Multifractal analysis of local entropies for amenable group actions

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Abstract. In this paper, we give the multifractal analysis of the weighted local entropies for arbitrary invariant measures for amenable group actions.

Keywords and phrases: amenable group actions, multifractal analysis .

1 Background and Introduction.

Let (X, d, T) be a dynamical system, where (X, d) is a compact metric space and $T : X \to X$ is a continuous map. The set $M(X)$ of all Borel probability measures is compact under the weak* topology. Denote by $M(X, T) \subset M(X)$ the subset of all T-invariant measures and $E(X, T) \subset M(X, T)$ the subset of all ergodic measures. Multifractal analysis is concerned with the study of pointwise dimension of a Borel measure μ (provided the limit exists):

$$d_\mu(x) = \lim_{\epsilon \to 0} \frac{\log \mu(B(x, \epsilon))}{\log \epsilon},$$

where $B(x, \epsilon)$ is an open ϵ-neighborhood of x. Set

$$X_\alpha := \{x \in X : d_\mu(x) = \alpha\}.$$

The purpose is to describe the set X_α. It is worthwhile to mention that the multifractal analysis of Birkhoff average is closely related to the pointwise dimension of the Borel

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2010 Mathematics Subject Classification: 37B40, 37C45

measure. We refer the reader to the references [4, 14, 20, 21, 22]. Here, we introduce the general form of Pesin's multifractal formalism in [12], or [2] as follows. Consider a function $g : Y \to [-\infty, +\infty]$ in a subset Y of X. The level set

$$K_\alpha^g = \{x \in Y : g(x) = \alpha\}$$

are pairwise disjoint, and we obtain a *multifractal decomposition* of X given by

$$X = (X \setminus Y) \cup \bigcup_{\alpha \in [-\infty, +\infty]} K_\alpha^g.$$

Let G be a function defined in the set of subsets of X. The *multifractal spectrum* : $\mathcal{F} : [-\infty, +\infty] \to \mathbb{R}$ of the pair (g, G) is defined by

$$\mathcal{F}(\alpha) = G(K_\alpha^g),$$

where g may denote the Birkhoff averages, Lyapunov exponents, pointwise dimension or local entropies and G may denote the topological entropy, topological pressure or Hausdorff dimension.

Let (X, G) be a $G-$action topological dynamical system, where X is a compact metric space with metric d and G a topological group. In this paper, we assume G is a discrete countable amenable group. Recall that a group G is *amenable* if it admits a left invariant mean(a state on $\ell^\infty(G)$ which is invariant under left translation by G). This is equivalent to the existence of a sequence of finite subsets $\{F_n\}$ of G which are asymptotically invariant, i.e.,

$$\lim_{n \to +\infty} \frac{|F_n \vartriangle gF_n|}{|F_n|} = 0, \text{ for all } g \in G.$$

Such sequences are called Følner sequences. For the detail of amenable group actions, one may refer to Ornstein and Weisss pioneering paper [11].

The topological entropy of (X, G) is defined in the following way.

Let $\mathcal{U}$ be an open cover of X, the topological entropy of $\mathcal{U}$ is

$$h_{top}(G, \mathcal{U}) = \lim_{n \to +\infty} \frac{1}{|F_n|} \log N\big(\mathcal{U}_{F_n}\big),$$

where $\mathcal{U}_{F_n} = \bigvee_{g \in F_n} g^{-1}\mathcal{U}$. It is shown that $h_{top}(G, \mathcal{U})$ is not dependent on the choice of the Følner sequences $\{F_n\}$. And the topological entropy of (X, G) is

$$h_{top}(X, G) = \sup_{\mathcal{U}} h_{top}(G, \mathcal{U}),$$

where the supremum is taken over all the open covers of X.

Bowen [1] introduced a definition of topological entropy on subsets inspired by Hausdorff dimension. For an amenable group action dynamical system (X, G), we define the Bowen topological entropy in the following way.

Let $\{F_n\}$ be a Følner sequence in G and $\mathcal{U}$ be a finite open cover of X. Denote $diam(\mathcal{U}) := \max\{diam(U) : U \in \mathcal{U}\}$. For $n \geq 1$ we denote by $\mathcal{W}_{F_n}(\mathcal{U})$ the collection of families $\mathbf{U} = \{U_g\}_{g \in F_n}$ with $U_g \in \mathcal{U}$. For $\mathbf{U} \in \mathcal{W}_{F_n}(\mathcal{U})$ we call the integer $m(\mathbf{U}) = |F_n|$ the length of $\mathbf{U}$ and define

$$\begin{aligned} X(\mathbf{U}) &= \bigcap_{g \in F_n} g^{-1} U_g \\ &= \{x \in X : gx \in U_g \text{ for } g \in F_n\}. \end{aligned}$$

For $Z \subset X$, we say that $\Lambda \subset \bigcup_{n \geq 1} \mathcal{W}_{F_n}(\mathcal{U})$ covers Z if $\bigcup_{\mathbf{U} \in \Lambda} X(\mathbf{U}) \supset Z$. For $s \in \mathbf{R}$, define

$$\mathcal{M}(Z, \mathcal{U}, N, s, \{F_n\}) = \inf_{\Lambda} \{\sum_{\mathbf{U} \in \Lambda} \exp(-sm(\mathbf{U}))\}$$

and the infimum is taken over all $\Lambda \subset \bigcup_{j \geq N} \mathcal{W}_{F_j}(\mathcal{U})$ that covers Z. We note that $\mathcal{M}(\cdot, \mathcal{U}, N, s, \{F_n\})$ is a finite outer measure on X, and

$$\mathcal{M}(Z, \mathcal{U}, N, s, \{F_n\}) = \inf\{\mathcal{M}(C, \mathcal{U}, N, s, \{F_n\}) : C \text{ is an open set that contains } Z\}.$$

