Arithmetic Moving Average and Optimization of Expected Utility of Wealth

Abstract

From the perspective of asset allocation, the moving average trading method is studied by providing the complete optimal investment solution for the expected log-utility of wealth under the arithmetic moving average (AMA) rule. The technical analysis adds value to the practical fixed allocation rules if stock returns are not predictable. We also show that the implement approximation for the optimal strategy can be constructed explicitly and is convergent to the theoretical optimal investment solution for the AMA. We illustrate numerically that the geometric moving average (GMA) rule can either overestimate or underestimate the practical AMA rule.

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1 Introduction

The disparity between academic finance and industrial practice is the disaffiliation between technical analysis and academic financial theory. For the technical analysis, it is lack of theoretic foundation; For the academic analysis, it rules out any predictability from technical charting by assuming geometric Brownian motions for the stock prices. The empirical data falls inconclusive between the technical analysis and the academics. Lo, Mamaysky and Wang [1] propose computational algorithms and statistical inference to recognize the effectiveness of technical analysis on a large number of U. S. stocks from 1962 to 1996. They find out, over 34-year sample period, several technical indicators do provide incremental information and may have some practice value. Zhu and Zhou [2] further indicate that all major brokerage firms issue technical commentary on the market and many advisory services come from technical analysis. Covel [3] advocates the technical analysis exclusively by quoting examples of large and successful hedge funds, without using any fundamental information on the market. However, those facts mentioned above are not easy to convince the academics. Among academics, the joint distribution (chartings) of prices and volumes contribute important information, but is lack of the financially theoretical support. Lo and MacKinlay [4] have shown that past prices may be used to forecast future returns to some degree, a fact that all technical analysts take for granted. Zhu and Zhou [2] realize it is theoretically reasonable to use technical analysis in a standard asset allocation problem. They show that the use of geometric moving average (GMA) rule combined with the fixed rule can help increase the investor's utility of wealth by solving the the problem of allocating the optimal amount of stock. They analyze the GMA with explicit solutions under log-utility due to the complexity and difficulty of the distribution of the arithmetic moving average (AMA), though the GMA rule is not a widely used strategy in real investment world. Their solution for the GMA is not complete since the optimal amount of the expected utility can be reached for other constraints.

The AMA rule is a common market timing strategy that shifts investments between cash and stocks. The use of the AMA rule can not only help increase the investors' utility of wealth enormously, but interlink the gulf between quantitative finance and technical analysis through a systemic and scientific approach to the technical analysis practice and by using the now-standard empirical analysis to gauge the efficacy of technical indicators over time and across securities.

In this present paper, the utility of the most popular technical trading method, AMA, is analyzed. And we solve the optimal problem that investors buy the stock when its current stock price is moving above the arithmetic average price over a given period L for both constraints on the ratio of pure AMA rule and the ratio without AMA rule. The constraints on the ratio of pure AMA rule and the ratio without AMA rule determine the allocation of the wealth. Our analysis indicates that the optimal trading strategy is a linear combination of a fixed strategy and a pure AMA strategy. The technical analysis from the AMA therefore can be adopted to increase the expected utility and improve the popular fixed strategy in Markowitz [5] portfolio theory. Our complete solution for improving the fixed strategy with the AMA should have the practical importance in the real world. In particular, when there are more ambiguous models for the stock price, the AMA method provides more advantage in real world for investors. It is model-free and easy to compute for the optimal trading strategy. For the log-utility function of wealth, explicitly the approximated investment strategy is constructed. And we prove that the approximated investment strategy indeed converges to the optimal investment solution. The implementation is based on four sets of parameters. One set of parameters is from Huang and Liu [6]. The other three sets of parameters are those used in Zhu and Zhou [2]. The optimal values and the optimal strategy choices under the AMA rule are given and compared with the counterparts under the GMA rule given in Zhu and Zhou [2]. Zhu and Zhou [2] mention that the optimal value for the expected log-utility can be achieved in all possibilities from various constraints, though they only give the optimal value for the expected log-utility under GMA from two constraints. The complete optimal value for the expected log-utility under AMA and the associated optimal strategy choices are given in this paper, which include the optimal values on all boundaries and at the interior part. We find out that the optimal investment strategy from the GMA rule can be misleading in practice for the AMA rule. The optimal log-utility function of wealth from the GMA can either overestimate or underestimate that from the practical AMA rule[1].

The paper is organized as follows. In Section 2, the background of the mathematical problem is introduced. The main results of this paper are presented in Section 3. We define a discrete time and discrete state approximation from practice point of view for the optimal choice of the strategy and show this approximation converges to the continuous time counterpart in section 4. Section 5 gives numerical examples of the technical analysis. We conclude in Section 6, and the proof is given in Appendix.

2 Optimal Investment Problem

We set the theoretical background for the optimal investment problem with the expected utility of wealth subject to the budget constraint following from the general model developed by Merton [7]. We assume the following dynamics for the cum-dividend stock price following Kim and Omberg [8] and Huang and Liu [6],

$$\frac{dS_t}{S_t} = (\mu_0 + \mu_1 X_t)dt + \sigma_s dB_t, \tag{1}$$

$$dX_t = (\theta_0 + \theta_1 X_t)dt + \rho\sigma_x dB_t + \sqrt{1-\rho^2}\sigma_x dZ_t, \tag{2}$$

[1] Zhu and Zhou [2] provide an excellent literature review on the technical analysis related to the arithmetic moving average in section 2. We refer to their paper for further references.

where $\mu_0,\ \mu_1,\ \theta_0,\ \theta_1(<0),\ \sigma_s,\ \sigma_x$ are parameters, $\rho \in [-1,1]$, X_t is the stationary predictable variable from the Ornstein-Uhlenbeck process, B_t and Z_t are independent Brownian motions, and $\theta_1 < 0$ is a mean reverting process.

Suppose that W_0 is the initial wealth, T is the investment horizon and X_0 is normally distributed with mean M_{0^-} and variance $V(0^-)$. Let $\mathcal{F}_t$ be the filtration at time t generated by $\{S_u\}_{0\le u\le t}$ and the prior $(M_{0^-}, V(0^-))$. Assume that an investor has HARA preference over $[0,T]$. The standard allocation problem of an investor is to decide a portfolio strategy ξ_t to maximize the expected utility of wealth,

$$\max_{\xi_t} \mathbb{E}[u(W_T)] \tag{3}$$

subject to the budget constraint

$$dW_t = rW_t dt + \xi_t(\mu_0 + \mu_1 X_t - r)W_t dt + \xi_t \sigma_s W_t dB_t. \tag{4}$$

The solution to (3) and (4) is the optimal investment strategy.

For a power type utility, $u(W_T) = \frac{\gamma}{1-\gamma}\left(\frac{\lambda W}{\gamma} + \eta\right)^{1-\gamma}$, where γ is the investor's risk aversion parameter ($\gamma \neq 0, 1$), Huang and Liu [6] gave an implicit form of the optimal dynamic strategy, ξ_t^*, in the Proposition 1 therein. If the stock returns are assumed to be independently and identically distributed, Markowitz [5] suggests the optimal strategy is:

$$\xi^*_{fix1} = \frac{\mu_s - r}{\gamma\sigma_s^2}, \tag{5}$$

where $\mu_s = \mu_0 + \mu_1\overline{X} = \mu_0 + \mu_1 EX_t = \mu_0 - \frac{\mu_1\theta_0}{\theta_1}$ is the long term mean of the stock return. In practice, this is an important benchmark model on the investment strategy. Apparently, ξ^*_{fix1} is no longer optimal if stock returns are not i.i.d. The constant ξ^*_{fix1} indicates a fixed portion of wealth invested into the stock all the time and ignores any predictability completely. If the investor knows the true predictable process but not the state variable, Zhu and Zhou [2] give the optimal constant strategy for the power utility, $u(W_T) = \frac{W_T^{1-\gamma}}{1-\gamma}$(See formula (15)-(17) therein). If the log-utility function, $u(W_T) = \log(W_T)$, is considered, Zhu and Zhou [2] obtain the same optimal solution as that in (5). That is, $\xi^*_{fix1} = \frac{\mu_s - r}{\sigma_s^2}$.

3 Optimal Strategy under the Arithmetic Moving Average

In this section, we study a time-varying strategy, Arithmetic moving average (AMA) strategies, for the log-utility. The complete and explicit solution to the optimal AMA strategy is provided. The difference between the optimal AMA strategy and the optimal fixed allocation is given. Our analysis indicates that the optimal trading strategy is a linear combination of a fixed strategy and a pure AMA strategy. The technical analysis from the AMA strategy therefore can be used to maximize the expected utility and improve the popular fixed strategy in Markowitz [5] portfolio theory.

