Analogous to Ramanujan's Remarkable product of Theta Function of degree 9 and Their Explicit Evaluation

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Abstract

In this paper, We study the Analogous of Ramanujan's Remarkable product of theta-function of degree 9 of and their explicit values.

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1. Introduction

In Chapter 16 of his second notebook [2], Ramanujan develops the theory of theta-function and is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, |\ ab\ |< 1, \tag{1.1}$$

$$= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty$$

where $(a;q)_0 = 1$ and $(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots$.

Following Ramanujan, we defined

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{1.2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.3}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty \tag{1.4}$$

and

$$\chi(q) := (-q; q^2)_\infty. \tag{1.5}$$

On page 338 in his first notebook [11, p.338], Ramanujan defines

$$a_{m,n} = \frac{n e^{\frac{-(n-1)\pi}{4}\sqrt{\frac{m}{n}}} \psi^2(e^{-\pi\sqrt{mn}})\varphi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(e^{-\pi\sqrt{\frac{m}{n}}})\varphi^2(-e^{-2\pi\sqrt{\frac{m}{n}}})}. \tag{1.6}$$

He then, on pages 338 and 339, offers a list of eighteen particular values. All these eighteen values have been established by Berndt, Chan and Zhang [1]. An account of these can be found in Berndt's book [3], M. S. Mahadeva Naika and B. N. Dharmendra [4], also established some general theorems for explicit evaluations of the product of $a_{m,n}$ and found some new explicit values therefrom. Further results on $a_{m,n}$ can be found by Mahadeva Naika, Dharmendra and K. Shivashankar [5], and Mahadeva Naika and M. C. Mahesh Kumar [6].

In [7], Mahadeva Naika et al. defined the product

$$b_{m,n} = \frac{n e^{\frac{-(n-1)\pi}{4}\sqrt{\frac{m}{n}}} \psi^2(-e^{-\pi\sqrt{mn}})\varphi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(-e^{-\pi\sqrt{\frac{m}{n}}})\varphi^2(-e^{-2\pi\sqrt{\frac{m}{n}}})}. \tag{1.7}$$

They established general theorems for explicit evaluation of $b_{m,n}$ and obtained some particular values. Mahadeva Naika et al. [8] established general formulas for explicit values of Ramanujan's cubic continued fraction $V(q)$ in terms of the products $a_{m,n}$ and $b_{m,n}$ defined above, where

$$V(q) := \frac{q^{1/3}}{1} {}_{+} \frac{q+q^2}{1} {}_{+} \frac{q^2+q^4}{1} {}_{+} \frac{q^3+q^6}{1} {}_{+} \cdots, \quad |q| < 1, \tag{1.8}$$

and found some particular values of $V(q)$

In [10], Nipen Saikia defined the product of theta-fuctions $I_{m,n}$ as

$$I_{m,n} = \frac{\psi(-q)\varphi(q^m)}{q^{(m-1)/8}\psi(-q^m)\varphi(q)}; \quad e^{-\pi\sqrt{\frac{n}{m}}} \tag{1.9}$$

where m and n are positive real numbers. We establish several properties of the product $I_{m,n}$. They prove general formulas for explicit evaluations of evaluation of $I_{m,n}$ and find its explicit values.

In this paper, we establish several General Theorems and explicit evaluation of $I_{9,n}$.

Now we define a modular equation in brief. The ordinary hypergeometric series

${}_2F_1(a, b; c; x)$ is defined by

$${}_2F_1(a, b; c; x) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n,$$

where $(a)_0 = 1, (a)_n = a(a+1)(a+2)\cdots(a+n-1)$ for any positive integer n, and $\mid x \mid < 1$.
Let

$$z := z(x) := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) \tag{1.10}$$

and

$$q := q(x) := \exp\left(-\pi \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)}\right), \tag{1.11}$$

where $0 < x < 1$.
Let r denote a fixed natural number and assume that the following relation holds:

$$r\frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)}. \tag{1.12}$$

Then a modular equation of degree r in the classical theory is a relation between α and β induced by (1.12). We often say that β is of degree r over α and $m := \dfrac{z(\alpha)}{z(\beta)}$ is called the multiplier. We also use the notations $z_1 := z(\alpha)$ and $z_r := z(\beta)$ to indicate that β has degree r over α.

