New significant results on Fermat numbers via elementary arithmetic methods

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Abstract

A *Fermat number* is a number of the form $F_n = 2^{2^n} + 1$, where n is an integer ≥ 0. In this paper, we show [via elementary arithmetic congruences] the following two results **(R.)** and **(R'.)**. **(R.)***: For every integer $n \geq 3$, $F_n - 1 \equiv 1 \, mod[j]$, where $j \in \{3, 5, 17\}$.* **(R'.)***: For every integer $n > 0$ such that $n \equiv 2 \, mod[6]$, we have $F_n - 1 \equiv 16 \, mod[19]$.* Result **(R.)** immediately implies that for every integer $d \geq 0$, there exists at most two primes of the form $2F_n + 1 + 10d$ [in particular, for every integer $d \geq 0$, the numbers of the form $2F_n + 1 + 10d$ (where $n \geq 2$) are all composites]; result **(R.)** also implies that there are infinitely many composite numbers of the form $2^n + F_n$ and for every $r \in \{-2, 16\}$, there exists only one prime of the form $r + F_n$. Result **(R'.)** immediately implies that there are infinitely many composite numbers of the form $2 + F_n$. That being said, we use the result **(R.)** and a special case of a Theorem of Dirichlet on arithmetic progression to explain why it is natural to conjecture that for every $r' \in \{0, 2\}$, there are infinitely many primes of the form $r' + F_n$.

Keywords. Fermat number, F_n, Dirichlet of Theorem On Arithmetic Progression, Primes Numbers, Modular Arithmetic, Congruences.
AMS **Classification** 2000: $05xx$, $11xx$, $11A07$, $11B30$, and $11F03$.

0. Introduction

A *Fermat number* is a number of the form $F_n = 2^{2^n} + 1$, where n is an integer ≥ 0. A *Fermat composite* (see [1] or [2] or [4] or [6] or [7] or [12] or [13] or [14] or [15]) is a non prime Fermat number and a *Fermat prime* is a prime Fermat number. Fermat composites and Fermat primes are characterized via divisibility in [4] and in [5]. It is known (see [4]) that for every $j \in \{0, 1, 2, 3, 4\}$, F_j is a Fermat prime ($F_0 = 2^{2^0} + 1 = 3$ and 3 is prime, $F_1 = 2^{2^1} + 1 = 5$ and 5 is prime, $F_3 = 2^{2^3} + 1 = 257$ and 257 is prime, and $F_4 = 2^{2^4} + 1 = 65537$ and 65537 is prime), and it is also known (see [2] or [3]or [8] or [9] or [10]or [11]) that F_5 and F_6 are Fermat composites ($F_5 = 2^{2^5} + 1 = 641 \times 6700417$, and since 2013, it is known that $F_{2747497} = 2^{2^{2747497}} + 1$ is Fermat composite number). Fermat numbers have importance and their application to other sciences such as cryptography, neural networks, electronic computer, polygons with straightedge, filtering, autocorrelation, and related areas with conventional computing have seen clearly. Factorization of Fermat numbers (specially for big one) is a very hard problem in number theory and cryptography too. The biggest known Fermat prime is $F_4 = 2^{2^4} + 1 = 65537$, and this number is used in cryptography because of the fact that this prime helps the cryptography less vulnerable to the public exponent attack lowly (as mentioned "Coppersmith's short pad attack" in the literature. Practical and efficient methods are still in need for factorization of such numbers, even there are many algorithms to factorize some of such composite numbers. That being so, in this paper, we show [via elementary arithmetic congruences] the following two results **(R.)** and **(R'.)**. **(R.)**: *For every integer* $n \geq 3$, $F_n - 1 \equiv 1 \, mod[j]$, *where* $j \in \{3, 5, 17\}$. **(R'.)**: *For every integer* $n > 0$ *such that* $n \equiv 2 \, mod[6]$, *we have* $F_n - 1 \equiv 16 \, mod[19]$. Result **(R.)** immediately implies that for every integer $d \geq 0$, there exists at most two primes of the form $2F_n + 1 + 10d$ [in particular, for every integer $d \geq 0$, the numbers of the form $2F_n + 1 + 10d$ (where $n \geq 2$) are all composites]; result **(R.)** also implies that there are infinitely many composite numbers of the form $2^n + F_n$ and for every $r \in \{-2, 16\}$, there exists only one prime of the form $r + F_n$. Result **(R'.)** immediately implies that there are infinitely many composite numbers of the form $2 + F_n$. That being said, we use the result **(R.)** and a special case of a Theorem of Dirichlet on arithmetic progression to explain why it is natural to conjecture that for every $r' \in \{0, 2\}$, there are infinitely many primes of the form $r' + F_n$.