$\mathcal{M}(Z, \mathcal{U}, N, s, \{F_n\})$ increases as N increases. Define

$$\mathcal{M}(Z, \mathcal{U}, s, \{F_n\}) = \lim_{N \to +\infty} \mathcal{M}(Z, \mathcal{U}, N, s, \{F_n\})$$

and

$$\begin{aligned} h_{top}^{B}(\{F_n\}, Z, \mathcal{U}) &= \inf\{s : \mathcal{M}(Z, \mathcal{U}, s, \{F_n\}) = 0\} \\ &= \sup\{s : \mathcal{M}(Z, \mathcal{U}, s, \{F_n\}) = +\infty\}. \end{aligned}$$

Set

$$h_{top}^{B}(\{F_n\}, Z) = \sup_{\mathcal{U}} h_{top}^{B}(\{F_n\}, Z, \mathcal{U}),$$

where $\mathcal{U}$ runs over all finite open covers of Z. We call $h_{top}^{B}(\{F_n\}, Z)$ the Bowen topological entropy of (X, G) restricted to Z or the Bowen topological entropy of Z(w.r.t. the Følner sequence $\{F_n\}$).

Similar to the Bowen topological entropy of subsets for $\mathbb{Z}-$actions(see, for example, Pesin [12]), it is easy to show that

$$h_{top}^{B}(\{F_n\}, Z) = \lim_{diam(\mathcal{U}) \to 0} h_{top}^{B}(\{F_n\}, Z, \mathcal{U}).$$

So the Bowen topological entropy can be defined in an alternative way.

For a finite subset F in G, we denote by

$$\begin{aligned} B_F(x, \epsilon) &= \{y \in X : d_F(x, y) < \epsilon\} \\ &= \{y \in X : d(gx, gy) < \epsilon, \text{ for any } g \in F\}. \end{aligned} \tag{1.1}$$

Definition 1.1. *For $Z \subseteq X, s \geq 0, N \in \mathbf{N}$, $\{F_n\}$ a Følner sequence in G and $\epsilon > 0$, define*

$$\mathcal{M}(Z, N, \epsilon, s, \{F_n\}) = \inf \sum_i \exp(-s|F_{n_i}|),$$

where the infimum is taken over all finite or countable families $\{B_{F_{n_i}}(x_i, \epsilon)\}$ such that $x_i \in X, n_i \geq N$ and $\bigcup_i B_{F_{n_i}}(x_i, \epsilon) \supseteq Z$. The quantity $\mathcal{M}(Z, N, \epsilon, s, \{F_n\})$ does not decrease as N increases and ϵ decreases, hence the following limits exists:

$$\mathcal{M}(Z, \epsilon, s, \{F_n\}) = \lim_{N \to +\infty} \mathcal{M}(Z, N, \epsilon, s, \{F_n\}), \mathcal{M}(Z, s, \{F_n\}) = \lim_{\epsilon \to 0} \mathcal{M}(Z, \epsilon, s, \{F_n\}).$$

Bowen topological entropy $h^B_{top}(Z, \{F_n\})$ can be equivalently defined as the critical value of the parameter s, where $\mathcal{M}(Z, s, \{F_n\})$ jumps from $+\infty$ to 0, i.e.,

$$\mathcal{M}(Z, s, \{F_n\}) = \begin{cases} 0, & s > h^B_{top}(Z, \{F_n\}), \\ +\infty, & s < h^B_{top}(Z, \{F_n\}). \end{cases}$$

In [1] Bowen showed that $h_{top}(X, T) = h^B_{top}(X, T)$ for any compact metric dynamical system (X, T). A Følner sequence $\{F_n\}$ in G is said to be *tempered* (see Shulman [17]) if there exists a constant C which is independent of n such that

$$|\bigcup_{k<n} F_k^{-1} F_n| \leq C|F_n|, \text{ for any } n. \tag{1.2}$$

In Lindenstrauss [6], (1.2) is also called **Shulman Condition**.

The increasing condition

$$\lim_{n \to +\infty} \frac{|F_n|}{\log n} = \infty. \tag{1.3}$$

In [23], the authors prove Brin-Katok's entropy formula [3] for amenable group action dynamical systems. The statement of this formula is the following.

Theorem 1.1 (Brin-Katok's entropy formula: ergodic case)**.** *Let (X, G) be a compact metric $G-$action topological dynamical system and G a discrete countable amenable group. Let μ be a $G-$ergodic Borel probability measure on X and $\{F_n\}$ a tempered Følner sequence in G with the increasing condition* (1.3)*, then for μ almost everywhere $x \in X$,*

$$\begin{aligned} &\lim_{\delta \to 0} \liminf_{n \to +\infty} -\frac{1}{|F_n|} \log \mu(B_{F_n}(x, \delta)) \\ =&\lim_{\delta \to 0} \limsup_{n \to +\infty} -\frac{1}{|F_n|} \log \mu(B_{F_n}(x, \delta)) = h_\mu(X, G). \end{aligned}$$

Since this formula gives an alternative definition for metric entropy(known as local entropy), we give the following definition of local entropy in amenable group action case.

Definition 1.2. *Let (X, G) be a compact metric $G-$action topological dynamical system and G a discrete countable amenable group. Denote by $M(X)$ the collection of Borel probability measures on X. For any $\mu \in M(X)$, $x \in X, n \in \mathbf{N}$, $\epsilon > 0$ and $\{F_n\}$ any Følner sequence in G, denote by*

$$\underline{h}_\mu^{loc}(x, \epsilon, \{F_n\}) = \liminf_{n \to +\infty} -\frac{1}{|F_n|} \log \mu(B_{F_n}(x, \epsilon)).$$

Then the (lower) local entropy of μ at x (along $\{F_n\}$) is defined by

$$\underline{h}_\mu^{loc}(x, \{F_n\}) = \lim_{\epsilon \to 0} \underline{h}_\mu^{loc}(x, \epsilon, \{F_n\})$$

and the (lower) local entropy of μ is defined by

$$\underline{h}_\mu^{loc}(\{F_n\}) = \int_X \underline{h}_\mu^{loc}(x, \{F_n\}) d\mu.$$

Similarly, we can define the upper local entropy.

