**HOMOTOPY ANALYSIS METHOD FOR SOLVING VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND**

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**Abstract**

In this research, we presents Homotopy Analysis Method to volterra integral equations of the second kind. The method is an analytical method for solving linear and nonlinear equations. The Homotopy Analysis Method provides us with an infinite solution series which usually converges to the exact solution of considered equations. The method allows us to choose an initial guess where the considered equation is iteratively deforms starting with an initial guess to the exact solution. The method provides an auxiliary parameter h to analyze strongly linear and nonlinear problems. Application of the method to Volterra integral equations of the second kind is analyzed which gives a rapid convergence for the solutions.

**Keywords**: Homotopy Analysis Method; Volterra integral equations.

**Introduction**

In this thesis,we apply homotopy analysis method (HAM)which was proposed by

Liao in 1992. In this method, the solution is considered as the summation of an

infinite series, which usually converges rapidly to the exact solution. The HAM is

based on homotopy, a fundamental concept in topology and differential geometry.

Briey speaking, by means of the HAM one constructs a continuous mapping of

an initial guess approximation to the exact solution of considered equations. An

auxiliary linear operator is chosen to construct such kind of continuous mapping

and an auxiliary parameter is used to ensure the convergence of solution series.

The method enjoys great freedom in choosing initial approximation and auxiliary

linear operators. The approximations obtained by the HAM are uniformly valid

not only for small parameters, but also for very large parameters.In this paper, we

present an iterative scheme based on the HAM for the second kind of linear and

non-linear volterra integral equations.

y(x) = g(x) + (1.1)

Where the upper limit may be either variable or fixed, ƛ is a complex number,

the kernel H(x,t) and g(x) are known functions, whereas y is to be determined.

**2.1 Description of the method**

Consider the following equation

 N[y(x)]=0

Where N is an operator, y(x) is unknown function and x the independent vari-

able. Let y0(x) denote an initial guess of the exact solution y(x),h0 an auxiliary

parameter,H(x)≠0 an auxiliary function,and L an auxiliary linear operator with

the property L[r(x)]=0 when r(x)= 0. Then using q[0,1] as an embedding parameter,

 we construct such a homotopy

(1 - q)L[ɸ (x : q) – y0(x)] – qhH(x)N[ɸ(x : q)] = 0 (2.2)

It should be emphasized that we have great freedom to choose the initial guess

y0(x), the auxiliary linear operator L, the non-zero auxiliary parameter h, and the

auxiliary function H(x). We have the so-called zero-order deformation equation

(1-q)L[ɸ (x : q) – y0(x)] = qhH(x)N[ɸ (x : q)]: (2.3)

When q=0, the zero-order deformation equation (2.3) becomes ɸ (x:0)= y0(x) And

when q=1, since h0 and H(x)0, the zero order deformation equation (2.2) is

equivalent to ɸ (x:1)= y(x) Thus according to (2.2) and (2.3) as the embedding

parameter increases from 0 to 1, ɸ (x:q) varies continuously from the initial ap-

proximation y0(x) to the exact solution y(x). such a kind of continuous variation

is called deformation in homotopy.

By Taylor's theorem, ɸ (x;q) can be expanded in a power series of q as follows

0(x)+n(x)qn (2.4)

where

 yn(x)=q=0 (2.5)

If the initial guess y0(x), the auxiliary linear parameter L, the non-zero auxiliary

parameter h, and the auxiliary function H(x) are properly chosen so that the

power series (3.4) of (x:q) converges at q=1.Then,we have under these assumptions the solution series

 y(x)=0(x)+n(x) (2.6)

where the vector is defined

 yn(x)=y0(x), y1(x), y2(x),… yn(x) (2.7)

L[yn(x) - nyn-1(x)]=hH(x)Rn((x)) , yn(0)=0 (2.8)

Where

Rn((x)=q=0 (2.9)

And

n

Note that the high-order deformation equation(2.8) is governing by the linear

operator L, and the term Rn((x) can be expressed simply by (2.9) for any

nonlinear operator N. Therefore, yn(x) can be easily gained especially by means of

computational software such as MATLAB. The solution y(x) given by the above

approach is dependent of L,h,H(x) and y0(x). Thus, unlike all previous analytical

techniques, the convergence region and rate of solution series given by the above

approach might not be uniquely determined. If n(x) tends uniformly to

a limit as n→, this limit is the required solution (Vahdati, Zulkifly Abbas,

Ghasemi,2010).

3.0 HAM's solution to Volterra integral equations

Let consider the equation

h(t)u(t)=g(x)+ (3.10)

where the solution to equation (3.10) of

Volterra integral equations of the second kind.

3.1 Volterra integral equations of the second kind

If h(t)=1 is substituted into equation(3.10),we have

U(t)=g(t)+ b (3.11)

We construct the zeroth-order deformation for this kind of integral equations as

(1-p)(u(t,p,h)-g(x))=hp(u(t,p,h)-g(t)-) (3.12)

For p=0 and p=1,we have

u(t,0,h)=g(t)

u(t,1,h)=u(t)

For Maclaurin series of u(t,p,h) corresponding to p,we have

U(t,p,h)=u(t,0,h)+ (3.13)

Which

p=0 (3.14)

Substituting p=1 into (3.13) give

u(t)=g(t)+ (3.15)

where we obtain the nth-order deformation equation

L[]=h (3.16)

And the solution of the nth-order deformation equation for n1 yields

 (3.17)

And

 (3.18)

Choosing h = -1,the solution of the problem is similar to the Homotopy Perturbation Method, (Vahdati,et al,2010)

 Applying the HAM

In this section,we apply the HAM for solving Volterra integral

equations.

