ANALYTICAL APPROXIMATION SOLUTION OF PSEUDO-PARABOLIC FRACTIONAL EQUATION USING A MODIFIED DOUBLE LAPLACE DECOMPOSITION METHOD

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Abstract

In this paper, the modification of double Laplace decomposition method is proposed for the analytical approximation solution of pseudo-parabolic fractional equation with initial conditions. Some examples are given to support the validity and applicability of the presented method. In addition, we check the convergence of double Laplace method applied to our problem.

Keywords double Laplace transform; inverse double Laplace transform; singular pseudo-parabolic fractional equation; single Laplace transform; decomposition methods

1 INTRODUCTION

Fractional differential equations can describe many phenomena in various fields of engeneering and scientific displines such as control theory, physics,chemistry, biology, economic, mechanics and electromagnetic.

In theoretical physics, it is usually very important to seek and construct explicit solutions of linear and nonlinear partial differential equations (PDEs). Therefore, the solution helps the researchers to understand the physical phenomena. The parabolic equation occurs in several areas of applied mathematics, such as heat conduction, the phenomenea of turbulence and flow through a shock wave traveling in a viscous fluid such as the modeling of dynamics. In recent years, several studies for the linear and nonlinear initial value of fractionnals problems arise in the literature. The pseudo-parabolic equation models a variety of physical processes. The one dimensional pseudo-parabolic equation was derived in [12]. In general, some of the nonlinear models of real-life problems are still very difficult to solve either theoretically or numerically. Recently, many authors have proposed analytical solution to one dimensional system of parabolic equation (Burgers equation) eg.[13, 14] using a domain decomposition method. In [5] the author used Laplace transform and homotopy perturbation method to obtain approximate solutions of homogeneous and inhomogeneous coupleg

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Berger's equation. The author in [1] used a modified double Laplace decomposition method to solve coupled pseudo-parabolic equations. Our aim in this work, is to use modified double Laplace decoposition for solving a singular one dimensional pseudo-parabolic fractional equation.

We will now recall the following definitions which are given by [1, 15]. The double Laplace transform of the functions $u(x,t)$ and $f(x,t)$ are defined as

$$L_xL_t\left[u(x,t)\right] = U(p,s) = \int_0^\infty \int_0^\infty e^{-px}e^{-st}u(x,t)dtdx, \tag{1}$$

$$L_xL_t\left[f(x,t)\right] = F(p,s) = \int_0^\infty \int_0^\infty e^{-px}e^{-st}f(x,t)dtdx, \tag{2}$$

where $x, t > 0$ and p, s complex values.
The Caputo derivative of the function $u(x,t)$ is defined as follow:

$$\mathtt{D}_t^\alpha v(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial v(x,\tau)}{\partial \tau} \times \frac{1}{(t-\tau)^\alpha} d\tau. \tag{3}$$

The single Laplace transform of Caputo derivative of α order $(0 < \alpha < 1)$ of the function $f(t)$ is defined as

$$L\left[\mathtt{D}_t^\alpha f(t)\right](s) = s^\alpha F(s) - s^{\alpha-1}f(0) \tag{4}$$

where $F(S)$ is Laplace transforme of $f(t)$.
Then, the double Laplace transform of the Caputo derivative of the function $u(x,t)$ with respect to x and t is given by

$$\begin{aligned} L_xL_t\left[\mathtt{D}_t^\alpha u(x,t)\right] &= \int_0^\infty \int_0^\infty e^{-px}e^{-st}\mathtt{D}_t^\alpha u(x,t)dtdx \\ &= \int_0^\infty e^{-px}\left[\int_0^\infty e^{-st}\mathtt{D}_t^\alpha u(x,t)dt\right]dx \\ &= \int_0^\infty e^{-px}\left[s^\alpha \int_0^\infty e^{-st}u(x,t)dt - s^{\alpha-1}u(x,0)\right]dx \\ &= s^\alpha U(p,s) - s^{\alpha-1}U(p,0). \end{aligned} \tag{5}$$

The following basic lemma of the double Laplace transform is given and used in this paper.