In this case the common value will be denoted by

$$h_\mu^{loc}(x, \{F_n\}) := \underline{h}_\mu^{loc}(x, \{F_n\}) = \overline{h}_\mu^{loc}(x, \{F_n\}).$$

And then, for any G-invariant Borel probability measure μ, and $\alpha \geq 0$, define

$$\widehat{K}_\alpha(\mu) = \{x \in X : h_\mu^{loc}(x, \{F_n\}) = \alpha\}.$$

In [20], Takens and Verbitski defined the (q, μ)-entropy $h_\mu(T, q, \cdot)$ by extending the definition of generalized Hausdorff dimension $\dim_\mu^q(\cdot)$ and showed the following formula

$$h_{top}(\widehat{K}_\alpha(\mu)) = q\alpha + h_\mu(T, q, \widehat{K}_\alpha(\mu)),$$

where $h_{top}(\cdot)$ denotes the topological entropy. Later, in 2007, Yan and Chen [15] considered the multifractal spectra associated with Poincaré recurrences and established an exact formula on multifractal spectrum of local entropies for recurrence time.

2 Preliminaries and main results

Let $\mu \in M(X_1, T_1)$ be an invariant Borel measure. For $\alpha \geq 0$, define

$$K_\alpha(\mu) = \{x \in X_1 : h_\mu^{loc}(x, \{F_n\}) = \alpha\}.$$

In this paper, we are interested in local entropies and spectra associated for amenable group actions, we study the size of the set $K_\alpha(\mu)$.

Next, we will try to give our result by defining the weighted (G, q, t)-energy. Let μ be an invariant non-atomic Borel measure. Without loss of generality we may assume that μ is positive on any non-empty open set. For any at most countable collection $\mathcal{G} = \{B_{\{F_n\}}(x, \epsilon)\}$, any $q, t \in \mathbb{R}$ define the (G, q, t)-free energy of $\mathcal{G}$ by

$$F_\mu(\mathcal{G}, q, t) = \sum_{B_{F_n}(x,\epsilon)\in\mathcal{G}} \mu(B_{F_n}(x, \epsilon))^q \exp(-t|F_n|).$$

For any given set $Z \subset X_1, Z \neq \emptyset$, and numbers $q, t \in \mathbb{R}, \epsilon > 0, N \in \mathbb{N}$, put

$$M_{\mu,c}(Z, q, t, \epsilon, N) = \inf_{\mathcal{G}} F_\mu(\mathcal{G}, q, t)$$

where the infimum is taken over all finite or countable collections $\mathcal{G} = \{B_{F_{n_i}}(x_i, \epsilon)\}$ with $x_i \in Z$ and $n_i \geq N$ such that $Z \subset \bigcup_{B_{F_n}(x,\epsilon)\in\mathcal{G}} B_{F_{n_i}}(x_i, \epsilon)$. To complete the definition, we assume that

$$M_{\mu,c}(\emptyset, q, t, \epsilon, N) = 0$$

for any q, t, ϵ and N. The quantities $M^{\mathbf{a}}_{\mu,c}(Z, q, t, \epsilon, N)$ are non-decreasing in N, hence the following limit exists:

$$M_{\mu,c}(Z, q, t, \epsilon) = \lim_{N\to\infty} M_{\mu,c}(Z, q, t, \epsilon, N) = \sup_{N>1} M_{\mu,c}(Z, q, t, \epsilon, N).$$

Since we consider covers with centers in a given set, the qualities $M_{\mu,c}(Z, q, t, \epsilon)$ are not necessarily monotonic with respect to the set Z. We enforce monotonicity by putting

$$M_\mu(Z, q, t, \epsilon) = \sup_{Z'\subset Z} M_{\mu,c}(Z', q, t, \epsilon).$$

We now state (without proof) some basic facts. And these are standard proofs of Hausdorff dimension type and similar to the properties of topological entropy in [1], topological pressure in [13].

Lemma 2.1. *For any $t \in \mathbb{R}$ the set function $M_\mu(Z, q, t, \epsilon)$ has the following properties:*

(1) $M_\mu(\emptyset, q, t, \epsilon) = 0$;
(2) $M_\mu(Z_1, q, t, \epsilon) \leq M_\mu(Z_2, q, t, \epsilon)$ *for any* $Z_1 \subset Z_2$;
(3) $M_\mu(\bigcup_{i=1}^{\infty} Z_i, q, t, \epsilon) \leq \sum_{i=1}^{\infty} M_\mu(Z_i, q, t, \epsilon)$ *for any* $Z_i \subset X, i = 1, 2, \cdots$.