Let $L > 0$ be the lag or the lookback period. The AMA of the stock price $\{S_t\}_{t\geq 0}$ at time t is given by

$$A_t = \frac{1}{L}\int_{t-L}^{t} S_u du. \tag{6}$$

The simplest moving average trading rule is the following stock allocation strategy:

$$\eta_t = \eta(S_t,\ A_t) = \begin{cases} 1,\ if\ S_t > A_t \\ 0,\ \text{otherwise.} \end{cases} \tag{7}$$

When $t > L$, η_t is well-defined and $\eta_t = 0$ for $t \leq L$. The Arithmetic Moving Average (AMA) rule is

$$\xi_t = \xi_{fix} + \xi_{mv}\eta(S_t,\ A_t), \tag{8}$$

where ξ_{fix} and ξ_{mv} are constant. The strategy consists of a fixed strategy and a pure moving strategy. If $\xi_{mv} = 0$ and $\xi_{fix} = \xi^*_{fix1}$, then the strategy invests a fixed portion of wealth into the stock all the time. If $\xi_{mv} = 1$ and $\xi_{fix} = 0$, then this strategy is commonly adapted to the pure AMA strategy which positions in the stock or nothing with portfolio weight η_t. All these strategies are commonly used and may not be optimal almost surely due to the irrelevance to the investor's tolerance γ to stock risk and to the degree of predictability. The optimal choice of ξ_{fix} and ξ_{mv} is the goal of this study for the AMA and the log-utility function, $u(W_T) = \log W_T$.

Recall (4). By Itô formula, we have that

$$
\begin{aligned}
\log W_T &= \log W_0 + rT + \int_0^L dt \left[\xi^*_{fix1}\left(\mu_0+\mu_1 X_t - r - \frac{\sigma_s^2}{2}\xi^*_{fix1}\right)\right] \\
&\quad + \int_L^T \left[\xi_{fix}\left(\mu_0+\mu_1 X_t - r - \frac{\sigma_s^2}{2}\xi_{fix}\right)\right] dt + \xi_{mv}\mu_1 \int_L^T \hat{X}_t \eta_t dt \\
&\quad + \int_L^T \left[\xi_{mv}(\mu_0+\mu_1\bar{X} - r) - \frac{\sigma_s^2}{2}\xi_{mv}^2 - \sigma_s^2 \xi_{fix}\xi_{mv}\right]\eta_t dt \\
&\quad + \sigma_s \int_0^T (\xi_{fix} + \xi_{mv}\eta_t) dB_t, \qquad (9)
\end{aligned}
$$

where $\hat{X}_t = X_t - \bar{X}$ with $\bar{X} = -\theta_0/\theta_1$. Then the expected log-utility is

$$
\begin{aligned}
U_{AMA} &= \mathbb{E}\log W_T = \log W_0 + rT + \frac{(\mu_0+\mu_1\bar{X}-r)^2}{2\sigma_s^2} L \\
&\quad + \xi_{fix}\left(\mu_0+\mu_1\bar{X} - r - \frac{\sigma_s^2}{2}\xi_{fix}\right)(T-L) + \xi_{mv}\mu_1 \int_L^T b_1(t) dt \\
&\quad + \left[\xi_{mv}(\mu_0+\mu_1\bar{X}-r) - \frac{\sigma_s^2}{2}\xi_{mv}^2 - \sigma_s^2\xi_{fix}\xi_{mv}\right]\int_L^T b_2(t)dt, \qquad (10)
\end{aligned}
$$

where

$$
b_1(t) \equiv \mathbb{E}[\hat{X}_t \eta_t(S_t,\ A_t)],\ b_2(t) \equiv \mathbb{E}[\eta_t(S_t,\ A_t)] = P(S_t > A_t). \qquad (11)
$$

Remark 3.1. *Both $b_1(t)$ and $b_2(t)$ proposed by Zhu and Zhou [2] are proved to be constant in their GMA model. However, neither $b_1(t)$ nor $b_2(t)$ is a constant in our AMA model. In the real world, the probability of $(S_t > A_t)$ at any given time t and $\mathbb{E}[\hat{X}_t\eta_t(S_t,\ A_t)]$ both vary according to the time t.*

We first answer the question that what the optimal choices of ξ_{fix} and ξ_{mv} are in the interior part of the region $\xi_{fix} \in [0, \xi^*_{fix1}]$ and $\xi_{mv} \in [0,1]$ for AMA in Proposition 3.2 below. Then we restrict to four different boundaries to find other optimal choices of ξ_{fix} and ξ_{mv} in Proposition 3.3. By the end, we compare all these choices to obtain the globally optimal choice to be the optimal investment strategy for AMA in Theorem 3.5.

Proposition 3.2. *In the class of strategies ξ_t, the interior optimal choice of ξ_{fix} and ξ_{mv} under the log-utility is*

$$
\xi^*_{fix} = \frac{\mu_s - r}{\sigma_s^2} - \frac{\mu_1 A}{(T-L-B)\,\sigma_s^2},\ \xi^*_{mv} = \frac{\mu_1(T-L)A}{B\,(T-L-B)\,\sigma_s^2}, \qquad (12)
$$

and

$$U^*_{AMA0} = U^*_{fix1} + \frac{\mu_1^2 (A)^2 (T-L)}{2B(T-L-B)\sigma_s^2} \geq U^*_{fix1}, \tag{13}$$

for $\xi_{fix} \in (0, \xi^*_{fix1})$ *and* $\xi_{mv} \in (0,1)$. *Here and later on,* $A = \int_L^T b_1(t)dt$, $B = \int_L^T b_2(t)dt$ *and the value associate with* ξ^*_{fix1}, $U^*_{fix1} = \log W_0 + rT + \frac{(\mu_0+\mu_1\overline{X}-r)^2}{2\sigma_s^2}T$.

Proposition 3.2 is a direct generalization of Proposition 1 of Zhu and Zhou [2]. From Proposition 3.2, it follows that the improvement over ξ^*_{fix1} is positive by combining a suitable fixed strategy with the AMA one (this is a similar verification as (35) of Zhu and Zhou [2]). However, the interior optimal strategy may not be the optimal in general since the AMA rule can happen to be along the fixed strategies on those boundaries, $\xi_{fix} = 0$ or ξ^*_{fix1} and $\xi_{mv} = 0$ or 1. For the GMA case, Zhu and Zhou [2] only give the optimal solutions from the boundaries of $\xi_{fix} = \xi^*_{fix1}$ and $\xi_{fix} = 0$. For the AMA case, we give the optimal solutions from all of the four boundaries in the following proposition.

Proposition 3.3. *(1) In the class of strategies* ξ_t *with restriction* $\xi_{mv} = 0$, *the optimal choice of* ξ_{fix} *under the log-utility is* $\xi^*_{fix} = \xi^*_{fix1} = \frac{\mu_s - r}{\sigma_s^2}$ *and the associated value function is* $U^*_{AMA1} = U^*_{fix1}$.
(2) In the class of strategies ξ_t *with restriction* $\xi_{mv} = 1$, *the optimal choice of* ξ_{fix} *under the log-utility is* $\xi^*_{fix} = \frac{\mu_s - r}{\sigma_s^2} - \frac{B}{T-L}$ *if* $\frac{\mu_s - r}{\sigma_s^2} - \frac{B}{T-L} > 0$ *and the associated value function is*

$$U^*_{AMA2} = U^*_{fix1} + \mu_1 A - \frac{\sigma_s^2 B}{2(T-L)}(T-L-B). \tag{14}$$

(3) In the class of strategies ξ_t *with restriction* $\xi_{fix} = 0$, *the optimal choice of* ξ_{mv} *under the log-utility is* $\xi^*_{mv} = \frac{\mu_1 A + (\mu_s - r)B}{\sigma_s^2 B}$ *if* $\frac{\mu_1 A + (\mu_s - r)B}{\sigma_s^2 B} \in (0,\ 1)$ *and the associated value function is*

$$U^*_{AMA3} = U^*_{fix1} + \frac{(\mu_1 A + (\mu_s - r)B)^2}{2\sigma_s^2 B} - \frac{(\mu_s - r)^2 (T-L)}{2\sigma_s^2}. \tag{15}$$

(4) In the class of strategies ξ_t *with restriction* $\xi_{fix} = \xi^*_{fix1}$, *the optimal choice of* ξ_{mv} *under the log-utility is* $\xi^*_{mv} = \frac{\mu_1 A}{\sigma_s^2 B}$ *if* $\frac{\mu_1 A}{\sigma_s^2 B} \in (0,\ 1)$ *and the associated value function is*

$$U^*_{AMA4} = U^*_{fix1} + \frac{\mu_1^2 (A)^2}{2\sigma_s^2 B} \geq U^*_{fix1}. \tag{16}$$

Comparing with proposition 2 and proposition 3 of Zhu and Zhou [2], proposition 3.3 is a complete version with respect to AMA. Proposition 2 of Zhu and Zhou [2] only considers $\xi_{fix} = \xi^*_{fix1}$ as Case (4) in our proposition 3.3, and proposition 3 of Zhu and Zhou [2] as Case (3) in our proposition 3.3.