Theorem 1. *The following are satisfied.*

(R.). *For every integer* $n \geq 3$, $F_n - 1 \equiv 1 \, mod[j]$, *where* $j \in \{3, 5, 17\}$.

(R.1). *For every fixed integer* $d \geq 0$, *there exists at most two primes of the form* $2F_n + 1 + 10d$.

(R.2). *For every fixed integer* $d \geq 0$, *the numbers of the form* $2F_n + 1 + 10d$ *(where* $n \geq 2$*) are all composites.*

(R.3). *There are infinitely many composite numbers of the form* $2^n + F_n$.

(R.4). *The only prime of the form* $-2 + F_n$ *is* $-2 + F_1 = 3$ *and the only prime of the form* $16 + F_n$ *is* $16 + F_0 = 19$.

(R.5) *For every integer* $n \geq 2$, $-2 + F_n$ *is composite; and for every integer* $n \geq 1$, $16 + F_n$ *is composite.*

Theorem 2. *The following are satisfied.*

(R'.). *For every integer* $n > 0$ *such that* $n \equiv 2 \, mod[6]$, *we have* $F_n - 1 \equiv 16 \, mod[19]$.

(R'.1). *There are infinitely many composite numbers of the form* $2 + F_n$.

1. Proof of Theorem 1.

To prove Theorem **1**, we need the following remarks and Propositions.

Remark 1.0. *Let* n *be an integer* ≥ 4. *If for every* $j \in \{3, 5, 17\}$ *we have* $2^{2^{n-1}} \equiv 1 \, mod[j]$, *then for every* $j \in \{3, 5, 17\}$, $2^{2^{n-1}} \times 2^{2^{n-1}} \equiv 1 \, mod[j]$.

(**Proof** Immediate [via elementary arithmetic congruences]. □)

Proposition 1.1. *Let* n *be an integer* ≥ 3. *Then for every* $j \in \{3, 5, 17\}$, *we have* $2^{2^n} \equiv 1 \, mod[j]$.

$\Big($ **Proof** Otherwise

$$let\, n\, be\, minimum\, such\, that\, there\, exists\, j \in \{3,5,17\}\, with\, 2^{2^n} \not\equiv 1\, mod[j] \qquad (1.1).$$

Clearly

$$n \geq 4 \qquad (1.2)$$

(since $2^{2^3} = 256$ and $256 \equiv 1\, mod[j]$ where $j \in \{3,5,17\}$). It is immediate to see that

$$2^{2^n} = 2^{2^{n-1}} \times 2^{2^{n-1}} \qquad (1.3).$$

Now using equality *(1.3)* and inequality *(1.2)*, we easily deduce that *(1.1)* clearly implies that

$$there\, exists\, j \in \{3,5,17\}\, with\, 2^{2^{n-1}} \times 2^{2^{n-1}} \not\equiv 1\, mod[j]\, and\, 2^{2^{n-1}} \equiv 1\, mod[j];\ (n \geq 4) \quad (1.4).$$

(1.4) clearly contradicts Remark **1.0**. $\square\Big)$

Proposition 1.2. *Let n be an integer ≥ 3 and let d be a fixed integer ≥ 0 [d is fixed once and for all, so d does not move anymore]. Then $2F_n + 1 + 10d \equiv 0\, mod[5]$ and $2F_n + 1 + 10d$ is composite.*

$\Big($ **Proof (i)**. $2F_n + 1 \equiv 0\, mod[5]$ *and* $2F_n + 1$ *is composite.* Clearly

$$2(2^{2^n} + 1) + 1 \equiv 0\, mod[5] \qquad (1.5)$$

[indeed observe (via Proposition **1.1** and the fact that $F_n - 1 = 2^{2^n}$) that $F_n - 1 \equiv 1\, mod[5]$ and use elementary arithmetic congruences]. So $2F_n + 1 \equiv 0\, mod[5]$ and $2F_n + 1$ is composite [use congruence *(1.5)* and observe that $2(2^{2^n} + 1) + 1 = 2F_n + 1$ and $2F_n + 1 > 5$ (note that $n \geq 3$)].
(**ii**). $2F_n + 1 + 10d \equiv 0\, mod[5]$ *and* $2F_n + 1 + 10d$ *is composite.* Immediate (use (**i**) and observe that $10d \equiv 0\, mod[5]$). Proposition **1.2** immediately follows [use (**i**) and (**ii**)]$\Big).\square$