Lemma 0.1 *Double Laplace transform of the non constant coefficient Caputo derivative $x\mathtt{D}_t^\alpha u(x,t)$ and the function $xf(x,t)$ are given by*

$$\begin{aligned} L_xL_t\left[x\mathtt{D}_t^\alpha u(x,t)\right] &= -\frac{d}{dp}\left[L_xL_t\left[x\mathtt{D}_t^\alpha u(x,t)\right]\right] \\ &= -\frac{d}{dp}\left[s^\alpha U(p,s) - s^{\alpha-1}U(p,0)\right] \end{aligned} \tag{6}$$

$$
\begin{aligned}
L_xL_t\left[xf(x,t)\right] &= -\frac{d}{dp}\left[L_xL_t\left[f(x,t)\right]\right] \\
&= -\frac{dF(p,s)}{dp}
\end{aligned} \tag{7}
$$

One can prove this lemma by the definition of double Laplace transform in Eq.(2) and Eq.(6).

1 One Dimensional Pseudo-Parabolic Fractional Equation

In this section we will use modified double Laplace decomposition method to solve singular one dimensional pseudo-parabolic fractional equation. We consider singular one dimensional pseudo- parabolic fractional equation with initial conditions in the form

$$
\mathtt{D}_t^{\alpha}u - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) - \frac{1}{x}\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial u}{\partial x}\right) = f(x,t), \tag{8}
$$

subject to Cauchy condition

$$
u(x,0) = g(x), \quad x \geq 0, \quad t > 0, \tag{9}
$$

where, the linear term, $\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)$ is called Bessel's operator and $f(x,t)$, $g(x)$ are known functions.In order to obtain the solution of Eq. (9), we apply the following steps.

Step 1 Multiplying both sides of Eq.(9) by x.

Step 2 Using Lemma 0.1 and definition of the double Laplace transform of Caputo derivative for equations in Step 1 and single Laplace transform for initial condition, we get

$$
\frac{dU(p,s)}{dp} = \frac{1}{s}\frac{dG(p)}{dp} + \frac{1}{s^{\alpha}}\frac{dF(p,s)}{dp} - \frac{1}{s^{\alpha}}L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) + \frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial u}{\partial x}\right)\right]. \tag{10}
$$

where $G(p), U(p,s), F(p,s)$ are Laplace transform of the functions $g(x)$, $u(x,t)$ and $f(x,t)$ respectivily.

Step 3 Applaying the integral for both sides of Eq.(10), from 0 to p with respectively to p, we have

$$
U(p,s) = \frac{G(p)}{s} + \frac{1}{s^{\alpha}}F(p,s) - \frac{1}{s^{\alpha}}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) + \frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial u}{\partial x}\right)\right]dp. \tag{11}
$$

Step 4 Using double Laplace Adomain decomposition methods to define the solution of the singular one dimensional pseudo-parabolic fractional equation as

$$
u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{12}
$$

Step 5 By applying inverse double Laplace transform on both sides of Eq. (11) and use Eq.(12), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(x,t) &= g(x) + L_p^{-1}L_s^{-1}\left[\frac{1}{s^\alpha}F(p,s)\right] \\ &- L_p^{-1}L_s^{-1}\left[\frac{1}{s^\alpha}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}u_n(x,t)\right)\right]dp\right] \\ &- L_p^{-1}L_s^{-1}\left[\frac{1}{s^\alpha}\int_0^p L_xL_t\left[\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}u_n(x,t)\right)\right]dp\right].\end{aligned} \tag{13}$$

In particular, we have

$$u_0 = g(x) + L_p^{-1}L_s^{-1}\left[\frac{1}{s^\alpha}\int_0^p dF(p,s)\right] \tag{14}$$

and the rest terms can be written as follows

$$\begin{aligned}u_{n+1} &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s^\alpha}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}u_n(x,t)\right)\right]dp\right] \\ &- L_p^{-1}L_s^{-1}\left[\frac{1}{s^\alpha}\int_0^p L_xL_t\left[\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}u_n(x,t)\right)\right]dp\right],\end{aligned} \tag{15}$$

where $L_p^{-1}L_s^{-1}$ is called double inverse Laplace transform with respect to p and s. Here we provide double inverse Laplace transform with respect to p and s exist for each term in the right hand side of Eqs. (14) and (15).
In order to confirm our method for solving the singular one dimensional pseudo-parabolic fractional equation, we consider the following examples.