Remark 3.4. *From Proposition 3.3, it follows that* U^*_{AMA2} *and* U^*_{AMA3} *could be either greater than or less than* U^*_{fix1}, $U^*_{AMA1} = U^*_{fix1}$, *and that* U^*_{AMA0} *and* U^*_{AMA4} *are both greater than* U^*_{fix1} *if each corresponding* $(\xi^*_{fix},\ \xi^*_{mv})$ *satisfies the corresponding restrictions. Hence, the optimal value function is* $U^*_{AMA} = \max\{U^*_{AMA0},\ U^*_{AMA1},\ U^*_{AMA2},\ U^*_{AMA3},\ U^*_{AMA4}\}$. *In the next theorem, we will show that* U^*_{AMA0} *is the greatest among all those interior and boundary optimal choices if its corresponding* $(\xi^*_{fix},\ \xi^*_{mv})$ *satisfies* $\xi^*_{fix} \in (0, \xi^*_{fix1})$ *and* $\xi^*_{mv} \in (0, 1)$. *I.e.,* U^*_{AMA} *is actually equal to* U^*_{AMA0} *if the corresponding* $(\xi^*_{fix},\ \xi^*_{mv})$ *given in (12) do lies in the interior part. On the other hand, the optimal expected log-utility value must be obtained on one of the four boundaries if the* $(\xi^*_{fix},\ \xi^*_{mv})$ *with respect to* U^*_{AMA0} *does not lie in the interior part.*

Theorem 3.5. *The overall optimal value function is given by* $U^*_{AMA} = U^*_{AMA0}$, *where* U^*_{AMA0} *is the one given in (13), if the optimal choice given in (12) satisfies* $\xi^*_{fix} \in (0, \xi^*_{fix1})$ *and* $\xi^*_{mv} \in (0, 1)$. *Otherwise,* $U^*_{AMA} = \max\{U^*_{AMA0},\ U^*_{AMA1},\ U^*_{AMA2},\ U^*_{AMA3},\ U^*_{AMA4}\}$.

Proofs of Proposition 3.2, Proposition 3.3 and Theorem 3.5 are given in the appendix.

4 Approximation of the Optimal Strategy

As we mentioned in Remark 3.1, $b_1(t)$ and $b_2(t)$ proposed by Zhu and Zhou [2] are showed to be constant because their definitions are based on the GMA model and the stock price is log-normal. Furthermore, the closed form formulas for $b_1(t)$ and $b_2(t)$ are given in Zhu and Zhou [2] accordingly. However, it is not the case in the real world for AMA. Even the stock price usually does not follow any log-normal distribution. Maller, Solomon and Szimayer [9] define the stock price as an exponential of a Lévy process. As a result, one cannot find a closed form formula for the optimal expected utility of wealth for those general models of the stock price.

In this section, we evaluate $b_1(t)$, $t \in [L,\ T]$, and $b_2(t)$, $t \in [L,\ T]$, defined in (11) for AMA instead of GMA by giving a discrete time and discrete value approximation from practice point of view. It is easy to implement and give us insight on extending the model based on a Brownian motion to any process that has a convergence discrete approximation under the uniform topology.

Firstly, we give the explicit expressions of the closed forms of X_t, S_t and A_t/S_t, for $t \geq 0$. By (2), we have the solution for the Ornstein-Uhlenbeck process

$$X_t = X_0 e^{\theta_1 t} - \frac{\theta_0}{\theta_1}\left(1 - e^{\theta_1 t}\right) + \rho\sigma_x \int_0^t e^{\theta_1(t-s)} dB_s + \sqrt{1-\rho^2}\sigma_x \int_0^t e^{\theta_1(t-s)} dZ_s. \tag{17}$$

Hence, X_t is normally distributed with $EX_t = -\frac{\theta_0}{\theta_1} = M_{0^-}$, and $VarX_t = -\frac{\sigma_x^2}{2\theta_1} = V(0^-)$.

By (1) and (17), we obtain, for any $t \geq 0$,

$$S_t = C_t \exp\{SE(t)\}, \tag{18}$$

where $C_t = S_0 \exp\left\{(\mu_0 - \frac{\sigma_s^2}{2} - \frac{\mu_1\theta_0}{\theta_1})t\right\}$ and

$$\begin{aligned} SE(t) &= \mu_1\left(\frac{X_0}{\theta_1} + \frac{\theta_0}{\theta_1^2}\right)\left(e^{\theta_1 t} - 1\right) \\ &+ \sigma_s B_t + \mu_1\sigma_x\rho \int_0^t e^{\theta_1(t-u)} B_u du + \mu_1\sigma_x\sqrt{1-\rho^2} \int_0^t e^{\theta_1(t-u)} Z_u du. \end{aligned}$$

The proof of Equation (18) is given in the appendix. It follows from the basic Itó Lemma and Fubini theorem. By (6), the definition of the arithmetic average over the period L, and (18), we get the expression, for any $t \geq L$,

$$\frac{A_t}{S_t} = \frac{1}{L}\int_{t-L}^t \frac{S_u}{S_t} du = \frac{1}{L}\int_{t-L}^t \exp\left\{\left(\mu_0 - \frac{\sigma_s^2}{2} - \frac{\mu_1\theta_0}{\theta_1}\right)(u-t)\right\}\exp\{E(u)\}du, \tag{19}$$

where

$$\begin{aligned} E(u) &= \mu_1\left(\frac{X_0}{\theta_1} + \frac{\theta_0}{\theta_1^2}\right)\left(e^{\theta_1 u} - e^{\theta_1 t}\right) + \sigma_s(B_u - B_t) \\ &+ \mu_1\sigma_x\rho\left(\int_0^u e^{\theta_1(u-v)} B_v dv - \int_0^t e^{\theta_1(t-v)} B_v dv\right) \\ &+ \mu_1\sigma_x\sqrt{1-\rho^2}\left(\int_0^u e^{\theta_1(u-v)} Z_v dv - \int_0^t e^{\theta_1(t-v)} Z_v dv\right). \end{aligned}$$

The cum-dividend stock price given in (18) is log-normal. Whereas, the arithmetic average over a period is no longer log-normal. Li and Chen [10] study a few properties of the arithmetic average of the log-normal stock. Note that $\frac{A_t}{S_t}$ is equal to an integral of log-normal distribution, which is no longer log-normal distributed. Hence, we could not evaluate exactly. We look for a discrete approximation of this term $\frac{A_t}{S_t}$ in such a way that it is easy to implement. First of all, for the Brownian motion $\{B_t, t \geq 0\}$, we take the discrete approximation, $\{B_t(n), t \geq 0\}$, proposed in the proof of Theorem 3.3 in Szimayer and Maller [11], which can be viewed as a modification of the approximation proposed by Itó and McKean [12]. Here, for the convenience of the readers, we repeat the construction of the approximation for the Brownian motion $\{B_t, t \geq 0\}$. For any $n \in N$, let $0 = t_0(n) < t_1(n) < \cdots < t_{\lfloor nT \rfloor}(n) \leq T$ be an equal interval partition of $[0,\ T]$, such that $t_j(n) - t_{j-1}(n) = \Delta t(n) = \frac{1}{n},\ j = 1, 2, \ldots, \lfloor nT \rfloor$. Define stopping times by: $e_0(n) = 0$, and for $j = 1, 2, \ldots,$

$$e_j(n) = \inf \left\{ t > e_{j-1}(n) :\ |B_t - B_{e_{j-1}(n)}| \geq \sqrt{\Delta t(n)} \right\},\ j = 1, 2, \ldots.$$

Let $B_t^{(n)},\ t \in [0,\ T]$, be a step function valued random variable defined by

$$B_t^{(n)} = B_{e_{j-1}(n)},\ t_{j-1}(n) \leq t < t_j(n).$$

By the arguements in Szimayer and Maller [11] and Itó and McKean [12],

$$\sup_{0 \leq t \leq T} \left| B_t^{(n)} - B_t \right| \longrightarrow 0 \ \textit{almost surely},\ \textit{as}\ n \to \infty. \tag{20}$$

Similarly, for the Brownian motion $\{Z_t,\ t \geq 0\}$, an approximation denoted by $\{Z_t^{(n)}),\ t \geq 0\}$ can be defined, such that

$$\sup_{0 \leq t \leq T} |Z_t^{(n)} - Z_t| \longrightarrow 0 \ \textit{almost surely},\ \textit{as}\ n \to \infty. \tag{21}$$

Remark 4.1. *From the definitions of $B_t^{(n)}$ and $Z_t^{(n)}$, we have that*

$$B_t^{(n)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} V_i, \tag{22}$$

where $V_i,\ i = 1, 2, \ldots$ are i.i.d with distribution $P(V_i = 1) = P(V_i = -1) = \frac{1}{2}$ and that

$$Z_t^{(n)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} Y_i, \tag{23}$$

where $Y_i,\ i=1,2,\dots$ *are also i.i.d with the same distribution with* V_i. *Note that,* $\{Y_i,\ i=1,2,\dots\}$ *are independent of* $\{V_i,\ i=1,2,\dots\}$ *because* $\{Z_t,\ t\in[0,\ T]\}$ *is independent of* $\{B_t,\ t\in[0,\ T]\}$.