Proposition 1.3. *Let n be an integer ≥ 3. Then $F_n - 2 \equiv 0\, mod[17]$ and $F_n - 2$ is composite.*

$\Big($ **Proof** Clearly

$$(2^{2^n} + 1) - 2 \equiv 0\, mod[17] \qquad (1.6)$$

[observe (via Proposition **1.1** and the fact that $F_n - 1 = 2^{2^n}$) that $F_n - 1 \equiv 1\, mod[17]$ and use elementary arithmetic congruences]. So $F_n - 2 \equiv 0\, mod[17]$ and $F_n - 2$ is composite [use congruence *(1.6)* and observe that $(2^{2^n} + 1) - 2 = F_n - 2$ and $F_n - 2 > 17$ (note that $n \geq 3$)].Proposition **1.3** immediately follows$\Big).\square$

Proposition 1.4. *Let n be an integer ≥ 3. Then $F_n + 16 \equiv 0\, mod[3]$ and $F_n + 16$ is composite.*

$\Big($ **Proof** Clearly

$$(2^{2^n} + 1) + 16 \equiv 0\, mod[3] \qquad (1.7)$$

[observe (via Proposition **1.1** and the fact that $F_n - 1 = 2^{2^n}$) that $F_n - 1 \equiv 1\, mod[3]$ and use elementary arithmetic congruences]. So $F_n + 16 \equiv 0\, mod[3]$ and $F_n + 16$ is composite [use congruence *(1.7)* and observe that $(2^{2^n} + 1) + 16 = F_n + 16$ and $F_n + 16 > 3$ (note that $n \geq 3$)].Proposition **1.4** immediately follows$\Big).\square$

Remark 1.5. *Let n be an integer ≥ 4. If $2 \times 2^{n-1} \equiv 0\, mod\, [3]$, then $2^{n-1} \equiv 0\, mod\, [3]$.* (**Proof** Immediate [via elementary arithmetic congruences and the fact that $2 \equiv 2\, mod\ [3]$]. □)

Proposition 1.6. *Let n be an integer ≥ 3; then $2^n \not\equiv 0\, mod[3]$.*
(**Proof** Otherwise

$$let\, n\, be\, minimum\, such\, that\, 2^n \equiv 0\, mod[3] \qquad (1.8).$$

Clearly

$$n \geq 4 \qquad (1.9)$$

(since $2^4 = 16$ and $16 \not\equiv 0\, mod[3]$). It is immediate to see that

$$2 \times 2^{n-1} = 2^n \qquad (1.10).$$

Now using equality *(1.10)* and inequality *(1.9)*, we easily deduce that *(1.8)* clearly implies that

$$2 \times 2^{n-1} \equiv 0\, mod[3],\ where\, 2^{n-1} \not\equiv 0\, mod[3];\ n \geq 4 \qquad (1.11).$$

(1.11) clearly contradicts Remark **1.5**. □)

Proposition 1.7. *Let n be an integer ≥ 3 and let $B_n = 2^n + F_n$; then there exists $j \in \{0, 1\}$ such that B_{n+j} is composite.*
(**Proof** **(i')**. *If $2^n \equiv 2\, mod[3]$, then the number B_{n+j} is composite, where $j = 1$.* Indeed if $2^n \equiv 2\, mod[3]$, clearly

$$2 \times 2^n \equiv 1\, mod[3] \qquad (1.12)$$

[use elementary arithmetic congruences] and so

$$2^{n+1} \equiv 1\, mod[3] \qquad (1.13)$$

[use *(1.12)* and observe that $2 \times 2^n = 2^{n+1}$]. Observe (via Proposition **1.1**) that