Remark 2.1. *It is easily to check that $M_\mu(\cdot, q, t, \epsilon)$ is an outer measure. And $M_\mu(Z, q, t, \epsilon)$ plays a similar role with the $\mathcal{M}(Z, \epsilon, s)$ in Definition 1.1.*

Lemma 2.2. *There exists a critical value $h_\mu(\{F_n\}, q, Z, \epsilon) \in [-\infty, \infty]$ such that*

$$M_\mu(Z, q, t, \epsilon) = \begin{cases} 0 & if\ t > h_\mu(\{F_n\}, q, Z, \epsilon) \\ \infty & if\ t < h_\mu(\{F_n\}, q, Z, \epsilon). \end{cases}$$

Lemma 2.3. *The following holds:*

(1) $h_\mu(\{F_n\}, q, \emptyset, \epsilon) = -\infty$;

(2) $h_\mu(\{F_n\}, q, Z_1, \epsilon) \leq h_\mu(\{F_n\}, q, Z_2, \epsilon)$ *for* $Z_1 \subset Z_2$;

(3) $h_\mu(\{F_n\}, q, \bigcup_{i=1}^{\infty} Z_i, \epsilon) = \sup_i h_\mu(\{F_n\}, q, Z_i, \epsilon)$ *where* $Z_i \subset X_1, i = 1, 2, \cdots$.

Definition 2.1. *The* $(\{F_n\}, q, \mu)$*-entropy of* Z *is*

$$h_\mu(\{F_n\}, q, Z) = \limsup_{\epsilon \to 0} h_\mu(\{F_n\}, q, Z, \epsilon).$$

Similar to Lemma 2.3, we state (without proof) some basic properties of $h_\mu(G, q, \cdot)$.

Proposition 2.1. *The following holds:*

(1) $h_\mu(\{F_n\}, q, \emptyset) = -\infty$;

(2) $h_\mu(\{F_n\}, q, Z_1) \leq h_\mu(\{F_n\}, q, Z_2)$ *for* $Z_1 \subset Z_2$;

(3) $h_\mu(\{F_n\}, q, \bigcup_{i=1}^{\infty} Z_i) = \sup_i h_\mu(\{F_n\}, q, Z_i)$ *where* $Z_i \subset X_1, i = 1, 2, \cdots$.

In this paper, we will prove

Theorem 2.1. *Let* μ *be a non-atomic* G*-invariant measure and positive on any non-empty open set. For any* $\alpha \geq 0$ *and every* $q \in \mathbb{R}$*, we have*

$$h_{top}^B(\{F_n\}, K_\alpha(\mu)) = q\alpha + h_\mu(\{F_n\}, q, K_\alpha(\mu)).$$

3 Proof of Main results

Proposition 3.1. *Let* μ *be non-atomic* G*-invariant measure and positive on any non-empty open set. For any subset* $Z \subset X$ *one has* $h_\mu(\{F_n\}, 0, Z) = h_{top}^B(\{F_n\}, Z)$.

Proof. If $Z = \emptyset$, the statement is obvious, since both sides are equal to $-\infty$. Suppose that $Z \neq \emptyset$, we start by showing $h_\mu(\{F_n\}, 0, Z) \geq h_{top}^B(\{F_n\}, Z)$. Let $\mathcal{U}$ be an open cover of X and choose any $\epsilon < \frac{\gamma(\mathcal{U})}{2}$ with $\gamma(\mathcal{U})$ denotes the Lebesgue number of $\mathcal{U}$. Consider an arbitrary collection $\mathcal{G} = \{B_{F_{n_i}}(x_i, \epsilon)\}$ with $n_i > N$ such that $x_i \in Z$ and $Z \subset \bigcup_{B_{F_{n_i}}(x_i,\epsilon) \in \mathcal{G}} B_{F_{n_i}}(x_i, \epsilon)$. For the fixed $\mathcal{U}$, we can choose $\mathbf{U}_{n_i} \in \mathcal{W}_{F_{n_i}}$ such that $B_{F_{n_i}}(x_i, \epsilon) \subset \mathbf{U}_{n_i}$. Let $\Gamma_{\mathcal{G}} = \{\mathbf{U}_{n_i}\}$. Obviously, $\Gamma_{\mathcal{G}}$ covers Z and

$$F_\mu(\mathcal{G}, 0, t) = \sum_{B_{F_{n_i}}(x_i,\epsilon) \in \mathcal{G}} \exp(-t|F_{n_i}|) = \sum_{\mathbf{U}_{n_i} \in \Gamma_{\mathcal{G}}} \exp(-t|F_{n_i}|).$$

Since $\mathcal{G}$ is arbitrary, we conclude that

$$M_{\mu,c}(Z,0,t,\epsilon,N) = \inf_{\mathcal{G}} F_\mu(\mathcal{G},0,t) \geq \mathcal{M}(Z,\mathcal{U},t,N).$$

Taking limits as $N \to \infty$,

$$\mathcal{M}(Z,\mathcal{U},t) \leq M_{\mu,c}(Z,0,t,\epsilon) \leq M_\mu(Z,0,t,\epsilon).$$

Therefore,

$$h^B_{top}(\{F_n\},Z,\mathcal{U}) \leq h_\mu(\{F_n\},0,Z,\epsilon)$$

for any $\epsilon < \dfrac{\gamma(\mathcal{U})}{2}$. Let $\epsilon \to 0$, we have

$$h^B_{top}(\{F_n\},Z,\{\mathcal{U}_i\}_{i=1}^k) \leq \limsup_{\epsilon\to 0} h_\mu(\{F_n\},0,Z,\epsilon) = h_\mu(\{F_n\},0,Z),$$

which yields that

$$h^B_{top}(\{F_n\},Z) \leq h_\mu(\{F_n\},0,Z).$$

Let us now show the opposite inequality. Assume that $h_\mu(\{F_n\},0,Z) - h^B_{top}(\{F_n\},Z) > 3\gamma > 0$. Then there exists $\epsilon > 0$ such that

$$h_\mu(\{F_n\},0,Z,\epsilon) - h^B_{top}(\{F_n\},Z) > 2\gamma.$$

By definition of topological entropy, there exists an open cover $\mathcal{U}$ with $\mathrm{diam}(\mathcal{U}) < \epsilon$ such that