Recall that X_0 is normally distributed with expectation $-\frac{\theta_0}{\theta_1}$ and variance $-\frac{\sigma_x^2}{2\theta_1}$. We take the classic approximation of X_0, denoted by $X_0^{(n)}$, such that $X_0^{(n)}$ converges in distribution to X_0. That is, For any $n\in N$, let

$$X_0^{(n)} = -\frac{\theta_0}{\theta_1} + \sqrt{-\frac{\sigma_x^2}{2\theta_1 n}}\sum_{i=1}^{n} R_i, \tag{24}$$

where $R_i,\ i\geq 1$ are i.i.d random variables with distribution: $P(R_i=1)=P(R_i=-1)=\frac{1}{2}$ (see Page 357 of Billingsley [13]). Notice that, R_i is independent of V_i and Y_i for all $i\geq 1$ and $j\geq 1$ and that the distribution of $X_0^{(n)}$ is given by

$$P\left(X_0^{(n)} = -\frac{\theta_0}{\theta_1} + (2k-n)\sqrt{-\frac{\sigma_x^2}{2\theta_1 n}}\right) = \frac{n!}{k!(n-k)!2^n}.$$

Now we construct the approximated the cum-dividend stock price $S_t^{(n)}$ and its arithmetic average $A_t^{(n)}$ by simply replace B_t, Z_t and X_0 in (18) with $B_t^{(n)}$, $Z_t^{(n)}$ and $X_0^{(n)}$ respectively. That is, let the approximated stock price at $t\geq 0$ be

$$S_t^{(n)} = C_t \exp\left\{\mu_1\left(\frac{X_0^{(n)}}{\theta_1}+\frac{\theta_0}{\theta_1^2}\right)\left(e^{\theta_1 t}-1\right)+Q_t^{(n)}\right\}, \tag{25}$$

where

$$Q_t^{(n)} = \sigma_s B_t^{(n)} + \mu_1\sigma_x\rho\int_0^t e^{\theta_1(t-u)}B_u^{(n)}du + \mu_1\sigma_x\sqrt{1-\rho^2}\int_0^t e^{\theta_1(t-u)}Z_u^{(n)}du, \tag{26}$$

and its corresponding arithmetic average

$$A_t^{(n)} = \frac{1}{L}\int_{t-L}^{t} S_u^{(n)}du. \tag{27}$$

The following three theorems state the convergence results of the approximations, and further indicate that the corresponding optimal choice of ξ_{fix} and ξ_{mv} under the log-utility have such approximations that the desired optimal expected log-utility function is approximated by the construction.

Theorem 4.2. *For any* $t\in[L,T]$, $\frac{A_t^{(n)}}{S_t^{(n)}} \longrightarrow \frac{A_t}{S_t}$ *in distribution as* $n\to\infty$.

Assume that $P(\frac{A_t}{S_t}=1)=0$[2]. Then, the following corollary is an immediate result of Theorem 4.2.

Corollary 4.3. *For any* $t\in[L,T]$, $b_2^{(n)}(t)=P\left(\frac{A_t^{(n)}}{S_t^{(n)}}<1\right)\longrightarrow P\left(\frac{A_t}{S_t}<1\right)=b_2(t)$ *as* $n\to\infty$.

Theorem 4.4. *For any* $t\in[L,T]$, $b_1^{(n)}(t)=E\left(\widehat{X}_t^{(n)}\eta_{\{S_t^{(n)}>A_t^{(n)}\}}\right)\longrightarrow E\left(\widehat{X}_t\eta_{\{S_t>A_t\}}\right)=b_1(t)$ *as* $n\to\infty$, *where*

$$\widehat{X}_t=X_t-\overline{X}=X_t+\frac{\theta_0}{\theta_1}=\left(X_0+\frac{\theta_0}{\theta_1}\right)e^{\theta_1 t}+\rho\sigma_x\int_0^t e^{\theta_1(t-s)}dB_s+\sqrt{1-\rho^2}\sigma_x\int_0^t e^{\theta_1(t-s)}dZ_s,$$

$$\begin{aligned}\widehat{X}_t^{(n)} &= \left(X_0^{(n)}+\frac{\theta_0}{\theta_1}\right)e^{\theta_1 t}+\rho\sigma_x\int_0^t e^{\theta_1(t-s)}dB_s^{(n)}+\sqrt{1-\rho^2}\sigma_x\int_0^t e^{\theta_1(t-s)}dZ_s^{(n)}\\ &= \left(X_0^{(n)}+\frac{\theta_0}{\theta_1}\right)e^{\theta_1 t}+\rho\sigma_x\left(B_t^{(n)}+\theta_1\int_0^t e^{\theta_1(t-s)}B_s^{(n)}ds\right)\\ &\quad +\sigma_x\sqrt{1-\rho^2}\left(Z_t^{(n)}+\theta_1\int_0^t e^{\theta_1(t-s)}Z_s^{(n)}ds\right).\end{aligned}$$

By Corollary 4.3 and Theorem 4.4, we can obtain the approximation for $A=\int_L^T b_1(t)dt$ and $B=\int_L^T b_2(t)dt$.

Theorem 4.5. $A_n=\int_L^T b_1^{(n)}(t)dt\longrightarrow A,\ B_n=\int_L^T b_2^{(n)}(t)dt\longrightarrow B,$ *as* $n\to\infty$.

Proofs of Theorem 4.2, Theorem 4.4 and Theorem 4.5 are given in the appendix. The implementation in next section is to compute the approximated values of $U^*_{AMAi},\ i=0,1,2,3,4$, and their corresponding ξ^*_{fix}'s, ξ^*_{mv}'s constructed in Proposition 3.2 and 3.3 by replacing A and B with A_n and B_n, respectively.

5 Empirical Analysis

In this section, the evaluation of the approximated optimal expected log-utility of wealth under the AMA is illustrated. Meanwhile, the optimal log-utilities under the AMA and those under the GMA are compared.

[2]For geometric Brownian motions and exponential Lévy process of the stock price, this assumption is valid. In fact, this should be true for any nontrivial model of the stock price.

We outline the procedure of evaluating the optimal expected log-utility of wealth under the AMA, U^*_{AMA0}, with a fixed $L < T$. The procedure can be implemented by the following steps:

(1) Simulate the paths of the discrete time and discrete states processes $V = \{V_i,\ i \in \mathbb{N}\}$ and $Y = \{Y_i,\ i \in \mathbb{N}\}$ with probabilities;

(2) Obtain the simple random walk $\{B_t^{(n)},\ 0 \le t \le T\}$ and $\{Z_t^{(n)},\ 0 \le t \le T\}$ by plugging the paths of V as well as Y into terms (22) and (25);

(3) Evaluate $S_t^{(n)},\ 0 \le t \le T$ and $\frac{A_t^{(n)}}{S_t^{(n)}},\ L \le t \le T$ by term (25), (26) and (27) in Section 4;

(4) Find the values of $b_2^{(n)}(t) = P\left(\frac{A_t^{(n)}}{S_t^{(n)}} < 1\right)$ and $b_1^{(n)}(t) = E(\widehat{X}_t^{(n)} \eta_{\{S_t^{(n)} > A_t^{(n)}\}})$ for any $t \in [L, T]$ and evaluate A_n, B_n given in Theorem 4.5;

(5) Approximate $U^*_{AMAi},\ i = 0,\ 1,\ 2,\ 3,\ 4$ and their corresponding ξ^*_{fix} and ξ^*_{mv} by simply replacing A and B with A_n and B_n in the Proposition 3.2 and the Proposition 3.3.

The most challenge part of the implementation is to simulate the paths of the discrete time and discrete states process $V = \{V_i,\ i = 1, 2, ..., n\}$ or $Y = \{Y_i,\ i = 1, 2, ..., n\}$ with probability of each path stated in step (1). By the definitions of processes V and Y defined in Section 4, the process $Y = \{Y_i,\ i = 1, 2, ..., n\}$ has the same distribution with the process $V = \{V_i,\ i = 1, 2, ..., n\}$. Because there are 2^n equally likely paths of V or Y, each path has the probability of $\frac{1}{2^n}$. To distinguish the paths, we construct a bijective correspondence between the set $\{1,\ 2,\ 3, ...,\ 2^n\}$ and the set of the paths as follows.

Each $j (\in \{0,\ 1,\ 2,\ 3, ...,\ 2^n - 1\})$ is converted into a binary numeral. If necessary, $-1's$ are attached to the front of the binary number such that there are n digits in total. The result is the $j + 1$th path of the sample process V. This finishes the step (1).

To process step (3), recall (22)-(24). Then, (25), (26) can be rewritten respectively as

$$S_t^{(n)} = C_t \exp\left\{\mu_1 \left(\frac{1}{\theta_1}\sqrt{-\frac{\sigma_x^2}{2\theta_1 n}} \sum_{i=1}^{n} R_i\right)\left(e^{\theta_1 t} - 1\right) + Q_t^{(n)}\right\},$$

if $t \in [t_i,\ t_{i+1}),\ i = 0,\ 1,\ \ldots,\ \lfloor nT \rfloor$, where

$$Q_t^{(n)} = \sigma_s B_{t_i}^{(n)} + \mu_1 \sigma_x \rho \sum_{j=0}^{\lfloor nt \rfloor} B_{t_j}^{(n)} \int_{t_j}^{t_j \wedge t} e^{\theta_1 (t-u)} du + \mu_1 \sigma_x \sqrt{1-\rho^2} \sum_{j=0}^{\lfloor nt \rfloor} Z_{t_j}^{(n)} \int_{t_j}^{t_j \wedge t} e^{\theta_1 (t-u)} du.$$

Our goal is to evaluate the optimal expected log-utility of wealth under the AMA and show the difference between the GMA strategy and the AMA strategy. The evaluations and comparisons are given with respect to the Term-spread, the Dividend yield, the Consumption-wealth ratio and the Payout ratio, respectively. The parameters are given in Table 1. The parameters with respect to the Consumption-wealth ratio are chosen as those in Huang and Liu [6]. They are based on the quarterly CRSP value-weighted return of stocks traded on the New York Stock Exchange (NYSE) and X_t that represents the estimated trend deviation variable, cay_t, for the consumption-wealth ratio from 1952 to 2001. The parameters with respect to the Payout ratio, the Term-spread and the Dividend yield are chosen as those in Zhu and Zhou [2]. Their stock return is the monthly return on the Standard and Poor 500. And, X_t represents the Payout ratio, the Term-spread and the Dividend yield, respectively. Their estimation period is from December 1926 to December 2004.