$$2^{2^{n+1}} \equiv 1\, mod[3] \qquad (1.14),$$

and so

$$2^{2^{n+1}} + 1 \equiv 2\, mod[3] \qquad (1.15)$$

[use *(1.14)* and elementary arithmetic congruences]. Clearly

$$2^{n+1} + (2^{2^{n+1}} + 1) \equiv 0 \mod [3] \qquad (1.16)$$

[use *(1.13)* and *(1.15)* and elementay arithmetic congruences]. Clearly

$$2^{n+j} + F_{n+j} \equiv 0\, mod[3]\, where\, j = 1 \qquad (1.17)$$

[use *(1.16)* and observe that $2^{n+1} + (2^{2^{n+1}} + 1) = 2^{n+j} + F_{n+j}$, where $j = 1$] and so B_{n+j} is composite, where $j = 1$ [use *(1.17)* and observe that $B_{n+1} = 2^{n+1} + F_{n+1}$ and $B_{n+1} > 3$ since $n \geq 3$].
(ii'). *If $2^n \not\equiv 2\, mod[3]$, then the number B_{n+j} is composite, where $j = 0$.* Indeed if $2^n \not\equiv 2\, mod[3]$, then

$$2^n \equiv 1\, mod[3] \qquad (1.18)$$

[use Proposition **1.6**, by observing that $2^n \equiv k\,mod[3]$ if and only if $k \in \{0,1,2\}$]. Now observe (by Proposition **1.1**) that

$$2^{2^n} \equiv 1\,mod[3] \qquad (1.19),$$

and so

$$2^{2^n} + 1 \equiv 2\,mod[3] \qquad (1.20)$$

[use *(1.19)* and elementary arithmetic congruences]. Clearly

$$2^n + (2^{2^n} + 1) \equiv 0 \mod [3] \qquad (1.21)$$

[use *(1.18)* and *(1.20)* and elementay arithmetic congruences]. Clearly

$$2^{n+j} + F_{n+j} \equiv 0\,mod[3]\,where\,j = 0 \qquad (1.22)$$

[use *(1.21)* and observe that $2^n + (2^{2^n}+1) = 2^{n+j} + F_{n+j}$, where $j = 0$] and so B_{n+j} is composite, where $j = 0$ [use *(1.22)* and observe that $B_n = 2^n + F_n$ and $B_n > 3$ since $n \geq 3$]. Proposition **1.7** immediately follows [use (**i'**) and (**ii'**)]$\Big)$.□

Remark 1.8. *There are infinitely many composite numbers of the form $2^n + F_n$ or there are infinitely many prime numbers of the form $2^n + F_n$.* $\Big($ **Proof** Immediate$\Big)$. □

Having made the previous Remarks and Propositions, then Theorem **1** becomes immediate to prove.

Proof of Theorem 1

(R.). Immediate [use Proposition **1.1** and observe that $2^{2^n} = F_n - 1$].
(R.1). Immediate [observe that $2F_0 + 1 + 10d = 7 + 10d$ and $2F_1 + 1 + 10d = 11 + 10d$ and $2F_2 + 1 + 10d = 35 + 10d$ (notice that $2F_2 + 1 + 10d$ is composite), and use Proposition **1.2**].
(R.2). Immediate [notice that $2F_2 + 1 + 10d = 35$ (so $2F_n + 1 + 10d$ is composite) and use Proposition **1.2**].
(R3). Immediate [use Proposition **1.7** and Remark **1.8**].
(R.4). Immediate [indeed let $r \in \{-2, 16\}$. **If** $r = -2$, clearly $-2 + F_0 = 1$ and $-2 + F_1 = 3$ and $-2 + F_2 = 15$ (observe that the only prime is $-2 + F_1 = 3$), and use Proposition **1.3**; now **if** $r = 16$, clearly $16 + F_0 = 19$ and $16 + F_1 = 21$ and $16 + F_2 = 33$ (observe that the only prime is $16 + F_0 = 19$), and use Proposition **1.4**].
(R.5). Immediate [indeed property **(R.5)** is only an immediate consequence of property **(R.4)**].□

2. Proof of Theorem 2.

To prove Theorem **2**, we need the following remarks and Propositions.