$$h_\mu(\{F_n\},0,Z,\epsilon) - h^B(\{F_n\},Z,\mathcal{U}) > \gamma. \tag{3.4}$$

Let Z' be an arbitrary subset of Z and $\Gamma = \{\mathbf{U}_{n_i}\}$ be an arbitrary collection of strings covering Z'. We may assume that $\mathbf{U}_{n_i} \cap Z' \neq \emptyset$ for $\mathbf{U}_{n_i} \in \Gamma$. Otherwise we just delete those strings and obtain a smaller collection of strings, which still covers Z'. For any $\mathbf{U}_{n_i} \in \Gamma$, we choose an arbitrary $x_{\mathbf{U}_{n_i}} \in \mathbf{U}_{n_i} \cap Z'$. Thus,

$$x_{\mathbf{U}_{n_i}} \in \mathbf{U}_{n_i} \subset B_{F_{n_i}}(x_{\mathbf{U}_{n_i}},\epsilon).$$

Therefore, the collection $\mathcal{G} = \{B_{F_n}(x_{\mathbf{U}_{n_i}},\epsilon)\}$ is a centered cover of Z'. From the definition of weighted free energies, we obtain

$$M_{\mu,c}(Z',0,s,\epsilon) \leq \mathcal{M}(Z',\mathcal{U},s)$$

for any $s \in \mathbb{R}$. Furthermore,

$$M_\mu(Z,0,s,\epsilon) = \sup_{Z'\subset Z} M_{\mu,c}(Z',0,s,\epsilon) \leq \mathcal{M}(Z,\mathcal{U},s).$$

The last inequality holds due to the monotonicity of $\mathcal{M}(.,\mathcal{U},s)$ with respect to the first argument. Finally, we get $h_\mu(\{F_n\},0,Z,\epsilon) \leq h(\{F_n\},Z,\mathcal{U})$ which is contradicted with (3.4). $\square$

Remark 3.1. *If $q = 0$, Theorem 2.1 can be showed by Proposition 3.1 easily. We will prove Theorem 2.1 for each $q \in \mathbb{R}$ in next section.*

Proof of Theorem 2.1 Consider $\alpha \geq 0$ and the corresponding level set

$$
\begin{aligned}
K_\alpha(\mu) &= \{x \in X_1 : \ h_\mu^{loc}(x, \{F_n\}) = \alpha\} \\
&= \{x \in X_1 : \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{-\log \mu(B^{\mathbf{a}}_{F_n}(x, \epsilon))}{|F_n|} = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{-\log \mu(B^{\mathbf{a}}_{F_n}(x, \epsilon))}{|F_n|} = \alpha\}.
\end{aligned}
$$

Choose some monotonic sequence $\epsilon_M \to 0$ as $M \to \infty$ and this sequence will be fixed for the rest of this section. Let $\delta > 0$ and put

$$
K_{\alpha,M} = \Big\{x \in K_\alpha(\mu) : \alpha - \delta < \liminf_{n \to \infty} \frac{-\log \mu(B_{F_n}(x, \epsilon_M))}{|F_n|}\Big\}.
$$

Obviously, $K_{\alpha,M} \subset K_{\alpha,M+1}$ and $K_\alpha(\mu) = \bigcup_{M=1}^{\infty} K_{\alpha,M}$. Due to the monotonicity of $\frac{-\log \mu(B_{F_n}(x, \epsilon))}{|F_n|}$ with respect to ϵ, for each $x \in K_\alpha(\mu)$ and every $\epsilon > 0$ one has

$$
\limsup_{n \to \infty} \frac{-\log \mu(B_{F_n}(x, \epsilon))}{|F_n|} \leq \alpha.
$$

Fix $x \in K_{\alpha,M}$, there exists $N_0 = N_0(x, \delta, \epsilon_M)$ such that

$$
\alpha - \delta < \frac{-\log \mu(B_{F_n}(x, \epsilon_M))}{|F_n|} \leq \alpha + \delta
$$

for all $n \geq N_0$. Put

$$
K_{\alpha,M,N} = \{x \in K_{\alpha,M} : N_0 = N_0(x, \delta, \epsilon_M) < N\}.
$$

Again, it is easy to see that $K_{\alpha,M,N} \subset K_{\alpha,M,N+1}$ and $K_{\alpha,M} = \bigcup_{N=1}^{\infty} K_{\alpha,M,N}$. Using the properties of weighted topological entropy, we conclude that

$$
h_{top}^B(\{F_n\}, K_\alpha(\mu), \mathcal{U}) = \lim_{M \to \infty} \lim_{N \to \infty} h_{top}^B(\{F_n\}, K_{\alpha,M,N}, \mathcal{U}).
$$

Lemma 3.1. *Suppose $\mathcal{U}$ is an open cover respect to X. Consider $K_{\alpha,M,N}$ for some $M, N \in \mathbb{N}$ such that $\epsilon_M < \frac{\gamma(\mathcal{U})}{2}$, where $\gamma(\mathcal{U})$ denotes the Lebesgue number of $\mathcal{U}$. Then for $s \geq q\alpha + |q|\delta + t$ one has*

$$
\mathcal{M}(\{F_n\}, K_{\alpha,M,N}, \mathcal{U}, s) \leq M_{\mu,c}(\{F_n\}, K_{\alpha,M,N}, q, t, \epsilon_M).
$$