Table 1. Parameters

	μ_0	μ_1	σ_s	θ_0	θ_1	σ_x	ρ
Term-spread	0.097	1.206	0.195	0.009	−0.527	0.013	0.001
Dividend yield	0.031	2.072	0.195	0.010	−0.253	0.012	−0.073
Consumption-wealth ratio	−1.301	2.040	0.0801	0.117	−0.180	0.00747	−0.620
Payout ratio	0.282	−0.292	0.194	0.014	−0.027	0.050	−0.003

Let the investment horizon and the lag in years, $(T,\ L)$, be $(12,\ 4)$, $(6,\ 2)$ and $(3,\ 2)$, respectively. Assume the risk-free interest rate is $r = 2\%$ and the initial wealth, $W_0 = \$100$. The approximations, A_n and B_n, defined in Theorem 4.5 are calculated for $n = 1,\ 2$, or 4. The optimal strategy choices and the corresponding optimal log-utilities for the interior case and case #1, #2 under GMA can be obtained directly from the closed form formulas in Proposition 1, 2, 3 of Zhu and Zhou [2]. The optimal values and optimal strategy choices under GMA for case #3 and #4 can be calculated by using our formula (15) and (16) with $A = b_1(T-L)$ and $B = b_2(T-L)$, where b_1 and b_2 are those given in (27) and (28) of Zhu and Zhou [2]. In Table 2, 3, 4, 5, the Certainty-Equivalent (CE) gains of each U^*_{AMAi} and U^*_{GMAi}, $i = 0,\ 1,\ 2,\ 3,\ 4$, compared with the risk free one and the corresponding optimal strategy choices, ξ^*_{fix}, ξ^*_{mv}, are given. The CE is the guaranteed amount of money that an individual would view as equally desirable as a risky asset. Mathematically, the certainty equivalent is the certain value, C, satisfying $\log(W_0 + C) + rT = U^*_{AMAi}$, $i = 0,\ 1,\ 2,\ 3,\ 4$. In each table, the interior values are based Proposition 3.2 and the 4 cases are corresponding to the four different boundaries reported in Proposition 3.3 in order. From now on in this section, each U^*_{AMAi} and U^*_{GMAi}, $i = 0,\ 1,\ 2,\ 3,\ 4$ means the CE gain of each U^*_{AMAi} and U^*_{GMAi}, $i = 0,\ 1,\ 2,\ 3,\ 4$.

TABLE 2. Comparison of the optimal U_1's with respect to Term-spread

	AMA			GMA		
	$\xi^*_{fix}(n)$	$\xi^*_{mv}(n)$	CE Gain	ξ^*_{fix}	ξ^*_{mv}	CE Gain
$T = 12,\ L = 4,\ n = 1$						
Interior	2.3837	0.1832	349.4966	2.5234	0.0535	349.5232
Case #1	2.5666	0	349.4929	2.5666	0	349.4929
Case #2	1.5682	1	349.4237	1.7586	1	340.1208
Case #3	0	2.5669	348.8748	0	2.5769	273.2539
Case #4	2.5666	2.9352e-04	349.4929	2.5666	0.0103	349.4987
$T = 6,\ L = 2,\ n = 2$						
Interior	2.5385	0.0338	112.0150	2.5320	0.0473	112.0196
Case #1	2.5666	0	112.0125	2.5666	0	112.0125
Case #2	1.7341	1	109.9265	1.8353	1	109.1637
Case #3	0	2.5723	95.3080	0	2.5793	85.9920
Case #4	2.5666	0.0057	112.0129	2.5666	0.0127	112.0144
$T = 3,\ L = 2,\ n = 4$						
Interior	2.5341	0.0399	45.6071	2.5320	0.0473	45.6077
Case #1	2.5666	0	45.6065	2.5666	0	45.6065
Case #2	1.7506	1	45.2245	1.8353	1	45.1149
Case #3	0	2.5740	42.3731	0	2.5793	40.9172
Case #4	2.5666	0.0073	45.6066	2.5666	0.0127	45.6068

Our discussion includes tree aspects.

Firstly, from Table 2 and Table 3, it follows that the interior optimal value, U^*_{AMA0}, is indeed the global maximum value when the corresponding $(\xi^*_{fix},\ \xi^*_{mv})$ satisfies $\xi^*_{fix} \in (0, \xi^*_{fix1})$ and $\xi^*_{mv} \in (0, 1)$, which exactly matches our conclusion in Theorem 3.5. However, if the restriction on ξ^*_{fix} or ξ^*_{mv} is not satisfied, the global maximum value will be obtained on one of the four boundaries that are reported in Proposition 3.3. For example, Table 4 shows that $U^*_{AMA} = U^*_{AMA3}$ for the Consumption-wealth ratio with $T = 12,\ L = 4,\ n = 1$ when considering the interior $\xi^*_{fix} = -0.0445 (< 0)$ and, on the boundary case #3, $\xi^*_{fix} = -0.0752 (< 0)$. Also, from Table 5, it follows that the maximum expected log-utility of wealth is obtained on the boundary case #2 with respect to the payout ratio, i.e., $U^*_{AMA} = U^*_{AMA2}$. In their numerical analysis, Zhu and Zhou [2] only give the interior optimal log-utility and optimal log-utility of case #1, #2 for GMA and ignore the rationality of the optimal strategy choices, ξ^*_{fix} and ξ^*_{mv}. Hence, the global optimal log-utility under GMA in Zhu

Table 3. Comparison of the optimal U_1's with respect to Dividend Yield

	AMA			GMA		
	$\xi^*_{fix}(n)$	$\xi^*_{mv}(n)$	CE Gain	ξ^*_{fix}	ξ^*_{mv}	CE Gain
$T=12,\ L=4,\ n=1$						
Interior	2.0011	0.4492	290.4789	2.2436	0.2519	290.9115
Case #1	2.4431	0	290.2898	2.4431	0	290.2898
Case #2	1.4591	1	290.1946	1.6513	1	285.4638
Case #3	0	2.4503	286.6822	0	2.4955	233.3012
Case #4	2.4431	0.0072	290.2928	2.4431	0.0525	290.4191
$T=6,\ L=2,\ n=2$						
Interior	2.2874	0.1894	97.6364	2.2988	0.2006	97.6796
Case #1	2.4431	0	97.5575	2.4431	0	97.5575
Case #2	1.6211	1	96.1961	1.7237	1	95.7495
Case #3	0	2.4768	84.1161	0	2.4993	76.5973
Case #4	2.4431	0.0337	97.5716	2.4431	0.0563	97.5918
$T=3,\ L=2,\ n=4$						
Interior	2.2504	0.2410	40.5801	2.2988	0.2006	40.5769
Case #1	2.4431	0	40.5552	2.4431	0	40.5552
Case #2	1.6438	1	40.3332	1.7237	1	40.2325
Case #3	0	2.4914	37.8887	0	2.4993	36.6688
Case #4	2.4431	0.0484	40.5602	2.4431	0.0563	40.5612

and Zhou [2] is always the interior optimal value, which will lead to wrong results when $\xi^*_{mv} \notin [0,\ 1]$ or $\xi^*_{fix} \notin [0,\ \xi^*_{fix1}]$.

Secondly, the results in the four tables (Table 2, 3, 4, 5) indicate the optimal log-utility of wealth under the AMA is greater than the optimal fixed log-utility of wealth, $U^*_{AMA} > U^*_{AMA1}$. That is, the AMA strategy can be adopted to improve the popular fixed strategy in Markowitz [5] portfolio theory. For instance, when $T=6,\ L=2,\ n=2$, with respect to the Consumption-wealth ratio, Table 4 shows that $U^*_{AMA} = U^*_{AMA0} = \$1.3067$ and $U^*_{AMA1} = \$1.1758$; When $T=12,\ L=4,\ n=1$, with respect to the Payout ratio, Table 5 shows that $U^*_{AMA} = U^*_{AMA2} = \$623.3218$ and $U^*_{AMA1} = \$602.7591$. However, with respect to the Consumption-wealth ratio, the optimal value $U^*_{GMA} = U^*_{GMA2}$. Therefore, the technical analysis from the GMA can not improve the fixed strategy in this case. Moreover, this indicate the GMA rule can not replace the AMA rule.