Example 1.1 *Consider the following nonhomogeneous form of a singular one dimensional pseudo-parabolic fractional equation*

$$\mathtt{D}_t^\alpha u - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) - \frac{1}{x}\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial u}{\partial x}\right) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} - 4, \quad t>0, \quad x \geq 0 \tag{16}$$

subject to the initial condition

$$u(x,0) = g(x) = x^2. \tag{17}$$

By applying the above steps, we obtain

$$u_0(x,t) = x^2 + t - 4\frac{t^\alpha}{\Gamma(\alpha+1)} \tag{18}$$

and

$$\begin{aligned} u_{n+1} &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s^\alpha}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}u_n(x,t)\right)\right]dp\right] \\ &- L_p^{-1}L_s^{-1}\left[\frac{1}{s^\alpha}\int_0^p L_xL_t\left[\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}u_n(x,t)\right)\right]dp\right]. \end{aligned} \tag{19}$$

By Eq.(18) and Eq.(19), we have

$$\begin{aligned} u_1(x,t) &= -L_P^{-1}L_S^{-1}\left[\frac{1}{S^\alpha}\int_0^P L_xL_t\left[\frac{\partial}{\partial x}(xu_{0x}) + \frac{\partial^2}{\partial x\partial t}(xu_{0x})\right]dP\right] \\ &= 4\frac{t^\alpha}{\Gamma(\alpha+1)}. \end{aligned}$$

In the same manner, we obtain that

$$u_2 = 0,\ u_3 = 0, ...$$

By adding the all terms together, we have

$$u(x,t) = u_0 + u_1 + ...$$

Therefore, the exact solution is given by

$$u(x,t) = x^2 + t$$

Example 1.2 *Consider the following nonhomogeneous form of a singular one dimensional pseudo-parabolic fractional equation*

$$\mathtt{D}_t^\alpha u - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) - \frac{1}{x}\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial u}{\partial x}\right) = x^2\frac{t^{1-\alpha}}{\Gamma(2-\alpha)},\quad x \geq 0,\quad t > 0 \tag{20}$$

subject to the initial condition

$$u(x,0) = g(x) = 0. \tag{21}$$

By applying the above steps, we obtain

$$u_0(x,t) = x^2t - 4\frac{t^\alpha}{\Gamma(\alpha+1)} - 4\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \tag{22}$$

$$\begin{aligned} u_1(x,t) &= -L_P^{-1}L_S^{-1}\left[\frac{1}{S^\alpha}\int_0^P L_xL_t\left[\frac{\partial}{\partial x}(xu_{0x}) + \frac{\partial^2}{\partial x\partial t}(xu_{0x})\right]dP\right] \\ &= 4\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 4\frac{t^\alpha}{\Gamma(\alpha+1)}. \end{aligned}$$

In the same manner, we obtain that

$$u_2 = 0,\ u_3 = 0, ...$$

By adding the all terms together, we have

$$u(x,t) = u_0 + u_1 + ...$$

Therefore, the exact solution is given by

$$u(x,t) = x^2 t.$$

Example 1.3 *Consider the following nonhomogeneous form of a singular one dimensional pseudo-parabolic fractional equation*

$$\mathtt{D}_t^\alpha u - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) - \frac{1}{x}\frac{\partial^2}{\partial x \partial t}\left(x\frac{\partial u}{\partial x}\right) = (x^2+1)\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} - 8 - 4t,\quad x \geq 0,\quad t > 0 \tag{23}$$