Remark 2.0. *Let n be an integer > 2 such that $n \equiv 2\,mod[6]$. If $2^{2^{n-6}} \equiv 16\,mod[19]$, then $(2^{2^{n-6}})^{2^6} \equiv 16\,mod[19]$.* $\Big($**Proof** Indeed observe that $(2^{2^{n-6}})^{2^6} = (2^{2^{n-6}})^{64}$ and

$$(2^{2^{n-6}})^{64} \equiv (-3)^{64}\,mod[19] \qquad (2.1)$$

(since $2^{2^{n-6}} \equiv 16\,mod[19]$ and $16 \equiv -3\,mod[19]$). Observing that

$$(-3)^{64} = (81)^{16}\ and\ 81 \equiv 5\,mod[19] \qquad (2.2)$$

and using *(2.2)*, then it becomes immediate to deduce that congruence *(2.1)* clearly says that

$$(2^{2^{n-6}})^{64} \equiv (5)^{16}\,mod[19] \qquad (2.3).$$

Noticing that

$$(5)^{16} = (625)^4 \; and \; 625 \equiv 17 \, mod[19] \; and \; 17 \equiv -2 \, mod[19] \qquad (2.4)$$

and using *(2.4)*, then it becomes immediate to deduce that congruence *(2.3)* clearly says that

$$(2^{2^{n-6}})^{64} \equiv (-2)^4 \, mod[19] \qquad (2.5).$$

So

$$(2^{2^{n-6}})^{64} \equiv 16 \, mod[19] \qquad (2.6)$$

[use congruence *(2.5)* and observe that $(-2)^4 = 16$] and clearly $(2^{2^{n-6}})^{2^6} \equiv 16 \, mod[19]$ [use congruence *(2.6)* and observe that $(2^{2^{n-6}})^{2^6} = (2^{2^{n-6}})^{64}$]. □)

Proposition 2.1. *Let n be an integer > 0 such that $n \equiv 2 \mod [6]$; then $2^{2^n} \equiv 16 \, mod[19]$.*

(**Proof** Otherwise

$$let \; n \; be \; minimum \; such \; that \; 2^{2^n} \not\equiv 16 \, mod[19] \; (\, n \equiv 2 \, mod[6] \; and \; n > 0) \qquad (2.7).$$

Clearly

$$n \geq 8 \qquad (2.8)$$

(since $2^{2^2} = 16$ and $16 \equiv 16 \, mod[19]$). It is immediate to see that

$$2^{2^n} = (2^{2^{n-6}})^{2^6} \qquad (2.9).$$

Now using equality *(2.9)* and inequality *(2.8)*, we easily deduce that *(2.7)* clearly implies that

$$(2^{2^{n-6}})^{2^6} \not\equiv 16 \, mod[13]; \; 2^{2^{n-6}} \equiv 16 \, mod[19] \; (\, n \equiv 2 \, mod[6]; \; n > 2) \qquad (2.10).$$

(2.10) clearly contradicts Remark **2.0**. □)

Proposition 2.2. *Let n be an integer > 2 such that $n \equiv 2 \, mod[6]$. Then $2 + F_n \equiv 0 \, mod[19]$ and $2 + F_n$ is composite.*

(**Proof** Clearly

$$2 + (2^{2^n} + 1) \equiv 0 \, mod[19] \qquad (2.11)$$

[observe (via Proposition **2.1**) that $2^{2^n} \equiv 16 \, mod[19]$ and use elementary arithmetic congruences]. So $2 + F_n \equiv 0 \, mod[19]$ and $2 + F_n$ is composite [use congruence *(2.11)* and observe that $2+(2^{2^n}+1) = 2+F_n$ and $2+F_n > 19$ (note that $n > 2$)]. Proposition **2.2** immediately follows).□

Remark 2.2. *There are infinitely many composite numbers of the form $2 + F_n$ or there are infinitely many prime numbers of the form $2 + F_n$.*

(**Proof** Immediate). □

Having made the previous Remarks and Propositions, then Theorem **2** becomes immediate to prove.

Proof of Theorem 2

(R'.). Immediate [use Proposition **2.1** and observe that $2^{2^n} = F_n - 1$].
(R'.1). Immediate [use Proposition **2.2** and Remark **2.3**].□

3.Epilogue

In this section, we explain why is natural and not surprising to conjecture that for every $r' \in \{0,2\}$, there are infinitely many primes of the form $r' + F_n$, by using result **(R.)** [use Theorem **1**] and a special case of a Theorem of Dirichlet on arithmetic progression [observe that for every $n \in \{0,1,2,4\}$, $2+F_n$ is prime and $2+F_3$ is not prime; and remark that for every $n \in \{0,1,2,3,4\}$, F_n is prime]. We recall:

Theorem 3. *(Theorem of Dirichlet on arithmetic progression). For any two positive coprime integers a and d, there are infinitely many primes of the form $a+nd$, where n is also a positive integer (In other words, there are infinitely many primes that are congruent to a modulo d)* .□

Observation. *It is natural to conjecture that there are infinitely many primes of the form F_n and there are infinitely many primes of the form $2+F_n$.*

Explanation. Indeed from Result **(R.)** of Theorem **1**, we have

$$F_n - 1 \equiv 1\, mod[3]\; and\, F_n - 1 \equiv 1\, mod[5]\; and\, F_n - 1 \equiv 1\, mod[17]\; (\, n\, is\, an\, integer\; \geq 3) \quad (1.12),$$

clearly

$$F_n \equiv 2\, mod[j]\, (\, j \in \{3,5,17\}),\; for\, every\, integer\, n \geq 3 \qquad (1.13)$$

[use *(1.12)* and elementary arithmetic congruences] and

$$2 + F_n \equiv 4\, mod[j']\, (j' \in \{5,17\})\, and\, 2 + F_n \equiv 1\, mod[3];\; for\, every\, integer\, n \geq 3 \qquad (1.14)$$

[use *(1.12)* and elementary arithmetic congruences].

Now let $A_{2.j} = \{e;\, e\, is\, prime\, and\, e \equiv 2\, mod\, [j]\}$ where $j \in \{3,5,17\}$, $B_{4.j'} = \{e;\, e\, is\, prime\, and\, e \equiv 4\, mod\, [j']\}$ where $j' \in \{5,17\}$ and $B_{1.3} = \{e;\, e\, is\, prime\, and\, e \equiv 1\, mod\, [3]\}$. Since it is immediate that

$$for\, every\, j \in \{3,5,17\}\, we\, have\, (2,j) = 1 \qquad (1.15)$$

and

$$for\, every\, j' \in \{5,17\}\, we\, have\, (4,j') = 1,\; and\; (1,3) = 1 \qquad (1.16),$$

then using (*(1.15)*,*(1.16)*) coupled with a special case of a Theorem of Dirichlet on arithmetic progression (use Theorem **3**), it follows that

$$for\, every\, j \in \{3,5,17\}\; card(A_{2.j})\, is\, infinite \qquad (1.17)$$

[use *(1.15)* and a special case of a Theorem of Dirichlet on arithmetic progression (use Theorem **3**)], and

$$for\, every\, j' \in \{5,17\}\, card(B_{4.j'})\, is\, infinite,\; and\, card(B_{1.3})\, is\, infinite \qquad (1.18)$$

[use *(1.16)* and a special case of a Theorem of Dirichlet on arithmetic progression (use Theorem **3**)].

Now using *(1.13)* and *(1.17)* and the fact that for every $n \in \{0,1,2,3,4\}$ F_n is prime, then it becomes naturel to conjecture the following.

Conjecture 1. *Union of the sets $A_{2.3}$, $A_{2.5}$, and $A_{2.17}$ (denoted by $A_{2.3} \bigcup A_{2.5} \bigcup A_{2.17}$) contains infinitely many numbers of the form F_n ($A_{2.3}$ and $A_{2.5}$ and $A_{2.17}$ are defined via the Observation placed just above).*

Note. It is trivial to see that the previous conjecture immediately implies that there are infinitely many Fermat primes.

That being said, using *(1.14)* and *(1.18)* and the fact that for every $n \in \{0, 1, 2, 4\}$, $2 + F_n$ is prime, then it becomes naturel to conjecture the following.

Conjecture 2. *Union of the sets $B_{4.5}$, $B_{4.17}$, and $B_{1.3}$ (denoted by $B_{4.5} \bigcup B_{4.17} \bigcup B_{1.3}$) contains infinitely many numbers of the form $2 + F_n$ ($B_{4.5}$ and $B_{4.17}$ and $B_{1.3}$ are defined via the Observation placed just above).*

Note. It is also trivial to see that Conjecture **2** immediately implies that there are infinitely many primes of the form $2 + F_n$

Conjecture **1** and Conjecture **2** immediately imply that there are infinitely many Fermat primes and there are infinitely many primes of the form $2 + F_n$.

Conclusion. In this work, we consider Fermat numbers and obtain useful new results on them. Basic results are obtained on Fermat primes and Fermat composites related with infinity. These results are new in the literature and will be useful for other sciences mentioned as above with mathematics too. The paper will be provide advantages for next works.

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