Proof. Suppose that $n > N$ and $\mathcal{G}_n = \{B_{F_{n_i}}(x_i, \epsilon_M)\}$ is an arbitrary cover of $K_{\alpha,M,N}$ with $x_i \in K_{\alpha,M,N}$ such that $n_i \geq n \geq N$ for all i. Then for every x_i, we can get some string $\mathbf{U}_{n_i}$ satisfying $B_{n_i}(x_i, \epsilon_M) \subset \mathbf{U}_{n_i}$, i.e., there exists $\Gamma_n := \{\mathbf{U}_{n_i}\}$ such that

$$
K_{\alpha,M,N} \subset \bigcup_{B_{F_{n_i}}(x_i, \epsilon_M) \in \mathcal{G}_n} B_{n_i}(x_i, \epsilon_M) \subset \bigcup_{\mathbf{U}_{n_i} \in \Gamma_n} \mathbf{U}_{n_i}.
$$

Since $x_i \in K_{\alpha,M,N}$ for all i and $n_i \geq n > N$, we get

$$\exp(-(\alpha+\delta)|F_{n_i}|) \leq \mu(B_{F_{n_i}}(x_i,\epsilon_M)) \leq \exp(-(\alpha-\delta)|F_{n_i}|).$$

If $q \geq 0$, then $\mu(B_{F_{n_i}}(x_i,\epsilon_M))^q \geq \exp(-q(\alpha+\delta)|F_{n_i}|)$ and

$$\begin{aligned}\sum_{B_{F_{n_i}}(x_i,\epsilon_M)\in\mathcal{G}_n} \mu(B_{F_{n_i}}(x_i,\epsilon_M))^q \exp(-t|F_{n_i}|) &\geq \sum_{B_{F_{n_i}}(x_i,\epsilon_M)\in\mathcal{G}_n} \exp(-|F_{n_i}|(q\alpha+q\delta+t)) \\ &\geq \sum_{\mathbf{U}_{n_i}\in\Gamma_n} \exp(-|F_{n_i}|s) \\ &\geq \mathcal{M}(K_{\alpha,M,N},\mathcal{U},s,n)\end{aligned} \tag{3.5}$$

for $s \geq q\alpha+q\delta+t$. On the other hand, if $q \leq 0$, then $\mu(B_{F_{n_i}}(x_i,\epsilon_M))^q \geq \exp(-(\alpha-\delta)q|F_{n_i}|)$ and

$$\begin{aligned}\sum_{B_{F_{n_i}}(x_i,\epsilon_M)\in\mathcal{G}_n} \mu(B_{F_{n_i}}(x_i,\epsilon_M))^q \exp(-t|F_{n_i}|) &\geq \sum_{B_{F_{n_i}}(x_i,\epsilon_M)\in\mathcal{G}_n} \exp(-|F_{n_i}|(q\alpha-q\delta+t)) \\ &\geq \sum_{\mathbf{U}_{n_i}\in\Gamma_n} \exp(-|F_{n_i}|s) \\ &\geq \mathcal{M}(K_{\alpha,M,N},\mathcal{U},s,n)\end{aligned} \tag{3.6}$$

for $s \geq q\alpha-q\delta+t$. Together (3.5) with (3.6), we have

$$\mathcal{M}(K_{\alpha,M,N},\mathcal{U},s,n) \leq M_{\mu,c}(K_{\alpha,M,N},q,t,\epsilon_M,n).$$

Let $n \to \infty$;

$$\mathcal{M}(K_{\alpha,M,N},\mathcal{U},s) \leq M_{\mu,c}(K_{\alpha,M,N},q,t,\epsilon_M).$$

□

Lemma 3.2. *Suppose $K_{\alpha,M,N}$ for some $M,N \in \mathbb{N}$ and $\mathcal{U}$ is an open cover of X satisfy* $\operatorname{diam}(\mathcal{U}) < \frac{\epsilon_M}{2}$. *Then for $s \leq q\alpha - |q|\delta + t$ one has*

$$M_\mu(K_{\alpha,M,N},q,t,\epsilon_M) \leq \mathcal{M}(K_{\alpha,M,N},\mathcal{U},s).$$

Proof. Fix some integers M,N and let $Z \subset K_{\alpha,M,N}$, Z be a nonempty set. Since the open cover $\mathcal{U}$ satisfy $\operatorname{diam}(\mathcal{U}) < \frac{\epsilon_M}{2}$, we can choose any $n > N$ and let $\Gamma_n = \{\mathbf{U}_{n_i}\}$ be an arbitrary collection of strings covering Z with $n_i \geq n$. Without loss of generality we may assume that $\mathbf{U}_{n_i} \cap Z \neq \emptyset$ for each $\mathbf{U}_{n_i} \in \Gamma_n$. Pick any $x_{\mathbf{U}_{n_i}} \in \mathbf{U}_{n_i} \cap Z$. It follows from $\operatorname{diam}(\mathcal{U}) < \frac{\epsilon_M}{2}$ that

$$\mathbf{U}_{n_i} \subset B_{F_{n_i}}(x_{\mathbf{U}_{n_i}},\epsilon_M).$$

The collection $B_{F_n}(x,\epsilon_M)$ is centered cover of Z. Since $x_{\mathbf{U}_{n_i}} \in Z \subset K_{\alpha,M,N}$ and $n > N$, one has

$$\exp(-|F_{n_i}|(\alpha+\delta)) \leq \mu(B_{F_n}(x_{\mathbf{U}_{F_{n_i}}},\epsilon)) \leq \exp(-|F_{n_i}|(\alpha-\delta))$$

For $q \geq 0$

$$\begin{aligned} M_{\mu,c}(Z,q,t,\epsilon_M,n) &\leq \sum_{\mathbf{U}_{n_i}\in\Gamma_n} \mu(B_n(x_{\mathbf{U}_{n_i}},\epsilon_M))^q \exp(-n_i t) \\ &\leq \sum_{\mathbf{U}_{n_i}\in\Gamma_n} \exp(-|F_{n_i}|(q\alpha - q\delta + t)) \\ &\leq \sum_{\mathbf{U}_{n_i}\in\Gamma_n} \exp(-|F_{n_i}|s) \end{aligned}$$

for $s \leq q\alpha - q\delta + t$. Since Γ_n is arbitrary, we get

$$M_{\mu,c}(Z,q,t,\epsilon_M,n) \leq \mathcal{M}(Z,\mathcal{U},s,n).$$

Let $n \to \infty$,

$$M_{\mu,c}(Z,q,t,\epsilon_M) \leq \mathcal{M}(Z,\mathcal{U},s) \leq \mathcal{M}(K_{\alpha,M,N},\mathcal{U},s).$$