subject to the initial

$$u(x,0) = g(x) = x^2. \tag{24}$$

By applying the above steps, we obtain

$$u_0(x,t) = x^2 + t - 4\frac{t^\alpha}{\Gamma(\alpha+1)} + x^2 t - 4\frac{t^\alpha}{\Gamma(\alpha+1)} - 4\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}.$$

$$\begin{aligned} u_1 &= -L_P^{-1}L_S^{-1}\left[\frac{1}{S^\alpha}\int_0^P L_x L_t\left[\frac{\partial}{\partial x}(xu_{0x}) + \frac{\partial^2}{\partial x \partial t}(xu_{0x})\right]dP\right] \\ &= 4\frac{t^\alpha}{\Gamma(\alpha+1)} + 4\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 4\frac{t^\alpha}{\Gamma(\alpha+1)} \end{aligned}$$

In the same manner, we obtain that

$$u_2 = 0,\ u_3 = 0, ...$$

By adding the all terms together, we have

$$u(x,t) = u_0 + u_1 + ...$$

Therefore, the exact solution is given by

$$u(x,t) = x^2 + t + x^2 t$$

Example 1.4 *Consider the following nonhomogeneous form of a singular one dimensional pseudo-parabolic fractional equation*

$$\mathtt{D}_t^\alpha u - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) - \frac{1}{x}\frac{\partial^2}{\partial x \partial t}\left(x\frac{\partial u}{\partial x}\right) = x^2 t,\quad x \geq 0,\quad t > 0, \tag{25}$$

subject to the initial condition

$$u(x,0) = g(x) = 0. \tag{26}$$

By applying the above steps, we obtain

$$u_0(x,t) = x^2 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)},$$

$$\begin{aligned} u_1 &= -L_P^{-1}L_S^{-1}\left[\tfrac{1}{S^\alpha}\int_0^P L_x L_t \left[\tfrac{\partial}{\partial x}(xu_{0x}) + \tfrac{\partial^2}{\partial x \partial t}(xu_{0x})\right] dP\right] \\ &= 4\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}. \end{aligned}$$

In the same manner, we obtain that

$$u_2 = 0,\ u_3 = 0, ...$$

By adding the all terms together, we have

$$u(x,t) = u_0 + u_1 + ...$$

Therefore, the exact solution is given by

$$u(x,t) = x^2 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 4\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}.$$

2 Convergence Analysis of the method

Finally, we discuss the convergence analysis of the modified double Laplace decomposition methods for the singular nonlinear one dimensional pseudo- parabolic fractional equation which is given by

$$\mathtt{D}_t^\alpha u - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) - \frac{1}{x}\frac{\partial^2}{\partial x \partial t}\left(x\frac{\partial u}{\partial x}\right) = f(u) \tag{27}$$

Consider the Hilbert space $H = L_\mu^2\left((a,b)\times[0,T]\right)$, defined by the set of applications

$$\left\{(u,v) : (a,b)\times[0,T] \quad with \quad L_s^{-1}L_t^{-1}\left[\frac{1}{s^\alpha}\int_0^p L_x L_t[u(x,t)](p,s)dp\right](x,t) < \infty\right\},$$

the scalar product

$$(u,v)_{L_\mu^2(Q)} = \int_Q xu(x,t)v(x,t)dxdt,$$

with the norm

$$|||u|_H^2 = \int_Q xu^2(x,t)dxdt,$$

where $Q = (a,b)\times[0,t]$. Multiplying both sides of Eq. (27) by x, and we write the equation in the operator form

$$L(u) = x\mathtt{D}_t^\alpha u = \frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^3 u}{\partial x^2 \partial t} - xv\frac{\partial u}{\partial x} + xf(u) \tag{28}$$

For L is hemicontinuous operator, consider the following hypotheses.
(H1) $\quad (L(u) - L(w), u - w) \geq k||u - w||^2,\ k > 0,\ \forall u, w \in H$

(**H2**) Whatever may be $M > 0$, there exist a constant $C(M) > 0$ such that for $u, w \in H$ with $||u|| \leq M$, and $||w|| \leq M$ we have

$$(L(u) - L(w), z_1) \leq C(M)||u - z_1||\, ||w||$$

for every $z_1 \in H$. In the next theorem we follow [1, 8, 10, 11]

