Locally conformally symplectic structures and Lie-Rinehart-Jacobi structures on Weil bundle

Servais Cyr GATSE*, Côme Chancel LIKOUKA†

Abstract

This paper is dedicated to a study of a locally conformally symplectic $\mathbb{A}$-structures in the case where the set, $\mathbb{A}$, is a Weil algebra. If M is a smooth manifold and $M^{\mathbb{A}}$ the associated Weil bundle, we shown that the $C^{\infty}\left(M^{\mathbb{A}},\mathbb{A}\right)$-module of first-order differential operators admits a Lie-Rinehart-Jacobi $\mathbb{A}$-structure.

Keywords: locally conformally symplectic manifold, Weil bundle, Lie-Rinehart algebra.

MSC(2010): 58A20, 58A32, 17D63, 13N10.

1 Introduction

1.1 first-order differential operators

Consider a commutative algebra Λ with unit 1_{Λ} over a commutative field $\mathbb{K}$ with characteristic zero. let $\mathbb{E}$ be an Λ-module. The action

$$\Lambda\times\mathbb{E}\longrightarrow\mathbb{E},(a,x)\longmapsto a\cdot x$$

denotes a multiplication by x.

A differential operator of order $\leq k,(k\in\mathbb{N})$, from Λ into $\mathbb{E}$, is a $\mathbb{K}$-linear map

$$\varphi:\Lambda\longrightarrow\mathbb{E}$$

such that for any $a\in\Lambda$, the map

$$\Lambda\longrightarrow\mathbb{E},b\longmapsto\varphi\left(a\cdot b\right)-a\cdot\varphi\left(b\right)$$

is a differential operator of order $\leq(k-1)$ from Λ into $\mathbb{E}$. Thus a $\mathbb{K}$-linear map

$$\varphi:\Lambda\longrightarrow\mathbb{E}$$

*Université Marien Ngouabi, BP: 69, Brazzaville, Congo. E-mail: servais.gatse@umng.cg

†Université Marien Ngouabi, BP: 69, Brazzaville, Congo. E-mail: côme.likouka@umng.cg

is a first-order differential operator from Λ into $\mathbb{E}$ if and only if for any $a, b \in \Lambda$,

$$\varphi(a \cdot b) = \varphi(a) \cdot b + a \cdot \varphi(b) - a \cdot b \cdot \varphi(1_\Lambda). \tag{1}$$

When $\varphi(1_\Lambda) = 0$, we have the usual notion of derivation from Λ into $\mathbb{E}$. Moreover the following assertions are equivalent:

1. A $\mathbb{K}$-linear map $\varphi\colon \Lambda \longrightarrow \mathbb{E}$ is a first-order differential operator;
2. A $\mathbb{K}$-linear map $\varphi - R_{\varphi(1_\Lambda)}\colon \Lambda \longrightarrow \mathbb{E}, a \longmapsto \varphi(a) - a \cdot \varphi(1_\Lambda)$ is a derivation.

1.2 $\mathbb{A}$-manifolds

In this part, M denotes a paracompact and connected smooth manifold, a first-order differential operator of the algebra of numerical functions of class C^∞ on M, will be said differential operator on M and the set, $\mathcal{D}(M)$, of this applications is a $C^\infty(M)$-module and admits a real Lie algebra structure. The set, $\mathbb{A}$, designs a Weil algebra i.e. a real commutative algebra with unit, of finite dimension, and with an unique maximal ideal $\mathfrak{M}$ of codimension 1 over $\mathbb{R}$. The quotient $\mathbb{A}/\mathfrak{M}$ is a field. We deduce that

$$\mathbb{A} = \mathbb{R} \bigoplus \mathfrak{M}. \tag{2}$$

In this case, there exists an integer h, called the height of $\mathbb{A}$ such that $\mathfrak{M}^{h+1} = (0)$ and $\mathfrak{M}^h \neq (0)$. For example, the algebra of dual numbers $\mathbb{D} = \mathbb{R}[T] / (T^2)$ is a Weil algebra with height 1.

A near point of $p \in M$ of kind $\mathbb{A}$ is a morphism of algebras

$$\xi : C^\infty(M) \longrightarrow \mathbb{A}$$

such that

$$\xi(f) = f(p)\,[mod \mathfrak{M}]$$

for any $f \in C^\infty(M)$.