Moreover,

$$M_\mu(K_{\alpha,M,N},q,t,\epsilon_M) \leq \mathcal{M}(K_{\alpha,M,N},\mathcal{U},s). \tag{3.7}$$

For $q \leq 0$, we have

$$\mu(B_{F_{n_i}}(x_{\mathbf{U}_{n_i}},\epsilon_M))^q \leq \exp(-|F_{n_i}|q(\alpha+\delta)).$$

Hence,

$$\begin{aligned} M_{\mu,c}(Z,q,t,\epsilon_M,n) &\leq \sum_{\mathbf{U}_{n_i}\in\Gamma_n} \mu(B_{F_n}(x_{\mathbf{U}^{\mathbf{a}}_{n_i}},\epsilon_M))^q \exp(-|F_{n_i}|t) \\ &\leq \sum_{\mathbf{U}_{n_i}\in\Gamma_n} \exp(-|F_{n_i}|(q\alpha + q\delta + t)) \\ &\leq \sum_{\mathbf{U}_{n_i}\in\Gamma_n} \exp(-|F_{n_i}|s) \end{aligned}$$

for $s \leq q\alpha + q\delta + t$. Similar to the case $q > 0$, we can get

$$M^{\mathbf{a}}_\mu(K_{\alpha,M,N},q,t,\epsilon_M) \leq m(K_{\alpha,M,N},\mathcal{U},s). \tag{3.8}$$

Together (3.7) with (3.8), we complete the proof. □

Finally, we prove Theorem 2.1. By the definition of Bowen topological entropy, we only need to show

$$h^B_{top}(\{F_n\},K_\alpha(\mu)) = q\alpha + h_\mu(\{F_n\},q,K_\alpha(\mu)).$$

We may assume that $K_\alpha(\mu) \neq \emptyset$. Otherwise, the statement is obvious, since both sides are equal to $-\infty$. When $K_\alpha(\mu) \neq \emptyset$, we divide the proof into two steps:
Step 1: $h^B_{top}(\{F_n\}, K_\alpha(\mu)) \leq q\alpha + h_\mu(\{F_n\}, q, K_\alpha(\mu))$. Suppose that the opposite is true: let

$$\gamma = \frac{1}{4}(h^B_{top}(\{F_n\}, K_\alpha(\mu)) - q\alpha - h_\mu(\{F_n\}, q, K_\alpha(\mu))) > 0.$$

Clearly,

$$h^B_{top}(K_\alpha(\mu)) = \lim_{diam(\mathcal{U})\to 0} h^B_{top}(\{F_n\}, K_\alpha(\mu), \mathcal{U}).$$

There exists a family of open covers $\mathcal{U}$ such that

$$h^B_{top}(\{F_n\}, K_\alpha(\mu), \mathcal{U}) > q\alpha + h_\mu(\{F_n\}, q, K_\alpha(\mu)) + 3\gamma.$$

Let $\delta > 0$ be an arbitrary positive number if $q = 0$ and $\delta = \frac{\gamma}{2|q|}$ if $|q| > 0$. Consider $K_{\alpha,M,N}$ defined above, choose sufficiently large M, N such that the following three conditions are satisfied:

$$\begin{aligned} &h^B_{top}(\{F_n\}, K_{\alpha,M,N}, \mathcal{U}) > q\alpha + h_\mu(\{F_n\}, q, K_\alpha(\mu)) + 2\gamma, \\ &\epsilon_M < \delta, h_\mu(\{F_n\}, q, K_\alpha(\mu)) + \frac{\gamma}{2} \geq h_\mu(\{F_n\}, q, K_\alpha(\mu), \epsilon_M). \end{aligned} \tag{3.9}$$

This is possible because

$$h^B_{top}(\{F_n\}, K_\alpha(\mu), \{\mathcal{U}_i\}_{i=1}^k) = \lim_{M\to\infty} \lim_{N\to\infty} h^B_{top}(\{F_n\}, K_{\alpha,M,N}, \{\mathcal{U}_i\}_{i=1}^k)$$

and

$$h_\mu(\{F_n\}, q, K_\alpha(\mu)) = \limsup_{\epsilon\to 0} h^B_\mu(\{F_n\}, q, K_\alpha(\mu), \epsilon).$$

By the definition of $h^B_{top}(\{F_n\}, K_{\alpha,M,N}, \{\mathcal{U}_i\}_{i=1}^k)$, the inequality (3.9) implies

$$\mathcal{M}(K_{\alpha,M,N}, \{\mathcal{U}_i\}_{i=1}^k, q\alpha + h_\mu(\{F_n\}, q, K_\alpha(\mu)) + 2\gamma)) = \infty.$$

It follows from $s = q\alpha + h_\mu(\{F_n\}, q, K_\alpha(\mu)) + 2\gamma$, $t = h_\mu(\{F_n\}, q, K_\alpha(\mu)) + \gamma - |q|\delta$ and Lemma 3.1 that

$$M_{\mu,c}(K_{\alpha,M,N}, q, h_\mu(\{F_n\}, q, K_\alpha(\mu)) + \gamma - |q|\delta, \epsilon_M) = \infty.$$