4.0 Volterra integral equation of the second kind

Let's consider the Volterra integral equation of the second kind, which reads

(x) = g(x) +(t)dt (4.19)

where H(x,t) is the kernel of the integral equation

**Example 1.**Consider the following Volterra integral equation

(x) =x+ (t)dt (4.20)

To begin,we choose

To begin with,we choose

(x)= x (4.21)

We choose the linear operator

L[ɸ(x,p)]= ɸ(x,p) (4.22)

Thus,we now define the nonlinear operator as

N[ɸ(x,p)]= ɸ(x,p)-x- (4.23)

where we construct the nth-order deformation equation

L[-]=h() (4.24)

And

=x+dy (4.25)

where the solution of the nth-order deformation equation(4.24)

=(x)+h[()] (4.26)

Finally,we have

ɸ(x)=(x)+(x) (4.27)

where

-h

 h

-h

 h

.

.

.

Hence

ɸ(x)=+++

 =x - h + h - h +

 If h= -1

 = x + - + -

 = (4.28)

Which is the exact solution of equation(4.20)



**Figure 4:1** Example 1.Exact solution to equation (4.20)

The following algorithm produces **figure 4.1** using the **Matlab** software.

function [x,sumc] = solplot1(x,n)

 sumc(1) = x;

 for i=1:n

 num = (-3)^i;

 den = factorial(2\*i+1);

 rsult = num/den; rsult = rsult\*x^(2\*i+1);

 sumc(i+1) = sumc(i) + rsult;

 end

 plot(1:n+1,sumc)

 %plot(1:n,sumc(2:end))

end

**Example 4.**Consider the following Volterra integral equation

ɸ(x)=2x - (4.29)

To solve equation (4.27),we choose

(x)=2x - (4.30)

We choose the linear operator

L[ (x; p)] = (x; p) (4.31)

Thus, we now define the nonlinear operator as

N[ (x; p)] = (x; p) - 2x + (4.32)

And we construct the nth-order deformation equation

L[-]=h() (4.33)

And

=x + (4.34)

where the solution of the nth-order deformation equation(4.33)

=(x)+h[()] (4.35)

Finally,we have

ɸ(x)=(x)+(x) (4.36)

where

(x)= 2x -

(x)= -)

(x)= )

(x)= )

(x)= )

.

.

.

Hence

ɸ(x) =+++

 = 2x - + h -) + + ) + )

If h= -1

 =2x - - +

 = = 2x - 2 (4.37)

Which is the exact solution to equation (4.29) ,(Issaka,2016)

**Example 3.**Let consider the following Volterra equation

ɸ(x)= x (4.38)

To solve equation(4.38),we choose

 (4.39)

We choose the linear operator

L[ɸ (x; p)] = ɸ(x; p) (4.40)

Thus,we now define the nonlinear operator as

N[ (x; p)] = (x; p) - x (4.41)

And we construct the nth-order deformation equation

L[-]=h() (4.42)

And

= x (4.43)

where the solution of the nth-order deformation equation (4.42)

=(x)+h[()] (4.44)

Finally,we have

ɸ(x)=(x)+(x) (4.45)

where

Hence

ɸ(x) =+++

 = x + +…

If h= -1

 = x + +…

 = (x) (4.46)

Which is the exact solution to equation(4.38)



**Figure 4.2:** Example 3.Exact solution to equation (4.38)

The following algorithm produces **figure 4.2** using the **Matlab** software.

function [x,sumc] = solplot4(x,n)

 sumc(1) = x; m=1;

 for i=1:n

 m = 2\*m+1;

 num = 1;

 den = factorial(m);

 rsult = (num\*x^m)/den;

 sumc(i+1) = sumc(i) + rsult;

 end

 plot(1:n,sumc(2:end))

end

**Example 6.** Let consider the Volterra integral equations

ɸ(x)= x (4.47)

To solve equation(4.39),we choose

 (4.48)

We choose the linear operator

L[ɸ (x; p)] = ɸ(x; p) (4.49)

Thus,we now define the nonlinear operator as

N[ (x; p)] = (x; p) - x (4.50)

And we construct the nth-order deformation equation

L[-]=h() (4.51)

And

= x (4.52)

where the solution of the nth-order deformation equation (4.51)

=(x)+h[()] (4.53)

Finally,we have

ɸ(x)=(x)+(x) (4.54)

where

(x) =

(x) =

(x) =

.

.

.

Hence

ɸ(x) =+++

 = x + + +

If h= - 1

= x + + +

Which is the exact solution to equation(4.47).



**Figure 4.3**: Example 4.Exact solution to equation (4.47),(Issaka,2016)

The following algorithm produces **figure 4.3** using the **Matlab** software.

function [x,sumc] = solplot5(x,n)

 sumc(1) = x; m=sqrt(2);

 for i=1:n

 m = m^2 + 1;

 num = x^m;

 den = 2\*m;

 rsult = (-1)^(i+1) \* (num/den);

 sumc(i+1) = sumc(i) + rsult;

 end

 plot(1:n,sumc(2:end))

end

% % Script to run

Solplot1(0.5,100)

Solplot2(1,100)

Solplot3(1,100)

\end{verbatim}

 **Conclusion**

 Volterra integral equation of the second kind has been solved successfully by Homotopy analysis method (HAM).The convergence control parameter h is introduced and it greatly influences the convergence of the solution series and the convergence rate. This paper analytically showed that the HAM is a powerful method for solving Volterra integral equations.

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