We recall that $M^\mathbb{A}$ is a manifold of infinitely near points on M of kind $\mathbb{A}$ or simply the Weil bundle of kind $\mathbb{A}$ [4]. For any $f \in C^\infty(M)$, the map

$$f^\mathbb{A} : M^\mathbb{A} \longrightarrow \mathbb{R}^\mathbb{A} \cong \mathbb{A}, \xi \longmapsto \xi(f)$$

is smooth. In [2], one showns that the set, $C^\infty(M^\mathbb{A}, \mathbb{A})$, of smooth functions on $M^\mathbb{A}$ with values in $\mathbb{A}$, is a commutative algebra over $\mathbb{A}$ with unit. Moreover the map

$$C^\infty(M) \longrightarrow C^\infty(M^\mathbb{A}, \mathbb{A}), f \longmapsto f^\mathbb{A}$$

is a monomorphism of algebras which satisfies for any $f, g \in C^\infty(M)$ and for any $\lambda \in \mathbb{R}$

$$\begin{aligned} (f+g)^\mathbb{A} &= f^\mathbb{A} + g^\mathbb{A}; \\ (\lambda f)^\mathbb{A} &= \lambda f^\mathbb{A}; \\ (f \cdot g)^\mathbb{A} &= f^\mathbb{A} \cdot g^\mathbb{A}. \end{aligned}$$

If $(\mathbf{U}, \beta)$ is a local chart of M with local coordinates $(x_1, ..., x_{2n})$ the map

$$\beta^{\mathbb{A}} : \mathbf{U}^{\mathbb{A}} \longrightarrow \mathbb{A}^{2n}, \xi \longmapsto (\xi(x_1), ..., \xi(x_{2n}))$$

is a bijection from $\mathbf{U}^{\mathbb{A}}$ into an open of $\mathbb{A}^{2n}$. Thus $M^{\mathbb{A}}$ is an $\mathbb{A}$-manifold of dimension $2n$.

2 Differential operators on $M^{\mathbb{A}}$

A differential operators on $M^{\mathbb{A}}$ is a $\mathbb{R}$-linear map $C^{\infty}\left(M^{\mathbb{A}}\right) \longrightarrow C^{\infty}\left(M^{\mathbb{A}}\right)$ fulfiling (1).

Proposition 1 *There is an equivalence between the following statements:*

1. *A differential operator on $M^{\mathbb{A}}$ is a differential operator of $C^{\infty}\left(M^{\mathbb{A}}\right)$;*
2. *A differential operator on $M^{\mathbb{A}}$ is a linear map*

$$\partial : C^{\infty}(M) \longrightarrow C^{\infty}\left(M^{\mathbb{A}}, \mathbb{A}\right)$$

such that

$$\partial(f \cdot g) = \partial(f) \cdot g^{\mathbb{A}} + f^{\mathbb{A}} \cdot \partial(g) - f^{\mathbb{A}} \cdot g^{\mathbb{A}} \cdot \partial\left(1_{C^{\infty}(M)}\right) \tag{3}$$

for any $f, g \in C^{\infty}(M)$;

3. *A differential operator on $M^{\mathbb{A}}$ is a differential operator of $C^{\infty}\left(M^{\mathbb{A}}, \mathbb{A}\right)$ which is $\mathbb{A}$-linear.*

Proof. We use the same technics that in [2]. ■

We denote, $\mathcal{D}_{\mathbb{A}}\left(M^{\mathbb{A}}\right)$, the $C^{\infty}\left(M^{\mathbb{A}}, \mathbb{A}\right)$-module of $\mathbb{A}$-linear differential operators. The skew-symmetric and $\mathbb{A}$-linear map

$$[\cdot, \cdot] : \mathcal{D}_{\mathbb{A}}\left(M^{\mathbb{A}}\right) \times \mathcal{D}_{\mathbb{A}}\left(M^{\mathbb{A}}\right) \longrightarrow \mathcal{D}_{\mathbb{A}}\left(M^{\mathbb{A}}\right), (\varphi, \psi) \longmapsto \varphi \circ \psi - \psi \circ \varphi$$

defines a $\mathbb{A}$-Lie algebra structure on $\mathcal{D}_{\mathbb{A}}\left(M^{\mathbb{A}}\right)$ and we verify that

$$\left[\varphi, f^{\mathbb{A}} \cdot \psi\right] = \left(\varphi\left(f^{\mathbb{A}}\right) - f^{\mathbb{A}} \cdot \varphi\left(1_{C^{\infty}(M^{\mathbb{A}}, \mathbb{A})}\right)\right) \cdot \psi + f^{\mathbb{A}} \cdot [\varphi, \psi] \tag{4}$$

for any $f^{\mathbb{A}} \in C^{\infty}\left(M^{\mathbb{A}}, \mathbb{A}\right)$.