Lastly, we will discuss the difference be tween the AMA strategy and the GMA strategy. Table 2 and Table 3 show that the optimal expected log-utility of wealth under the GMA is a good

Table 4. Comparison of the optimal U_1's with respect to Consumption-wealth ratio

	AMA			GMA		
	$\xi^*_{fix}(n)$	$\xi^*_{mv}(n)$	CE Gain	ξ^*_{fix}	ξ^*_{mv}	CE Gain
$T = 12,\ L = 4,\ n = 1$						
Interior	−0.0445	0.9640	2.6694	1.1257	−0.5056	2.5105
Case #1	0.7793	0	2.3654	0.7793	0	2.3654
Case #2	−0.0752	1	2.6690	0.0934	1	1.2317
Case #3	0	0.9196	2.6687	0	0.6200	1.4658
Case #4	0.7793	0.1403	2.4096	0.7793	−0.1593	2.4111
$T = 6,\ L = 2,\ n = 2$						
Interior	0.2904	0.6950	1.3067	1.0250	−0.3887	1.2214
Case #1	0.7793	0	1.1758	0.7793	0	1.1758
Case #2	0.0758	1	1.2815	0.1473	1	0.6405
Case #3	0	0.9853	1.2742	0	0.6363	0.7206
Case #4	0.7793	0.2060	1.2146	0.7793	−0.1430	1.1926
$T = 3,\ L = 2,\ n = 4$						
Interior	0.2170	0.8246	0.6338	1.0250	−0.3887	0.5975
Case #1	0.7793	0	0.5862	0.7793	0	0.5862
Case #2	0.0974	1	0.6316	0.1473	1	0.4529
Case #3	0	1.0416	0.6290	0	0.6363	0.4728
Case #4	0.7793	0.2623	0.6013	0.7793	−0.1430	0.5904

approximation of the optimal one under the AMA for those special parameters for the Term-spread and the Dividend yield. For instance, with respect to the Term-spread in Table 2, $U^*_{AMA} =$ \$112.0150, and $U^*_{GMA} =$ \$112.0196 when $T = 6,\ L = 2$; For the Dividend yield case in Table 3 with $T = 3,\ L = 4$, $U^*_{AMA} =$ \$40.5801, and $U^*_{GMA} =$ \$40.5769 when $T = 6,\ L = 2$. However, this will not be the case in general. Table 5 and Table 4 provide examples to show that there exists big difference between the optimal expected log-utilities of wealth under the AMA rule and that under the GMA rule. For the Payout ratio in Table 5 with $T = 12,\ L = 4$, $U^*_{AMA} = U^*_{AMA2} =$ \$623.3218 which is less than $U^*_{GMA} = U^*_{GMA3} =$ \$632.6934. The optimal expected log-utility of wealth under the GMA overestimates the one under the AMA. When it goes to $T = 6,\ L = 2$, $U^*_{AMA} = U^*_{AMA2} =$ \$170.8523 which is greater than $U^*_{GMA} = U^*_{GMA3} =$ \$169.1106. Thus, the optimal log-utility under GMA, in this case, under-estimates the actual AMA strategy. Also, Table 4 shows the optimal log-utilities under GMA under-estimate those under the actual AMA strategy for the Consumption-wealth ratio. Therefore, one cannot simply replace the AMA with the GMA. Zhu and Zhou [2] assume the GMA and the AMA would produce similar optimal values. Thus, they give a closed form answer for GMA

TABLE 5. Comparison of the optimal U_1's with respect to Payout ratio

	AMA			GMA		
	$\xi^*_{fix}(n)$	$\xi^*_{mv}(n)$	CE Gain	ξ^*_{fix}	ξ^*_{mv}	CE Gain
$T = 12,\ L = 4,\ n = 1$						
Interior	0.7174	2.3498	633.6688	1.8100	1.3955	635.3674
Case #1	2.9385	0	602.7591	2.9385	0	602.7591
Case #2	1.9933	1	623.3218	2.1298	1	632.6934
Case #3	0	3.0672	630.5616	0	3.2054	569.1591
Case #4	2.9385	0.1287	604.4182	2.9385	0.2670	608.8835
$T = 6,\ L = 2,\ n = 2$						
Interior	1.5902	1.5941	171.79241	2.1760	1.0245	169.1129
Case #1	2.9385	0	165.0960	2.9385	0	165.0960
Case #2	2.0927	1	170.85231	2.1942	1	169.1106
Case #3	0	3.1843	163.9314	0	3.2005	145.6667
Case #4	2.9385	0.2458	166.1177	2.9385	0.2620	166.1177
$T = 3,\ L = 2,\ n = 4$						
Interior	1.7558	1.4469	63.7781	2.1760	1.0245	63.4310
Case #1	2.9385	0	62.8177	2.9385	0	62.8177
Case #2	2.1211	1	63.6862	2.1942	1	63.4307
Case #3	0	3.2027	62.0519	0	3.2005	59.7487
Case #4	2.9385	0.2643	62.9927	2.9385	0.2620	62.9743

to avoid the difficult implementation for AMA. However, this will result in misleading conclusion or suboptimal log-utility value.

In summary, the optimal (or maximal) expected log-utility of wealth can be obtained either in the interior part or on one of the four boundaries reported in Proposition 3.3. In general, the technical analysis from the AMA strategy can be adopted to improve the popular fixed strategy in Markowitz [5]. The AMA cannot be replaced by the GMA in the real world investment.

6 Conclusion

In this paper, we assume the general model developed by Merton [7]and the dynamics for the cum-dividend stock price developed by Kim and Omberg [8]. The main contributions of this study are three-folds. First, the utility of the most popular technical trading method, the arithmetic moving average, is analyzed. We provide a theoretical justification for an investor to buy the stock when its current stock price is moving above the arithmetic average price over a given period L for both constraints on the ratio of pure AMA rule and the ratio without AMA rule. The technical analysis from the AMA therefore can be adopted to maximize the expected log-utility and improve the popular fixed strategy in Markowitz [5] portfolio theory. Second, an explicit implementation procedure for the optimal investment problem under the AMA rule is constructed. The explicit approximated optimal strategy is given from the approximated strategy choices to the approximated optimal value of the expected log-utility function. Third, by comparing the optimal log-utilities under the AMA with those under the GMA, we find the big discrepancy between AMA and GMA. The GMA rule can either overestimate or underestimate the practical AMA rule for the same set of parameters of the model. This indicates that the optimal investment strategy from the GMA rule can be misleading in practice.

Despite the vast literature on technical analysis and the numerous technical indicators following some traders in practice, our study is the first theoretic work to closely support the optimal strategy under the AMA rule, rather than the GMA rule which is not adopted in practice, and also the first work to provide a complete implementation procedure with approximated solutions under the AMA rule. Although our model is based on Brownian motions, the main optimal solution results and the convergence results of approximations proposed can be easily carried to more general stochastic processes, such as, Lévy process, when the weak convergence condition holds under the uniform topology. It is an interesting and quite challenge question to investigate whether the combination of the fixed rule with the AMA rule can outperform the the fixed rule when the utility function is of power type. We will leave it in a future study.

7 Appendix

Proof of Proposition 3.2: Recall (10)

$$\begin{aligned} U_{AMA} =& \log W_0 + rT + \frac{(\mu_0 + \mu_1 \bar{X} - r)^2}{2\sigma_s^2} L \\ &+ \xi_{fix} \left(\mu_0 + \mu_1 \bar{X} - r - \frac{\sigma_s^2}{2} \xi_{fix} \right) (T - L) + \xi_{mv} \mu_1 A \\ &+ \left[\xi_{mv} \left(\mu_0 + \mu_1 \bar{X} - r \right) - \frac{\sigma_s^2}{2} \xi_{mv}^2 - \sigma_s^2 \xi_{fix} \xi_{mv} \right] B, \end{aligned}$$

where $A = \int_L^T b_1(t)dt$, $B = \int_L^T b_2(t)dt$ are defined in (11).

To find the maximal value of $U_{AMA}(\xi_{fix},\ \xi_{mv})$, we take the first partial derivatives with respect to ξ_{fix} and ξ_{mv}, respectively, and set each of these two partial derivatives be zero:

$$\left.\frac{\partial U_{AMA}(\xi_{fix},\ \xi_{mv})}{\partial \xi_{fix}}\right|_{(\xi^*_{fix},\ \xi^*_{mv})} = 0, \tag{28}$$

$$\left.\frac{\partial U_{AMA}(\xi_{fix},\ \xi_{mv})}{\partial \xi_{mv}}\right|_{(\xi^*_{fix},\ \xi^*_{mv})} = 0. \tag{29}$$

From (28) and (29), it follows that

$$(\mu_0 + \mu_1 \overline{X} - r - \sigma_s^2 \xi_{fix})(T - L) - \sigma_s^2 \xi_{mv} B = 0 \tag{30}$$

and

$$\mu_1 A + (\mu_0 + \mu_1 \overline{X} - r)B - \sigma_s^2(\xi_{fix} + \xi_{mv})B = 0. \tag{31}$$

By solving the system of linear equation (30) and (31), we obtain that the optimal strategy choice:

$$\xi^*_{fix} = \frac{\mu_0 + \mu_1 \overline{X} - r}{\sigma_s^2} - \frac{\mu_1 A}{(T - L - B)\,\sigma_s^2},$$

$$\xi^*_{mv} = \frac{\mu_1 (T - L) A}{B\,(T - L - B)\,\sigma_s^2}.$$

Since the value function for log-utility associated with ξ^*_{fix1} is

$$U^*_{fix1} = \log W_0 + rT + \frac{(\mu_0 + \mu_1 \overline{X} - r)^2}{2\sigma_s^2} T,$$

the proposition is obtain directly by plugging U^*_{fix1}, ξ^*_{fix} and ξ^*_{mv} into (10).

Proof of Proposition 3.3: (1) From $\xi_{mv} = 0$ and (30), it follows that

$$\mu_s - r - \sigma_s^2 \xi_{fix} = 0,$$

which implies that $\xi^*_{fix} = \xi^*_{fix1} = \frac{\mu_s - r}{\sigma_s^2}$. By (10) and definition of U^*_{fix1}, $U^*_{AMA1} = U^*_{fix1}$.

(2) From $\xi_{mv} = 1$ and (30), it follows that

$$(\mu_s - r)(T - L) - \sigma_s^2 \xi_{fix}(T - L) - \sigma_s^2 B = 0,$$

which implies that $\xi^*_{fix} = \frac{\mu_s - r}{\sigma_s^2} - \frac{1}{T-L} B$. Then, by (10) and definition of U^*_{fix1}, we get (14).