2. Preliminary results

Lemma 2.1. [9] *If* $P := \dfrac{\psi(-q)}{q\psi(-q^9)}$ *and* $Q := \dfrac{\varphi(q)}{\varphi(q^9)}$, *then*

$$Q + PQ = 3 + P \tag{2.1}$$

Lemma 2.2. [9] *If* $P := \dfrac{\varphi(q)}{\varphi(q^9)}$ *and* $Q := \dfrac{\varphi(q^2)}{\varphi(q^{18})}$, *then*

$$\begin{aligned}&(-4Q^3 + 6Q^2 + Q^4 - 4Q + 1)P^4 + (8Q^3 - 8Q - 4Q^4 - 12)P^3\\&+ (-12Q^2 + 54 + 6Q^4)P^2 + (72Q - 4Q^4 - 8Q^3 - 108)P\\&- 108Q + 54Q^2 + Q^4 + 81 - 12Q^3.\end{aligned} \tag{2.2}$$

Lemma 2.3. [9] *If* $P = \dfrac{\varphi(q)}{\varphi(q^9)}$ *and* $Q = \dfrac{\varphi(q^3)}{\varphi(q^{27})}$, *then*

$$9P^2Q - 3P^2Q^2 - 9P^2 - 3P^3Q + P^3Q^2 + 3P^3 - 9PQ + 3PQ^2 + 9P - Q^3 = 0. \tag{2.3}$$

Lemma 2.4. [9] *If* $P = \frac{\varphi(q)}{\varphi(q^9)}$ *and* $Q = \frac{\varphi(q^5)}{\varphi(q^{45})}$, *then*

$$\begin{aligned} &P^6 + (-Q^5 - 15Q - 10Q^3 + 5Q^4 + 15Q^2)P^5 \\ &+ (-45Q^2 + 5Q^5 - 20Q^4 + 30Q^3 + 45Q)P^4 \\ &+ (30Q^4 - 40Q^3 - 90Q + 90Q^2 - 10Q^5)P^3 \\ &+ (90Q^3 + 15Q^5 - 45Q^4 - 180Q^2 + 135Q)P^2 \\ &+ (45Q^4 - 81Q - 15Q^5 - 90Q^3 + 135Q^2)P + Q^6 = 0. \end{aligned} \tag{2.4}$$

Lemma 2.5. [9] *If* $P := \frac{\varphi(q)}{\varphi(q^5)}$ *and* $Q := \frac{\varphi(q^7)}{\varphi(q^{63})}$, *then*

(2.5)
$$\begin{aligned} &P^8 + (-21Q^5 - 63Q^3 + 42Q^4 + 7Q^6 - 35Q - Q^7 + 63Q^2)P^7 \\ &+ (147Q^5 - 49Q^6 + 189Q + 7Q^7 - 413Q^2 - 294Q^4 + 441Q^3)P^6 \\ &+ (-1379Q^3 - 567Q - 441Q^5 - 21Q^7 + 147Q^6 + 1323Q^2 + 882Q^4)P^5 \\ &+ (-294Q^6 + 2646Q^3 + 1134Q - 2646Q^2 + 882Q^5 - 1694Q^4 + 42Q^7)P^4 \\ &+ (-3969Q^3 - 1701Q + 441Q^6 + 3969Q^2 + 2646Q^4 - 63Q^7 - 1379Q^5)P^3 \\ &+ (1701Q - 2646Q^4 + 63Q^7 + 3969Q^3 - 413Q^6 - 3969Q^2 + 1323Q^5)P^2 \\ &+ (1701Q^2 - 729Q + 1134Q^4 + 189Q^6 - 567Q^5 - 35Q^7 - 1701Q^3)P + Q^8 = 0. \end{aligned}$$