3 Differential $\mathbb{A}$-forms

Let ${}^{k}_{sks}\left(\mathcal{D}_{\mathbb{A}}\left(M^{\mathbb{A}}\right), C^{\infty}\left(M^{\mathbb{A}}, \mathbb{A}\right)\right) = \Omega^{k}\left(M^{\mathbb{A}}, \mathbb{A}\right)$ be the $C^{\infty}\left(M^{\mathbb{A}}, \mathbb{A}\right)$-module of skew-symmetric multilinear forms of degree k ($k \in \mathbb{N}$) on $\mathcal{D}_{\mathbb{A}}\left(M^{\mathbb{A}}\right)$.

We have

$$\Omega^{0}\left(M^{\mathbb{A}}, \mathbb{A}\right) = C^{\infty}\left(M^{\mathbb{A}}, \mathbb{A}\right).$$

One denotes

$$\Omega\left(M^{\mathbb{A}}, \mathbb{A}\right) = \bigoplus_{k=0}^{2n} \Omega^{k}\left(M^{\mathbb{A}}, \mathbb{A}\right).$$

Remark 2 *This algebra is canonically isomorph to* $\mathbb{A} \bigotimes \Omega\left(M^{\mathbb{A}}\right)$.

Theorem 3 *If η is a differential form of degree k on M, then there exists an unique differential $\mathbb{A}$-form on $M^{\mathbb{A}}$ of degree k such that*

$$\eta^{\mathbb{A}}\left(f_1^{\mathbb{A}} \cdot \theta_1^{\mathbb{A}}, ..., f_k^{\mathbb{A}} \cdot \theta_k^{\mathbb{A}}\right) = \left(f_1, ..., f_k\right)^{\mathbb{A}} \cdot \left[\eta\left(\theta_1, ..., \theta_k\right)\right]^{\mathbb{A}} \quad (5)$$

for any differential operators $\theta_1, ..., \theta_k \in \mathcal{D}(M)$ and for any $f_1, ..., f_k \in C^\infty(M)$. Moreover, for any η_1, η_2 elements of $\Omega(M)$, we have

$$\begin{aligned} \left(\eta_1 + \eta_2\right)^{\mathbb{A}} &= \eta_1^{\mathbb{A}} + \eta_2^{\mathbb{A}}; \\ \left(\eta_1 \wedge \eta_2\right)^{\mathbb{A}} &= \eta_1^{\mathbb{A}} \wedge \eta_2^{\mathbb{A}}. \end{aligned}$$

Proof. It is obvious. ■

The map $\Omega(M) \longrightarrow \Omega\left(M^{\mathbb{A}}, \mathbb{A}\right), \eta \longmapsto \eta^{\mathbb{A}}$ is a morphism of real graded algebras.

4 Locally conformally symplectic structures on Weil bundle

Recall that a locally conformally symplectic manifold is a triplet (M, α, ω) such that

- M is a smooth C^∞-manifold of dimension $2n$;
- $\alpha : \mathfrak{X}(M) \longrightarrow C^\infty(M), X \longmapsto \alpha(X)$ satisfying $d\alpha = 0$. Such as α called the Lee 1-form;
- $\omega : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow C^\infty(M), (X, Y) \longmapsto \omega(X, Y)$ satisfying $d\omega = -\alpha \wedge \omega$. For more details see [1].

We verify that for any X a vector field on M, then the map

$$\rho_\alpha(X) : C^\infty(M) \longrightarrow C^\infty(M), f \longmapsto X(f) + f \cdot \alpha(X)$$

is a differential operator. Moreover the map

$$\rho_\alpha : \mathfrak{X}(M) \longrightarrow \mathcal{D}(M), X \longmapsto X + \alpha(X)$$

is a representation and $\mathfrak{X}(M)$ admits a symplectic Lie-Rinehart-Jacobi structure. We denote d_α instead d_{ρ_α} the cohomology operator associated with the representation ρ_α.

For $X \in \mathfrak{X}\left(M^{\mathbb{A}}\right)$, considered as derivation of $C^\infty(M)$ into $C^\infty\left(M^{\mathbb{A}}, \mathbb{A}\right)$ in sense of [2], the application

$$\rho_{\alpha^{\mathbb{A}}} : \mathfrak{X}\left(M^{\mathbb{A}}\right) \longrightarrow \mathcal{D}_{\mathbb{A}}\left(M^{\mathbb{A}}\right), X \longmapsto \rho_{\alpha^{\mathbb{A}}}(X)$$

is $C^\infty\left(M^{\mathbb{A}}, \mathbb{A}\right)$-linear and is a morphism of $\mathbb{A}$-Lie algebras.