(3) By substituting $\xi_{fix} = 0$ into (31), we obtain that

$$\mu_1 A + (\mu_0 + \mu_1 \overline{X} - r)B - \sigma_s^2 \xi_{mv} B = 0,$$

which implies that

$$\xi_{mv}^* = \frac{\mu_1 A + (\mu_0 + \mu_1 \overline{X} - r)B}{\sigma_s^2 B} = \frac{\mu_1 A + (\mu_s - r)B}{\sigma_s^2 B}.$$

Then, by (10) and definition of U_{fix1}^*, (15) can be obtained.

(4) By $\xi_{fix} = \xi_{fix1}^* = \frac{\mu_s - r}{\sigma_s^2}$ and (31), we have that

$$\mu_1 A + (\mu_0 + \mu_1 \overline{X} - r)B - \sigma_s^2 \left(\frac{\mu_s - r}{\sigma_s^2} + \xi_{mv} \right) B = 0,$$

which implies that $\xi_{mv}^* = \frac{\mu_1 A}{\sigma_s^2 B}$. Thus, (16) can be obtained by simple algebra.

Proof of Theorem 3.5: We only need to show $U_{AMA0}^* \geq U_{AMAi}^*$, $i = 2,\ 3,\ 4$. As a probability, $b_2(t) \in [0,\ 1]$. So, $B \in [0,\ T - L]$. By comparing (13) and (14), we obtain that

$$\begin{aligned}
& U_{AMA0}^* - U_{AMA2}^* \\
= \quad & \frac{\mu_1^2 A^2 (T-L)}{2B(T-L-B)\sigma_s^2} - \mu_1 A + \frac{\sigma_s^2 B}{2(T-L)}(T-L-B) \\
= \quad & \frac{\mu_1^2 A^2 (T-L)^2 - 2\mu_1 A (T-L)\sigma_s^2 B (T-L-B) + \sigma_s^4 B^2 (T-L-B)^2}{2B(T-L-B)\sigma_s^2 (T-L)} \\
= \quad & \frac{\left[\mu_1 A (T-L) - \sigma_s^2 B (T-L-B)\right]^2}{2B(T-L-B)\sigma_s^2 (T-L)} \geq 0.
\end{aligned}$$

That is, $U_{AMA0}^* \geq U_{AMA2}^*$. Secondly, we compare (13) and (15).

$$\begin{aligned}
& U_{AMA0}^* - U_{AMA3}^* \\
= \quad & \frac{\mu_1^2 A^2 (T-L)}{2B\sigma_s^2 (T-L-B)} - \frac{(\mu_1 A + (\mu_s - r)B)^2}{2\sigma_s^2 B} + \frac{(\mu_s - r)^2 (T-L)}{2\sigma_s^2} \\
= \quad & \frac{[\mu_1 A - (\mu_s - r)(T-L-B)]^2}{2\sigma_s^2 (T-L-B)} \geq 0,
\end{aligned}$$

where the second equality is from some basic algebraic calculations. Moreover, $U^*_{AMA0} \geq U^*_{AMA3}$. At last, we are going to compare (13) and (16).

$$\begin{aligned} & U^*_{AMA0} - U^*_{AMA4} \\ = \ & \frac{\mu_1^2 A^2 (T-L)}{2B\sigma_s^2 (T-L-B)} - \frac{\mu_1^2 A^2}{2\sigma_s^2 B} \\ = \ & \frac{\mu_1^2 A^2 B}{2B\sigma_s^2 (T-L-B)} \geq 0. \end{aligned}$$

Hence, $U^*_{AMA0} \geq U^*_{AMA4}$. Thus, the theorem is proved.

Proof of equation (18): From (1), it follows that

$$\begin{aligned} S_t &= S_0 \exp\left\{ \int_0^t (\mu_0 + \mu_1 X_s - \frac{\sigma_s^2}{2}) ds + \sigma_s B_t \right\} \\ &= S_0 \exp\left\{ (\mu_0 - \frac{\sigma_s^2}{2}) t + \sigma_s B_t + \mu_1 \int_0^t X_s ds \right\}. \end{aligned}$$

By equation (17), we have

$$\begin{aligned} & \int_0^t X_s ds \\ = \ & \int_0^t \left[X_0 e^{\theta_1 s} - \frac{\theta_0}{\theta_1}\left(1 - e^{\theta_1 s}\right) + \rho\sigma_x \int_0^s e^{\theta_1 (s-u)} dB_u + \sigma_x \sqrt{1-\rho^2} \int_0^s e^{\theta_1(s-u)} dZ_u \right] ds \\ = \ & \left(\frac{X_0}{\theta_1} + \frac{\theta_0}{\theta_1^2} \right) \left(e^{\theta_1 t} - 1 \right) - \frac{\theta_0 t}{\theta_1} + \rho\sigma_x \int_0^t \int_0^s e^{\theta_1(s-u)} dB_u ds \\ & + \sigma_x \sqrt{1-\rho^2} \int_0^t \int_0^s e^{\theta_1(s-u)} dZ_u ds. \end{aligned}$$

Consider that

$$\begin{aligned}\int_0^t\int_0^s e^{\theta_1(s-u)}dB_u ds &= \int_0^t\int_u^t e^{\theta_1(s-u)}dsdB_u\\ &= \int_0^t \frac{1}{\theta_1}\left(e^{\theta_1(t-u)}-1\right)dB_u\\ &= \frac{e^{\theta_1 t}}{\theta_1}\int_0^t e^{-\theta_1 u}dB_u - \frac{1}{\theta_1}B_t\\ &= \frac{e^{\theta_1 t}}{\theta_1}\left(e^{-\theta_1 t}B_t + \theta_1\int_0^t e^{-\theta_1 u}B_u du\right) - \frac{1}{\theta_1}B_t\\ &= \int_0^t e^{\theta_1(t-u)}B_u du,\end{aligned}$$

where the fourth equality comes from the Itó formula. Similarly, we can obtain

$$\int_0^t\int_0^s e^{\theta_1(s-u)}dZ_u ds = \int_0^t e^{\theta_1(t-u)}Z_u du.$$

Hence,

$$\int_0^t X_s ds = \left(\frac{X_0}{\theta_1}+\frac{\theta_0}{\theta_1^2}\right)(e^{\theta_1 t}-1) - \frac{\theta_0 t}{\theta_1} + \rho\sigma_x\int_0^t e^{\theta_1(t-u)}B_u du + \sigma_x\sqrt{1-\rho^2}\int_0^t e^{\theta_1(t-u)}Z_u du.$$

Therefore, $S_t = C_t \exp\{SE(t)\}$ can be obtained directly.

Proof of Theorem 4.2: By (25), (26) and (27), we have

$$\frac{A_t^{(n)}}{S_t^{(n)}} = \frac{1}{L}\int_{t-L}^t \frac{S_u^{(n)}}{S_t^{(n)}}du = \frac{1}{L}\int_{t-L}^t \exp\left[\left(\mu_0 - \frac{\sigma_s^2}{2} - \frac{\mu_1\theta_0}{\theta_1}\right)(u-t)\right]\exp\left\{E^{(n)}(u)\right\}du,$$

where

$$\begin{aligned}E^{(n)}(u) &= \mu_1\left(\frac{X_0^{(n)}}{\theta_1}+\frac{\theta_0}{\theta_1^2}\right)\left(e^{\theta_1 u}-e^{\theta_1 t}\right) + \sigma_s\left(B_u^{(n)} - B_t^{(n)}\right)\\ &\quad +\mu_1\sigma_x\rho\left(\int_0^u e^{\theta_1(u-v)}B_v^{(n)}dv - \int_0^t e^{\theta_1(t-v)}B_v^{(n)}dv\right)\\ &\quad +\mu_1\sigma_x\sqrt{1-\rho^2}\left(\int_0^u e^{\theta_1(u-v)}Z_v^{(n)}dv - \int_0^t e^{\theta_1(t-v)}Z_v^{(n)}dv\right).\end{aligned}$$

Consider that

$$\begin{aligned}
& \sup_{0\le t\le T}\left|\int_0^t e^{\theta_1(t-u)}B_u^{(n)}du-\int_0^t e^{\theta_1(t-u)}B_u du\right| \\
= & \sup_{0\le t\le T}\left|\int_0^t e^{\theta_1(t-u)}\left(B_u^{(n)}-B_u\right)du\right| \\
\le & \sup_{0\le t\le T}\int_0^t e^{\theta_1(t-u)}\left|B_u^{(n)}-B_u\right|du \\
\le & \sup_{0\le t\le T} T\left|B_t^{(n)}-B_t\right|,
\end{aligned}$$

where the last inequality follows from that $\theta_1 < 0$. Then, by (20), we obtain that

$$\sup_{0\le t\le T}\left|\int_0^t e^{\theta_1(t-u)}B_u^{(n)}du-\int_0^t e^{\theta_1(t-u)}B_u du\right| \longrightarrow 0 \textit{ almost surely, as } n\to\infty. \tag{32}$$

Similarly, by (21), we get that

$$\sup_{0\le t\le T}\left|\int_0^t e^{\theta_1(t-u)}Z_u^{(n)}du-\int_0^t e^{\theta_1(t-u)}Z_u du\right| \longrightarrow 0 \textit{ almost surely, as } n\to\infty. \tag{33}$$

From (20), (32), (33) and the triangle inequality, it follows that

$$\begin{aligned}
& E^{(n)}(t)-\mu_1\left(\frac{X_0^{(n)}}{\theta_1}+\frac{\theta_0}{\theta_1^2}\right)\left(e^{\theta_1 t}-e^{\theta_1 T}\right) \\
\longrightarrow\ & \sigma_s(B_t-B_T)+\mu_1\sigma_x\rho\left(\int_0^t e^{\theta_1(t-u)}B_u du-\int_0^T e^{\theta_1(T-u)}B_u du\right) \\
& +\mu_1\sigma_x\sqrt{1-\rho^2}\left(\int_0^t e^{\theta_1(t-u)}Z_u du-\int_0^T e^{\theta_1(T-u)}Z_u du\right),
\end{aligned}$$

almost surely, as $n\to\infty$, under the uniform topology for $t\in[0,T]$.