Theorem 2.1 *(Sufficient condition of convergence) The modified double Laplace decom- position method applied to the singular nonlinear one dimensional pseudo-parabolic fractional equation Eq. (28) without initial and boundary conditions, converges towards a particular solution.*

Proof First, we verify the convergence hypothesis (H1) for the operator $L(u)$ of Eq. (28). We use the definition of our operator L, and then we have

$$\begin{aligned} L(u) - L(w) &= (\frac{\partial u}{\partial x} - \frac{\partial w}{\partial x}) + (x\frac{\partial^2 u}{\partial x^2} - x\frac{\partial^2 w}{\partial x^2}) + (\frac{\partial^2 u}{\partial x \partial t} - \frac{\partial^2 w}{\partial x \partial t}) + (x\frac{\partial^3 u}{\partial x^2 \partial t}) \\ &\quad -(xv\frac{\partial u}{\partial x} - xv\frac{\partial w}{\partial x}) + x(f(u) - f(w)) \\ &= \frac{\partial}{\partial x}(u - w) + x\frac{\partial^2}{\partial x^2}(u - w) + \frac{\partial^2}{\partial x \partial t}(u - w) \\ &\quad + x\frac{\partial^3}{\partial x^2 \partial t}(u - w) + x(f(u) - f(w)), \end{aligned} \tag{29}$$

Therefore,

$$\begin{aligned} (L(u) - L(w), u - w) &= \left(\frac{\partial}{\partial x}(u - w), u - w\right) + \left(x\frac{\partial^2}{\partial x^2}(u - w), u - w\right) \\ &\quad + \left(\frac{\partial^2}{\partial x \partial t}(u - w), u - w\right) + \left(x\frac{\partial^3}{\partial x^2 \partial t}(u - w), u - w\right) \\ &\quad - \left(xv\frac{\partial}{\partial x}(u - w), u - w\right) + (x(f(u) - f(w)), u - w) \end{aligned} \tag{30}$$

According to the properties of the differential operators $\frac{\partial}{\partial x}, \frac{\partial^2}{\partial x \partial t}$ in H, then there exist constants $\alpha, \beta, \delta, \gamma, k > 0$ such that

$$\left(\frac{\partial}{\partial x}(u - w), u - w\right) \geq \alpha||u - w||^2 \tag{31}$$

$$\begin{aligned} -\left(x\frac{\partial^2}{\partial x^2}(u - w), u - w\right) &\leq |x|\, ||\tfrac{\partial^2}{\partial x^2}(u - w)||\, ||u - w|| \\ &\leq b\beta||u - w||^2 \\ &\Longleftrightarrow \end{aligned}$$

$$\left(x\frac{\partial^2}{\partial x^2}(u - w), u - w\right) \geq -b\beta||u - w||^2, \tag{32}$$

$$\left(x\frac{\partial^2}{\partial x \partial t}(u - w), u - w\right) \geq \gamma||u - w||^2, \tag{33}$$

and

$$\begin{array}{rcl} -\left(x\dfrac{\partial^3}{\partial x^2\partial t}(u-w),u-w\right) & \leq & |x|\;||\frac{\partial^3}{\partial x^2\partial t}(u-w)||\;||u-w|| \\ & \leq & bk||u-w||^2 \\ & \Longleftrightarrow & \end{array}$$

$$\left(x\frac{\partial^3}{\partial x^2\partial t}(u-w),u-w\right)\geq -bk||u-w||^2. \tag{34}$$

According to the Schwartz inequality, we get

$$\begin{array}{rcl} \left(xv\dfrac{\partial}{\partial x}(u-w),u-w\right) & \leq & |x|\;||v||\;||\dfrac{\partial}{\partial x}(u-w)||\;||u-w|| \\ & \leq & bM\alpha||u-w||^2. \end{array}$$

Hence,

$$-\left(xv\frac{\partial}{\partial x}(u-w),u-w\right)\geq -bM\alpha||u-w||^2 \tag{35}$$

