_2 = dv - \sum_{j=0}^{m} v_j \ dx_j.$$

Equations (**??**) and (22) are called invariant under the prolonged Lie symmetry generators Z_{w_1} and Z_{w_2} if

$$L_{\check{Z}_{w_1}} F_1 = 0,$$
$$L_{\check{Z}_{w_2}} F_2 = 0. \tag{25}$$

L denotes the Lie derivative. $\check{Z}_{w_1}$ and $\check{Z}_{w_2}$ are found by prolonging the vertical generator Z_{w_1} and Z_{w_2}, i.e.,

$$\check{Z}_{w_1} = \sum_{j=0}^{m} D_j \ (U_1) \ \frac{\partial}{\partial u_j} + ... + \sum_{j_1,...,j_r=0}^{m} D_{j_1,...,j_r}(U_1)\frac{\partial}{\partial u_{j_1,...,j_r}},$$
$$\check{Z}_{w_2} = \sum_{j=0}^{m} D_j \ (U_2) \ \frac{\partial}{\partial v_j} + ... + \sum_{j_1,...,j_r=0}^{m} D_{j_1,...,j_r}(U_2)\frac{\partial}{\partial v_{j_1,...,j_r}}, \tag{26}$$

where

$$U_1 = (\eta_1 - \sum_{j=0}^{m} \xi_j \ u_j),$$
$$U_2 = (\eta_2 - \sum_{j=0}^{m} \xi_j \ v_j), \tag{27}$$

and D_j is the total derivative operator. We give the definition for conditional invariance of (**??**) and (22) as follows:

Definition: Equations (**??**) and (22) are called Q-conditionally invariant if

$$L_{\check{Z}_{w_1}} F_1 = 0,$$
$$L_{\check{Z}_{w_2}} F_2 = 0, \tag{28}$$

under the two conditions

$$Z_{w_1} \mid_{\theta_1} = 0 \text{ and } Z_{w_2} \mid_{\theta_2} = 0. \tag{29}$$

Z_{w_1} and Z_{w_2} are called the Q-symmetry generators. $\check{Z}_{w_1}$ and $\check{Z}_{w_2}$ are called the prolonged vertical Q-symmetry generators.

Let us study eqs. (**??**) and (1) by the used of the above definition. From the above definition it follows that the Lie derivative (**??**) and (28), for equations

$$
\begin{aligned}
F_1 &\equiv u_t + p\ v\ v_x = 0,\\
F_2 &\equiv v_t + q\ u_{xxx} + r\ u\ v_x + s\ v\ u_x = 0,
\end{aligned} \tag{30}
$$

under the two conditions

$$
\begin{aligned}
Z_{w_1} &\mid \ _{\theta_1} = \eta_1 - \xi_1\ u_x - \xi_2\ u_t = 0,\\
Z_{w_2} &\mid \ _{\theta_2} = \eta_2 - \xi_1\ v_x - \xi_2\ v_t = 0,
\end{aligned} \tag{31}
$$

have to be studied. Let us consider the Q-symmetry generator in the form:

$$
Z = \xi_1(x,t,u,v)\frac{\partial}{\partial x} + \xi_2(x,t,u,v)\frac{\partial}{\partial t} + \eta_1(x,t,u,v)\frac{\partial}{\partial u} + \eta_2(x,t,u,v)\frac{\partial}{\partial v}. \tag{32}
$$

By applying the Lie derivative (**??**) and condition (**??**) , we get :

$$
D_{_t}\ (U_1) + p\ \eta_2\ v_x + p\ v\ D_{_x}\ (U_2) = 0. \tag{33}
$$

Also, by applying the Lie derivative (28) and condition (30), we obtain :

$$
D_{_t}\ (U_2) + q\ D_{xxx}\ (U_1) + r\ \eta_1\ v_x + r\ u\ D_{_x}\ (U_2) + s\ \eta_2\ u_x + s\ v\ D_x\ (U_1) = 0, \tag{34}
$$

where

$$
\begin{aligned}
D_{_x} &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + v_x\frac{\partial}{\partial v} + v_{xx}\frac{\partial}{\partial v_x} + v_{xt}\frac{\partial}{\partial v_t} + ...\\
D_{_t} &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{xt}\frac{\partial}{\partial u_x} + v_t\frac{\partial}{\partial v} + v_{tt}\frac{\partial}{\partial v_t} + v_{xt}\frac{\partial}{\partial v_x} + ...
\end{aligned} \tag{35}
$$

The determining equations for the Q-symmetry generator Z are now obtained by equating the coefficients of the independent coordinates to zero.