Lemma 2.6. [10]

$$I_{m,1} = 1 \tag{2.6}$$

Lemma 2.7. [10]

$$I_{m,n} I_{m,1/n} = 1 \tag{2.7}$$

3. General Theorems and Explicit Evaluations of $I_{9,n}$

Theorem 3.1. *If* $X := I_{9,n}$ *and* $Y := I_{9,4n}$ *then*

$$\begin{aligned} &x^8 y^5 + (16y^4 - y^7 - 6y^6 + 12y^2 + y + 30y^3 + 12y^5)x^7 \\ &+ (-30y^2 - 132y^4 - 19y^3 + 12y - 96y^5 - 79y^6 - 6y^7)x^6 \\ &+ (60y^3 + 12y^7 - 96y^6 - 19y^2 + 30y + y^8 - 74y^5 - 78y^4)x^5 \\ &+ (-132y^2 - 78y^3 + 16y^7 - 132y^6 + 16y - 172y^4 - 78y^5)x^4 \\ &+ (12y + 30y^7 - 78y^4 - 74y^3 + 60y^5 + 1 - 19y^6 - 96y^2)x^3 \\ &+ (-19y^5 - 79y^2 - 96y^3 + 12y^7 - 30y^6 - 6y - 132y^4)x^2 \\ &+ (y^7 + 12y^6 + 30y^5 + 16y^4 + 12y^3 - y - 6y^2)x + y^3 = 0. \end{aligned} \tag{3.1}$$

Proof. Employing the definition of $I_{m,n}$ (1.9) with $m = 9$, we obtain

$$I_{9,n} = \frac{\psi(-q)\varphi(q^9)}{q\psi(-q^9)\varphi(q)}; \quad e^{-\pi\sqrt{\frac{n}{9}}} \tag{3.2}$$

By using lemma (2.1), we obtain

$$P = \frac{3-Q}{Q-1} \tag{3.3}$$

the above two equations (3.2) and (3.3) can be written as

$$I_{9,n} = \frac{3-Q}{Q^2-Q} \tag{3.4}$$

where $Q = \dfrac{\varphi(q)}{\varphi(q^9)}$, the above the equation (3.4) can be written as

$$Q = \frac{(a-1)+\sqrt{a^2+10a+1}}{2a} \tag{3.5}$$

where $a = I_{9,n}$
Employing the equation (3.5) in (2.2), we obtain (3.1) □

Corollary 3.1. *We have*

$$I_{9,2} = \frac{(9+5\sqrt{3})\sqrt{2}-2(5+3\sqrt{3})}{2}, \tag{3.6}$$

$$I_{9,1/2} = \frac{(9-5\sqrt{3})\sqrt{2}-2(5-3\sqrt{3})}{2}. \tag{3.7}$$

(3.8)

$$I_{9,4} = \frac{(9+5\sqrt{3})\sqrt{2}-2((16+9\sqrt{3})+2\sqrt{1197+693\sqrt{3}-(837+483\sqrt{3})\sqrt{2}}}{4},$$

(3.9)

$$I_{9,1/4} = \frac{(9+5\sqrt{3})\sqrt{2}-2((16+9\sqrt{3})-2\sqrt{1197+693\sqrt{3}-(837+483\sqrt{3})\sqrt{2}}}{4}.$$

Proof. Setting $n = 1/2$ in Theorem (3.1) and using the Lemma (2.7), we obtain

$$(I_{9,2}^4 + 20I_{9,2}^3 - 60I_{9,2}^2 + 20I_{9,2} + 1)(I_{9,2}^2 + I_{9,2} + 1)^4 = 0 \tag{3.10}$$

Since the root of the second and third factors are imaginary and $I_{5,2} < 10$ we deduce that

$$I_{9,2}^4 + 20I_{9,2}^3 - 60I_{9,2}^2 + 20I_{9,2} + 1 = 0 \tag{3.11}$$

On solving the above equation (3.11), we arrive at the equations (3.6) and (3.7).