Proposition 4 *The application*

$$d^{\mathbb{A}}_{\alpha^{\mathbb{A}}} : \Omega\left(M^{\mathbb{A}}, \mathbb{A}\right) \longrightarrow \Omega\left(M^{\mathbb{A}}, \mathbb{A}\right)$$

is $\mathbb{A}$*-linear and satisfies* $d^{\mathbb{A}}_{\alpha^{\mathbb{A}}}\left(\eta^{\mathbb{A}}\right) = \left(d_\alpha \eta\right)^{\mathbb{A}}$.

Proof. We verify that $d^{\mathbb{A}}_{\alpha^{\mathbb{A}}}$ is $\mathbb{A}$-linear. If $\eta \in \Omega^k(M)$, for any $\theta_1, ..., \theta_{k+1} \in \mathcal{D}(M)$, we get

$$\begin{aligned}\left[d^{\mathbb{A}}_{\alpha^{\mathbb{A}}}\left(\eta^{\mathbb{A}}\right)\right]\left(\theta^{\mathbb{A}}_1, ..., \theta^{\mathbb{A}}_{k+1}\right) &= \left[\left(d_\alpha \eta\right)\left(\theta_1, ..., \theta_{k+1}\right)\right]^{\mathbb{A}} \\ &= \left(d_\alpha \eta\right)^{\mathbb{A}}\left(\theta^{\mathbb{A}}_1, ..., \theta^{\mathbb{A}}_{k+1}\right).\end{aligned}$$

We deduce the assertion. ■

Theorem 5 *When the triplet* (M, α, ω) *is a locally conformally symplectic manifold, then* $\left(M^{\mathbb{A}}, \alpha^{\mathbb{A}}, \omega^{\mathbb{A}}\right)$ *is also a locally conformally symplectic* $\mathbb{A}$*-manifold.*

Proof. The theorem follows from the above proposition and see [2], [3]. ■

5 Symplectic Lie-Rinehart-Jacobi $\mathbb{A}$-algebra on Weil bundle

Proposition 6 *If* φ *is an element of* $\mathcal{D}_{\mathbb{A}}\left(M^{\mathbb{A}}\right)$ *and if* $\widetilde{f}$ *is an element of* $C^\infty\left(M^{\mathbb{A}}, \mathbb{A}\right)$*, we have*

$$\left[\varphi, \widetilde{f}\right] = \varphi\left(\widetilde{f}\right) - \widetilde{f} \cdot \varphi\left(1_{C^\infty(M^{\mathbb{A}}, \mathbb{A})}\right) \tag{6}$$

and the restriction of this bracket to $C^\infty\left(M^{\mathbb{A}}, \mathbb{A}\right)$ *is zero.*

Proof. It is obvious. ■

5.1 Lie-Rinehart $\mathbb{A}$-algebra structure on $\mathcal{D}_{\mathbb{A}}\left(M^{\mathbb{A}}\right)$

We denote δ the cohomology operator associated with the representation

$$id^{\mathbb{A}} : \mathcal{D}_{\mathbb{A}}\left(M^{\mathbb{A}}\right) \longrightarrow \mathcal{D}_{\mathbb{A}}\left(M^{\mathbb{A}}\right).$$

Proposition 7 *If*

$$\widetilde{\alpha^{\mathbb{A}}} : \mathcal{D}_{\mathbb{A}}\left(M^{\mathbb{A}}\right) \longrightarrow C^\infty\left(M^{\mathbb{A}}, \mathbb{A}\right)$$

is a linear $\mathbb{A}$*-form, then the* $C^\infty\left(M^{\mathbb{A}}, \mathbb{A}\right)$*-linear map*

$$\partial_{\widetilde{\alpha^{\mathbb{A}}}} : \mathcal{D}_{\mathbb{A}}\left(M^{\mathbb{A}}\right) \longrightarrow \mathcal{D}_{\mathbb{A}}\left(M^{\mathbb{A}}\right), \varphi \longmapsto \varphi + \widetilde{\alpha^{\mathbb{A}}}(\varphi)$$

is a representation of $\mathbb{A}$*-Lie algebra if and only if*

$$\delta\widetilde{\alpha^{\mathbb{A}}} = \left(\delta 1_{C^\infty(M^{\mathbb{A}}, \mathbb{A})}\right) \wedge \widetilde{\alpha^{\mathbb{A}}}.$$