Moreover,

$$M_\mu(K_{\alpha,M,N}, q, h_\mu(\{F_n\}, q, K_\alpha(\mu)) + \gamma - |q|\delta, \epsilon_M) = \infty. \tag{3.10}$$

Here, we arrive at a contradiction with the assumption above. Indeed,

$$\begin{aligned} h_\mu(\{F_n\}, q, K_\alpha(\mu)) + \gamma - |q|\delta \geq& h_\mu(\{F_n\}, q, K_\alpha(\mu)) + \frac{\gamma}{2} \\ \geq& h_\mu(\{F_n\}, q, K_\alpha(\mu), \epsilon_M) \\ \geq& h_\mu(\{F_n\}, q, K_{\alpha,M,N}, \epsilon_M) \end{aligned}$$

and therefore one must have

$$M_\mu(K_{\alpha,M,N}, q, h_\mu(T_1, q, K_\alpha(\mu)) + \gamma - |q|\delta, \epsilon_M) = 0$$

which contradicts (3.10).

Step 2: $h^B_{top}(\{F_n\}, K_\alpha(\mu)) \geq q\alpha + h_\mu(\{F_n\}, q, K_\alpha(\mu))$. Suppose that the opposite is true: let

$$\gamma = \frac{1}{4}(q\alpha + h_\mu(\{F_n\}, q, K_\alpha(\mu)) - h^B_{top}(\{F_n\}, K_\alpha(\mu))) > 0.$$

By $h_\mu(\{F_n\}, q, K_\alpha(\mu)) = \limsup_{\epsilon\to 0} h_\mu(\{F_n\}, q, K_\alpha(\mu), \epsilon)$, we can choose a decreasing sequence $\epsilon_M \to 0$ such that

$$h_\mu(\{F_n\}, q, K_\alpha(\mu)) = \lim_{M\to\infty} h_\mu(\{F_n\}, q, K_\alpha(\mu), \epsilon_M).$$

Let $\delta > 0$ be an arbitrary positive number if $q = 0$ and $\delta = \frac{\gamma}{2|q|}$ if $|q| > 0$. Choose sufficiently large M such

$$\epsilon_M < \delta, h_\mu(\{F_n\}, q, K_\alpha(\mu), \epsilon_M) > h_\mu(\{F_n\}, q, K_\alpha(\mu)) - \frac{\gamma}{2}.$$

Since

$$h^B_{top}(\{F_n\}, K_\alpha(\mu)) = \lim_{diam(\mathcal{U})\to 0} h^B(K_\alpha(\mu), \mathcal{U}).$$

One can find a family of open covers $\mathcal{U}$ such that

$$\text{diam}(\mathcal{U}) < \frac{\epsilon_M}{2}$$

and

$$q\alpha + h_\mu(\{F_n\}, q, K_\alpha(\mu)) > h(\{F_n\}, K_\alpha(\mu), \mathcal{U}) + 3\gamma.$$

Furthermore, consider $K_{\alpha,M,N}$ defined above, we can get

$$q\alpha + h_\mu(\{F_n\}, q, K_\alpha(\mu)) > h^B_{top}(\{F_n\}, K_{\alpha,M,N}, \mathcal{U}) + 2\gamma \tag{3.11}$$

$$h_\mu(\{F_n\}, q, K_\alpha(\mu), \epsilon_M) - \gamma \leq h_\mu(\{F_n\}, q, K_{\alpha,M,N}, \epsilon_M) \tag{3.12}$$

for M, N large enough. This is possible because

$$h^B_{top}(\{F_n\}, K_\alpha(\mu), \{\mathcal{U}_i\}^k_{i=1}) = \lim_{M\to\infty} \lim_{N\to\infty} h^B_{top}(\{F_n\}, K_{\alpha,M,N}, \{\mathcal{U}_i\}^k_{i=1})$$

and

$$h_\mu(\{F_n\}, q, K_\alpha(\mu), \epsilon_M) = \lim_{M\to\infty} \lim_{N\to\infty} h_\mu(\{F_n\}, q, K_{\alpha,M,N}, \epsilon_M).$$

By the definition of $h^B_{top}(\{F_n\}, K_{\alpha,M,N}, \mathcal{U})$, the inequality (3.11) implies

$$\mathcal{M}(K_{\alpha,M,N}, \mathcal{U}, q\alpha + h_\mu(\{F_n\}, q, K_\alpha(\mu)) - 2\gamma) = 0.$$

It follows from $s = q\alpha + h_\mu(T, q, K_\alpha(\mu)) - 2\gamma$, $t = h_\mu(\{F_n\}, q, K_\alpha(\mu)) - \gamma + |q|\delta$ and Lemma 3.2 that

$$M_\mu(K_{\alpha,M,N}, q, h_\mu(\{F_n\}, q, K_\alpha(\mu)) - \gamma + |q|\delta, \epsilon_M) = 0. \tag{3.13}$$

Here, we arrive at a contradiction with the assumption above. Indeed, by (3.12)

$$\begin{aligned} h_\mu(\{F_n\}, q, K_\alpha(\mu)) - \gamma + |q|\delta \leq & h_\mu(\{F_n\}, q, K_\alpha(\mu)) - \frac{\gamma}{2} \\ \leq & h_\mu(\{F_n\}, q, K_\alpha(\mu), \epsilon_M) - \gamma \\ \leq & h_\mu(\{F_n\}, q, K_{\alpha,M,N}, \epsilon_M). \end{aligned}$$

Therefore one must have

$$M_\mu(K_{\alpha,M,N}, q, h_\mu(\{F_n\}, q, K_\alpha(\mu)) - \gamma + |q|\delta, \epsilon_M) = \infty$$

which contracts (3.13).
Acknowledgements. We would like to thank the anonymous referees for valuable comments. The work was supported by NNSF of China (11671208 and 11431012).

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