By Cauchy Schwartz inequality, where $\sigma>0$ and f is Lipschitzian function, we have

$$\begin{array}{rcl} (-xf(u)-f(w),u-w) & \leq & |x|\;||f(u)-f(w)||\;||u-w|| \\ & \leq & b||f(u)-f(w)||\;||u-w|| \\ & \leq & b\sigma||u-w||^2 \\ & \Longleftrightarrow & \end{array}$$

$$(xf(u)-f(w),u-w)\geq -b\sigma||u-w||^2. \tag{36}$$

Substituting Eq.(31), Eq.(32), Eq.(33), Eq.(34), Eq.(35) and Eq.(36) into equation Eq.(30), gives

$$(L(u)-L(w),u-w)\geq(\alpha-b\beta+\gamma-bk-b\sigma-bM\alpha)\,||u-w||^2$$

$$(L(u)-L(w),u-w)\geq k_1||u-w||^2,$$

where $k_1=\alpha-b\beta+\gamma-bk-b\sigma-bM\alpha>0$. Hence hypothesis (H1) holds. Now, we verify the convergence hypothesis $(H2)$ for the operators $L(u)$. For every $M>0$, there exists a constant $C(M)>0$ such that for all $u,w\in H$ with $||u||\leq M$, we have

$$(L(u)-L(w),z_1)\leq C(M)||u-w||\;||z_1||$$

for every $z_1\in H$. For that we have,

$$\begin{array}{rcl} (L(u)-L(w),z_1) & = & \left(\dfrac{\partial}{\partial x}(u-w),z_1\right)+\left(x\dfrac{\partial^2}{\partial x^2}(u-w),z_1\right) \\ & & +\left(\dfrac{\partial^2}{\partial x\partial t}(u-w),z_1\right)+\left(x\dfrac{\partial^3}{\partial x^2\partial t}(u-w),z_1\right) \\ & & -\left(xv\dfrac{\partial}{\partial x}(u-w),z_1\right)+(x(f(u)-f(w)),z_1) \end{array}$$

By the Cauchy Schwartz inequality and the fact that$u$ and w are bounded, we obtain

$$\left(\frac{\partial}{\partial x}(u-w), z_1\right) \leq \alpha_1 ||u-w||\ ||z_1||,$$

$$\left(x\frac{\partial^2}{\partial x^2}(u-w), z_1\right) \leq b\beta_1 ||u-w||\ ||z_1||,$$

$$\left(\frac{\partial^2}{\partial x \partial t}(u-w), z_1\right) \leq \gamma_1 ||u-w||\ ||z_1||,$$

$$\left(x\frac{\partial^3}{\partial x^2 \partial t}(u-w), z_1\right) \leq bk_1 ||u-w||\ ||z_1||,$$

$$\begin{array}{rcl} -\left(xv\frac{\partial}{\partial x}(u-w), z_1\right) & \leq & \alpha_1 |x\ ||v|||\ ||u-w||\ ||z_1|| \\ & \leq & b\alpha_1 M ||u-v||\ ||z_1||, \end{array}$$

$$(x(f(u)-f(w)), z_1) \leq b\sigma_1 ||u-w||\ ||z_1||,$$

where the constants $\alpha_1, \beta_1, \sigma_1, \gamma_1$ and k_1 are positive. Then we have

$$\begin{array}{rcl} (L(u)-L(w), z_1) & \leq & (\alpha_1 + b\beta_1 + \gamma_1 + bk_1 + b\alpha_1 M + b\sigma_1)||u-w||\ ||z_1|| \\ & = & C_1(M)||u-w||\ ||z_1||, \end{array}$$

where $C_1(M) = \alpha_1 + b\beta_1 + \gamma_1 + bk_1 + b\alpha_1 M + b\sigma_1$.

■

Conclusion

In this paper, we have proposed new modified double Laplace decomposition method to solve singular one dimensional pseudo-parabolic fractinal equation. The efficiency and accuracy of the present scheme are validated through examples. This method is much simpler and more efficient in the study of linear and nonlinear PDEs.

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