Since $X_0^{(n)}\longrightarrow X_0$ in distribution as $n\to\infty$,

$$\mu_1\left(\frac{X_0^{(n)}}{\theta_1}+\frac{\theta_0}{\theta_1^2}\right)\left(e^{\theta_1 t}-e^{\theta_1 T}\right)\xrightarrow{\pounds}\mu_1\left(\frac{X_0}{\theta_1}+\frac{\theta_0}{\theta_1^2}\right)\left(e^{\theta_1 t}-e^{\theta_1 T}\right)$$

in distribution uniformly on $t\in[0,\ T]$, as $n\to\infty$.

Because $B^{(n)}(t)$, $Z^{(n)}(t)$ and $X_0(n)$ are mutually independent, $E^{(n)}(\cdot)\longrightarrow E(\cdot)$, in distribution under the uniform topology with respect to $t\in[0,\ T]$, as $n\to\infty$. Hence, $\frac{A_t^{(n)}}{S_t^{(n)}}\longrightarrow\frac{A_t}{S_t}$ in distribution, as $n\to\infty$ by the Theorem 1.2 in Berkes and Horváth [14].

Proof of Theorem 4.4: By the construction of $X_0^{(n)}$ in (24), we know that $E\left(\widehat{X}_t^{(n)}\right) = 0$ and that $Var\left(\widehat{X}_t^{(n)}\right) = E\left(\widehat{X}_t^{(n)}\right)^2 = -\frac{\sigma_x^2}{2\theta_1}$. Recall that $\widehat{X}_t$ is normally distributed with $E\left(\widehat{X}_t\right) = 0$ and $Var\left(\widehat{X}_t\right) = E\left(\widehat{X}_t\right)^2 = -\frac{\sigma_x^2}{2\theta_1}$. By the Cauchy-Schwartz inequality,

$$
\begin{aligned}
& E\left|\widehat{X}_t^{(n)}\eta_{\{S_t^{(n)}>A_t^{(n)}\}}\eta_{\{\left|\widehat{X}_t^{(n)}\right|\geq K\}}\right| + E\left|\widehat{X}_t\eta_{\{S_t>A_t\}}\eta_{\{\left|\widehat{X}_t\right|\geq K\}}\right| \\
\leq\ & \sqrt{E\left|\widehat{X}_t^{(n)}\right|^2 E(\eta^2_{\{S_t^{(n)}>A_t^{(n)}\}}\eta^2_{\{\left|\widehat{X}_t^{(n)}\right|\geq K\}})} + \sqrt{E\left|\widehat{X}_t\right|^2 E(\eta^2_{\{S_t>A_t\}}\eta^2_{\{\left|\widehat{X}_t\right|\geq K\}})} \\
\leq\ & \sqrt{-\frac{\sigma_x^2}{2\theta_1}P(\left|\widehat{X}_t^{(n)}\right|\geq K)} + \sqrt{-\frac{\sigma_x^2}{2\theta_1}P(\left|\widehat{X}_t\right|\geq K)},
\end{aligned}
$$

where the last inequality is from the fact that $\eta^2_{\{\cdot\}} \in [0,\ 1]$. Because $\widehat{X}_t$ is normally distributed with variance $-\frac{\sigma_x^2}{2\theta_1}$,

$$
P(\left|\widehat{X}_t\right| > K) = 1 - \Psi(K\sqrt{-\frac{2\theta_1}{\sigma_x^2}}) \to 0,
$$

as $K \to \infty$, where Ψ is the cumulative distribution function of a standard normal distribution. By the Chebyshev's Inequality,

$$
P(\left|\widehat{X}_t^{(n)}\right| > K) \leq \frac{1}{K^2}Var(\widehat{X}_t^{(n)}) = -\frac{\sigma_x^2}{2\theta_1 K^2} \to 0
$$

as $K \to \infty$. Hence, for any $\epsilon > 0$, there exists $K = K(\epsilon) > 0$, such that $P(\left|\widehat{X}_t\right| > K) < \epsilon$ and $P(\left|\widehat{X}_t^{(n)}\right| > K) < \epsilon$. For this K, let

$$
f(\widehat{X}_t,\ A_t/S_t) = \widehat{X}_t\eta_{\{A_t/S_t<1\}}\eta_{\{\left|\widehat{X}_t\right|<K\}} = \widehat{X}_t\eta_{\{S_t>A_t\}}\eta_{\{\left|\widehat{X}_t\right|<K\}}.
$$

Since $P(\frac{A_t}{S_t} = 1) = 0$, and $P(\left|\widehat{X}_t\right| = K) = 0$, $P((\widehat{X}_t,\ A_t/S_t) \in C) = 1$, where C is the continuity set of the bounded function f. Hence,

$$
f(\widehat{X}_t^{(n)},\ A_t^{(n)}/S_t^{(n)}) \xrightarrow{D} f(\widehat{X}_t,\ A_t/S_t)
$$

follows from Theorem 4.2, the Continuous Mapping Theorem and the fact that $\widehat{X}_t^{(n)} \xrightarrow{D} \widehat{X}_t$, where "$\xrightarrow{D}$" means convergent in distribution. Moreover,

$$
E\left[\widehat{X}_t^{(n)}\eta_{\{A_t^{(n)}/S_t^{(n)}<1\}}\eta_{\{\left|\widehat{X}_t^{(n)}\right|<K\}}\right] \longrightarrow E\left[\widehat{X}_t\eta_{\{A_t/S_t<1\}}\eta_{\{\left|\widehat{X}_t\right|<K\}}\right],
$$

as $n \to \infty$, because of the boundedness of f. Hence, there exists a positive N such that, whenever $n > N$,

$$\left| E\left[\widehat{X}_t^{(n)} \eta_{\{A_t^{(n)}/S_t^{(n)}<1\}} \eta_{\{\left|\widehat{X}_t^{(n)}\right|<K\}}\right] - E\left[\widehat{X}_t \eta_{\{A_t/S_t<1\}} \eta_{\{\left|\widehat{X}_t\right|<K\}}\right]\right| < \epsilon.$$

Above all,

$$\begin{aligned} & E\left[\widehat{X}_t^{(n)} \eta_{\{S_t^{(n)}>A_t^{(n)}\}}\right] - E\left[\widehat{X}_t \eta_{\{S_t>A_t\}}\right] \\ = \;& E\left[\widehat{X}_t^{(n)} \eta_{\{S_t^{(n)}>A_t^{(n)}\}} \eta_{\{\left|\widehat{X}_t^{(n)}\right|<K\}}\right] - E\left[\widehat{X}_t \eta_{\{S_t>A_t\}} \eta_{\{\left|\widehat{X}_t\right|<K\}}\right] \\ & + E\left[\widehat{X}_t^{(n)} \eta_{\{S_t^{(n)}>A_t^{(n)}\}} \eta_{\{\left|\widehat{X}_t^{(n)}\right|\geq K\}}\right] - E\left[\widehat{X}_t \eta_{\{S_t>A_t\}} \eta_{\{\left|\widehat{X}_t\right|\geq K\}}\right] \\ < \;& \epsilon + \sqrt{-\frac{\sigma_x^2 \epsilon}{2\theta_1}} + \sqrt{-\frac{\sigma_x^2 \epsilon}{2\theta_1}} \end{aligned}$$

as $n > N$. This proves Theorem 4.4.

Proof of Theorem 4.5: Consider that, for any $t \in [L,\ T]$, $\left|b_2^{(n)}(t)\right| = P\left(\frac{A_T^{(n)}}{S_T^{(n)}} < 1\right) \leq 1$ and that

$$\left|b_1^{(n)}(t)\right| = \left|E\left(\widehat{X}_t^{(n)} \eta_{\{S_t^{(n)}>A_t^{(n)}\}}\right)\right| \leq \sqrt{E\left|\widehat{X}_t^{(n)}\right|^2 E\left(\eta^2_{\{S_t^{(n)}>A_t^{(n)}\}}\right)} \leq \sqrt{E\left|\widehat{X}_t^{(n)}\right|^2} = \sqrt{-\frac{\sigma_x^2}{2\theta_1}},$$

where the first inequality is from the Chebyshev's Inequality, the sencond inequality from $\eta \in [0,\ 1]$ and the last equality from that $\widehat{X}_t^{(n)}$ is normally distributed with variance $-\frac{\sigma_x^2}{2\theta_1}$. By Corollary 4.3 and Theorem 4.4, Theorem 4.5 will be obtained immediately by the Fatou-Lebesgue theorem.

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