By solving this system of linear partial differential equations for the infinitesimal ξ_1, ξ_2, $\eta_1$and η_2, we obtain:

$$
\begin{aligned}
\eta_1(x,t,u,v) &= -\frac{2\ k_3\ u}{3},\\
\eta_2(x,t,u,v) &= -\frac{2\ k_3\ v}{3},\\
\xi_1(x,t,u,v) &= k_1 + \frac{k_3\ x}{3},\\
\xi_2(x,t,u,v) &= k_2 + k_3\ t.
\end{aligned} \tag{36}
$$

All of the similarity variables associated with the Lie symmetries (36) can be derived by solving the following characteristic equation:

$$\frac{dx}{\xi_1} = \frac{dt}{\xi_2} = \frac{du}{\eta_1} = \frac{dv}{\eta_2}. \tag{37}$$

Consequently

$$\frac{dx}{(k_1 + \frac{k_3\ x}{3})} = \frac{dt}{(k_2 + k_3\ t)} = \frac{du}{-\frac{2\ k_3\ u}{3}} = \frac{dv}{-\frac{2\ k_3\ v}{3}}. \tag{38}$$

To get the similarity variable z, we integrate the equation $\frac{dx}{(k_1+\frac{k_3\ x}{3})} = \frac{dt}{(k_2+k_3\ t)}v$. So, we get :

$$z = \frac{3\ k_1 + k_3\ x}{k_3\ (k_2 + k_3\ t)^{\frac{1}{3}}}. \tag{39}$$

From (38), we have :

$$\begin{aligned} \frac{dt}{(k_2 + k_3\ t)} &= \frac{du}{-\frac{2\ k_3\ u}{3}}, \\ \frac{dt}{(k_2 + k_3\ t)} &= \frac{dv}{-\frac{2\ k_3\ v}{3}}. \end{aligned} \tag{40}$$

By solving Eq.(40), we obtain the similarity solutions for the reduced ordinary differential equations as follows:

$$\begin{aligned} u(x,t) &= (k_2 + k_3\ t)^{\frac{-2}{3}} F_1(z), \\ v(x,t) &= (k_2 + k_3\ t)^{\frac{-2}{3}}\ F_2(z), \end{aligned} \tag{41}$$

where $z = \frac{3\ k_1+k_3\ x}{k_3\ (k_2+k_3\ t)^{\frac{1}{3}}}$, $F_1(z)$ and $F_2(z)$ are arbitrary functions of z.

Substituting (**??**) and (41) into equations (**??**) and (1), we finally obtain the system of nonlinear ordinary differential equations for $F_1(z)$ and $F_2(z)$ which take the form:

$$\begin{aligned} &-\frac{2\ F_1(z)}{3} - \frac{z\ F_1'(z)}{3} + \frac{p\ F_2(z)\ F_2'(z)}{k_3} = 0, \\ &-\frac{2\ F_2(z)}{3} + \frac{s\ F_2(z)\ F_1'(z)}{k_3} + \frac{r\ F_1(z)\ F_2'(z)}{k_3} - \frac{z\ F_2'(z)}{3} + \frac{q\ F_1'''(z)}{k_3} = 0, \end{aligned} \tag{42}$$

where $F_i' = \frac{d\varphi_i}{dz}, F_i'' = \frac{d^2\varphi_i}{dz^2}$ and $F_i''' = \frac{d^3\varphi_i}{dz^3}$; $(i = 1, 2)$.

Solving a system of an ordinary differential eqs.(**??**) and (42),we have three cases of solutions for $F_1(z)$ and $F_2(z)$:

Case 1.

$$\begin{aligned} F_1(z) &= a_1\ z \text{ and } F_2(z) = b_1\ z, \\ \text{with } a_1 &= \frac{k_3}{r+s} \text{ and } b_1 = \pm\frac{k_3}{\sqrt{p(r+s)}}. \end{aligned} \tag{43}$$

Case 2.

$$\begin{aligned} F_1(z) &= a_1\ z \text{ and } F_2(z) = b_1\ z, \\ \text{with } a_1 &= \pm\frac{b_1\sqrt{p}}{\sqrt{r+s}} \text{ and } k_3 = \pm b_1\sqrt{p(r+s)}. \end{aligned} \tag{44}$$

Case 3.

$$\begin{aligned} F_1(z) &= a_1\ z \text{ and } F_2(z) = b_1\ z, \\ \text{with } a_1 &= \frac{b_1^2\ p}{k_3} \text{ and } r = \frac{k_3^2}{b_1^2\ p} - s. \end{aligned} \tag{45}$$