Setting $n = 1$ in Theorem (3.1) and using the Lemma (2.6), we obtain

$$1 + 64I_{9,4} + 64I_{9,4}^7 - 350I_{9,4}^6 - 164I_{9,4}^3 - 560I_{9,4}^4 + I_{9,4}^8 - 164I_{9,4}^5 - 350I_{9,4}^2 = 0 \tag{3.12}$$

by above equation (3.12) can be written as

$$z^4 + 64z^3 - 354z^2 - 356z + 142 = 0, \quad z = I_{9,4} + I_{9,4}^{-1} \tag{3.13}$$

On solving the above equation (3.13), we arrive at the equations (3.8) and (3.9).

□

Theorem 3.2. *If* $X := I_{9,n}$ *and* $Y := I_{9,9n}$ *then*

$$\begin{aligned}
&(y^2 + 1 + y)x^6 + (-17y^2 - 10y^4 - 19y^3 + 1 - y^5 - 8y)x^5 \\
&+ (-10y^5 - 28y^4 - 46y^3 - 17y + 1 - 35y^2)x^4 + (-19y^5 \\
&- 19y - 46y^2 - 74y^3 - 46y^4)x^3 + (-28y^2 - 17y^5 - 46y^3 \\
&+ y^6 - 10y - 35y^4)x^2 + (y^6 - 17y^4 - 8y^5 - y - 10y^2 - 19y^3)x \\
&+ y^6 + y^4 + y^5 = 0.
\end{aligned} \tag{3.14}$$

Proof. Employing the equation (3.5) in (2.3), we obtain (3.14) □

Corollary 3.2. *We have*

$$I_{9,3} = 2^{2/3} + 2^{1/3} + 1, \tag{3.15}$$

$$I_{9,1/3} = 2^{1/3} - 1. \tag{3.16}$$

$$I_{9,9} = \frac{13\left[(11 + 7\sqrt{3})x + 13(3 + 2\sqrt{3})\right] + (51 + 23\sqrt{3})x^2}{169}, \tag{3.17}$$

$$I_{9,1/9} = \frac{13\left[(15 - 7\sqrt{3})x + 13(3 - 2\sqrt{3})\right] + (43 - 27\sqrt{3})x^2}{169}. \tag{3.18}$$

where $x = (47 + 2\sqrt{3})^{1/3}$.

Proof. Employing Theorem (3.2), Lemma (2.7) and (2.6), solving the resulting equation for $I_{9,3}$, $I_{9,9}$ and nothing that $I_{9,3} < 1$ and $I_{9,9} < 1$, we arrive (3.15)-(3.18). □

Theorem 3.3. *If* $X := I_{9,n}$ *and* $Y := I_{9,25n}$ *then*

$$\begin{aligned} &x^6 + (-45y - 60y^3 - 65y^2 - 15y^4 - y^5)x^5 \\ &+ (-65y + 75y^2 + 60y^3 - 15y^5 + 140y^4)x^4 \\ &+ (-60y + 60y^2 - 20y^3 - 60y^5 + 60y^4)x^3 \\ &+ (-65y^5 + 60y^3 + 140y^2 - 15y + 75y^4)x^2 \\ &+ (-y - 60y^3 - 15y^2 - 65y^4 - 45y^5)x + y^6 = 0. \end{aligned} \tag{3.19}$$

Proof. Employing the equation (3.5) in (2.4), we obtain (3.19) □

Corollary 3.3. *We have*

$$I_{9,5} = 4 + \sqrt{15}, \tag{3.20}$$

$$I_{9,1/5} = 4 - \sqrt{15}, \tag{3.21}$$

$$I_{9,25} = 2(23 + 6\sqrt{5}) + \sqrt{3(1425 + 368\sqrt{5})}, \tag{3.22}$$

$$I_{9,1/25} = 2(23 + 6\sqrt{5}) - \sqrt{3(1425 + 368\sqrt{5})}. \tag{3.23}$$

Proof. Employing Theorem (3.3), Lemma (2.7) and (2.6), solving the resulting equation for $I_{9,5}$, $I_{9,25}$ and nothing that $I_{9,5} < 1$ and $I_{9,25} < 1$, we arrive (3.20)-(3.23). □