Proof. It is clear that

$$\left[\partial_{\widetilde{\alpha^{\mathbb{A}}}}(\varphi), \partial_{\widetilde{\alpha^{\mathbb{A}}}}(\psi)\right] = \partial_{\widetilde{\alpha^{\mathbb{A}}}}([\varphi,\psi]) + \left(\delta\widetilde{\alpha^{\mathbb{A}}} - \left(\delta 1_{C^{\infty}(M^{\mathbb{A}},\mathbb{A})}\right)\wedge\widetilde{\alpha^{\mathbb{A}}}\right)(\varphi,\psi).$$

Hence the result following. ■

Theorem 8 *When $M^{\mathbb{A}}$ is a smooth $\mathbb{A}$-manifold, then*

1. *A Lie-Rinehart $\mathbb{A}$-algebra structure on $\mathcal{D}_{\mathbb{A}}\left(M^{\mathbb{A}}\right)$ is always of the form $\left(\mathcal{D}_{\mathbb{A}}\left(M^{\mathbb{A}}\right), \partial_{\widetilde{\alpha^{\mathbb{A}}}}\right)$;*
2. *The equation $\delta\widetilde{\alpha^{\mathbb{A}}} = \left(\delta 1_{C^{\infty}(M^{\mathbb{A}},\mathbb{A})}\right)\wedge\widetilde{\alpha^{\mathbb{A}}}$ is equivalent to $\widetilde{\alpha^{\mathbb{A}}}\left(1_{C^{\infty}(M^{\mathbb{A}},\mathbb{A})}\right)$ is a constante and $\widetilde{\alpha^{\mathbb{A}}}/\mathfrak{X}\left(M^{\mathbb{A}}\right)$ is $d^{\mathbb{A}}$-closed.*

5.2 Lie-Rinehart-Jacobi $\mathbb{A}$-algebra structure on $\mathcal{D}_{\mathbb{A}}\left(M^{\mathbb{A}}\right)$

We consider the results of the above theorem and we denote $\delta_{\widetilde{\alpha^{\mathbb{A}}}}$ the cohomology operator associated with the representation $\partial_{\widetilde{\alpha^{\mathbb{A}}}}$. For any $\eta \in^{k}_{sks}\left(\mathcal{D}_{\mathbb{A}}\left(M^{\mathbb{A}}\right), C^{\infty}\left(M^{\mathbb{A}},\mathbb{A}\right)\right)$, we verify that

$$\delta_{\widetilde{\alpha^{\mathbb{A}}}}\eta = \delta\eta + \widetilde{\alpha^{\mathbb{A}}}\wedge\eta. \tag{7}$$

Theorem 9 *The $C^{\infty}\left(M^{\mathbb{A}},\mathbb{A}\right)$-module $\mathcal{D}_{\mathbb{A}}\left(M^{\mathbb{A}}\right)$ admits a symplectic Lie-Rinehart-Jacobi $\mathbb{A}$-algebra is equivalent to the existence of a $C^{\infty}\left(M^{\mathbb{A}},\mathbb{A}\right)$-linear form $\widetilde{\alpha^{\mathbb{A}}}$ and a nondegenerate skew-symmetric $C^{\infty}\left(M^{\mathbb{A}},\mathbb{A}\right)$-bilinear form $\widetilde{\omega^{\mathbb{A}}}$ such that*

1. $$\delta\widetilde{\alpha^{\mathbb{A}}} = \left(\delta 1_{C^{\infty}(M^{\mathbb{A}},\mathbb{A})}\right)\wedge\widetilde{\alpha^{\mathbb{A}}};$$
2. $$\delta\widetilde{\omega^{\mathbb{A}}} = -\widetilde{\alpha^{\mathbb{A}}}\wedge\widetilde{\omega^{\mathbb{A}}}.$$

References

[1] Gatsé, S.C.,(2016). Hamiltonian Vector Field on Locally Conformally Symplectic Manifold, International Mathematical Forum, vol. 11, no. 19, (pp. 933-941). http://dx.doi.org/10.12988/imf.2016.6666

[2] Bossoto, B.G.R., Okassa, E.,(2012). *A-poisson structures on Weil bundles*, Int., J. Contemp. Math. Sciences, Vol. 7(16), (pp. 785-803).

[3] Bossoto, B.G.R., Okassa, E.,(2008). Champs de vecteurs et formes différentielles sur une variété des points proches, Arch. Math (Brno), 44, (pp. 159-171).

[4] Weil, A.,(1953).*Théorie des points proches sur les variétés différentiables*, Colloq. Géom. Diff. Strasbourg , (pp. 111-117).