Substituting from Eqs.(43)-(45) into Eqs.(**??**) and (41), we obtain the solutions for the Drinfel'd–Sokolov–Wilson (DSW) system (**??**) and (1) in the following:

Family 1.

$$\begin{aligned} u(x,t) &= a_1\ z\ (k_2 + k_3\ t)^{\frac{-2}{3}}, \\ v(x,t) &= b_1\ z\ (k_2 + k_3\ t)^{\frac{-2}{3}}, \end{aligned} \tag{46}$$

with $a_1 = \frac{k_3}{r+s}$ and $b_1 = \pm\frac{k_3}{\sqrt{p(r+s)}}$.

Family 2.

$$\begin{aligned} u(x,t) &= a_1\ z\ (k_2 + k_3\ t)^{\frac{-2}{3}}, \\ v(x,t) &= b_1\ z\ (k_2 + k_3\ t)^{\frac{-2}{3}}, \end{aligned} \tag{47}$$

with $a_1 = \pm\frac{b_1\sqrt{p}}{\sqrt{r+s}}$ and $k_3 = \pm b_1\sqrt{p(r+s)}$.

Family 3.

$$\begin{aligned} u(x,t) &= a_1\ z\ (k_2 + k_3\ t)^{\frac{-2}{3}}, \\ v(x,t) &= b_1\ z\ (k_2 + k_3\ t)^{\frac{-2}{3}}, \end{aligned} \tag{48}$$

with $a_1 = \frac{b_1^2\ p}{k_3}$ and $r = \frac{k_3^2}{b_1^2\ p} - s$.

References

[1] B. C. Jie, B.C. Lin, H. Ji-Guang, and Z. Hong, "On an Auto-Bäcklund transformation for (2+1)-Dimensional variable coefficient generalized KP equations and exact solutions", Communications in Theoretical Physics. (Beijing,China) (2005) pp. 677–680

[2] Z.Y. Yan, H.Q. Zhang, " On a new algorithm of constructing solitary wave solutions for systems of nonlinear evolution equations in mathematical physics", Journal of Applied Mathematics and Mechanics 21 (4) (2000) 383–388.

[3] L.M. Delves, J.L. Mohammed, Computational Methods for Integral Equations, Cambridge University Press, Oxford, 1983.

[4] Y. He, Y. Long and S. Li, "Exact solutions of the Drinfel'd-Sokolov-Wilson equation using the F-expansion method combined with Exp-function method", International mathematical forum, 5, 2010, no. 65, 3231 - 3242.

[5] Hassan Zedan,Symmetry analysis of an integrable Ito coupled system" Computers and Mathematics with Applications 60 (2010) 3088–3097.

[6] E. Yusufoglu, A. Bekir, "On the extended tanh method applications of nonlinear equations", International journal of nonlinear science, Vol. 4(2007) No.1, pp.10-16.

[7] Hassan Zedan and M. Alghamdi " Solution of (3 _ 1)-Dimensional Nonlinear Cubic Schrodinger Equation by Differential Transform Method" Hindawi Publishing Corporation ,Mathematical Problems in Engineering Volume (2012),
Article ID 531823, 14 pages.

[8] E. Misirli, Y. Gurefe, "Exact solutions of the Drinfel'd–Sokolov–Wilson equation using the Exp-function method", Applied mathematcis and Computation,216 (2010) 2623–2627.

[9] Y. Ugurlu, Dogan Kaya and Ibrahim E. Inan, "Generalized Jacobi elliptic function method for periodic wave solutions of SRLW equation and (1+1)-dimensional Dispersive long wave equation", C ankaya University Journal of Science and Engineering, Volume 8 (2011), No. 2, 205-223.

[10] Hassan A. Zedan and Eman El Adrous" The Application of the Homotopy Perturbation Method and the Homotopy Analysis Method to the Generalized
Zakharov Equations" Abstract and applied analysis, Volume 2012 (2012), Article ID 561252, 19 pages.

[11] B. Hong and D. Lu, "New Jacobi elliptic functions solutions for the higher-order nonlinear Schrodinger equation", International journal of nonlinear science, Vol.7(2009) No.3, pp.360-367.

[12] J. Hietarinta, "Hirota's bilinear method and soliton solutions", Physics USA, vol.15 (Part I), 31-37(2005).

[13] C. Liu and X. Liu, "Exact solutions of the classical Drinfel'd–Sokolov–Wilson equations and the relations among the solutions", Physics Letters A 303 (2002) 197–203.