Theorem 3.4. *If* $X := I_{9,n}$ *and* $Y := I_{9,49n}$ *then*

(3.24)

$$\begin{aligned} &x^8 + (-406y^4 - 357y^2 - 21y^6 - 147y^5 - 483y^3 - 105y - y^7)x^7 \\ &+ (-21y^7 - 1134y^4 - 2415y^3 + 903y^6 - 1757y^2 - 357y + 273y^5)x^6 \\ &+ (-6930y^4 - 7889y^3 + 273y^6 - 5145y^5 - 483y - 2415y^2 - 147y^7)x^5 \\ &+ (-406y - 1134y^6 - 7182y^4 - 6930y^5 - 406y^7 - 6930y^3 - 1134y^2)x^4 \\ &+ (-6930y^4 + 273y^2 - 483y^7 - 2415y^6 - 5145y^3 - 147y - 7889y^5)x^3 \\ &+ (273y^3 - 357y^7 - 21y - 1757y^6 + 903y^2 - 1134y^4 - 2415y^5)x^2 \\ &+ (-y - 406y^4 - 483y^5 - 21y^2 - 357y^6 - 105y^7 - 147y^3)x + y^8 = 0. \end{aligned}$$

Proof. Employing the equation (3.5) in (2.5), we obtain (3.24) □

Corollary 3.4. *We have*

$$I_{9,7} = \frac{(13 + 3\sqrt{21}) + \sqrt{6(57 + 13\sqrt{21})}}{4}, \tag{3.25}$$

$$I_{9,1/7} = \frac{(13 + 3\sqrt{21}) - \sqrt{6(57 + 13\sqrt{21})}}{4}. \tag{3.26}$$

Proof. Employing Theorem (3.4), Lemma (2.7), solving the resulting equation for $I_{9,7}$ and nothing that $I_{9,7} < 1$, we arrive (3.25)-(3.25). □

References

[1] B. C. Berndt, H. H. Chan and L.-C. Zhang, Ramanujan's remarkable product of the theta-function, Proc. Edinburgh Math. Soc., 40, 583-612 (1997).

[2] B. C. Berndt, Ramanujan's Notebooks, Part III, Springer-Verlag, New York (1991).

[3] B. C. Berndt, Ramanujan's Notebooks, Part V, Springer-Verlag, New York (1998).

[4] M. S. Mahadeva Naika and B. N. Dharmendra, On some new general theorems for the explicit evaluations of Ramanujan's remarkable product of theta-function Ramanujan J. 15(3), 349-366 (2008).

[5] M. S. Mahadeva Naika, B. N. Dharmendra and K. Shivashankara, On some new explicit evaluations of Ramanujan's remarkable product of theta-function, South East Asian J. Math. Math. Sci. 5(1), 107-119 (2006)

[6] M. S. Mahadeva Naika and M. C. Maheshkumar, Explicit evaluations of Ramanujan's remarkable product of theta-function, Adv. Stud. Contemp. Math., 13(2), 235-254 (2006).

[7] M. S. Mahadeva Naika, M. C. Maheshkumar and K. Sushan Bairy, On some remarkable product of theta-function, Aust. J. Math. Anal. Appl., 5(1), 1-15 (2008).

[8] M. S. Mahadeva Naika, M. C. Maheshkumar and K. Sushan Bairy, General formulas for explicit evaluations of Ramanujan's cubic continued fraction, Kyungpook Math. J., 49(3), 435-450 (2009)

[9] M. S. Mahadeva Naika, K. Sushan Bairy and S. Chandankumar, On Some Explicit evaluation of the ratios of Ramanujan's theta–function, (Communicated).

[10] Nipen Saikia, A Product of Theta-Functions Analogous to Ramanujan's Remarkable Product of Theta-Functions and Applications, Hindawi Publishing Corporation, Journal of Mathematics, Volume 2013.

[11] S. Ramanujan, Notebooks (2 volumes). Bombay. Tata Institute of Fundamental Research (1957).