Stability and boundedness analysis of Lotka-Volterra prey-predator model with prey refuge and predator cannibalism

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Abstract

In this paper, a prey-predator system incorporating prey refuge and predator cannibalism is studied. The stability and ultimate boundedness of the analyzed state parameters (y_1, y_2) defining the system are obtained using the Lyapunov's second or direct method. We construct a suitable complete Lyapunov function for the nonlinear system and demonstrate its efficacy. The method is built upon applying various theoretical Lyapunov functions. By constructing a Lyapunov function which possesses a functional relationship to the the original model system, we give sufficient conditions which ensure the stability and ultimate boundedness of the state parameters describing the nonlinear prey-predator system. We give a numeric example to illustrate the result obtained.

Mathematics Subject Classification: 34C27, 34D23, 34C25.

Key words: Prey-predator system, prey refuge, predator cannibalism, stability, boundedness, Lyapunov's method.

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1 Introduction

In the last two decades, studies on the dynamic behaviors of predator-prey species have received more remarkable attention in [5, 6, 7, 8, 12, 15, 17, 27, 29, 30] and the references cited therein. This dynamic relationship between predator and prey is one of the most crucial relationship that has existed between two populations in ecological systems because of its universal existence and significance. See Berryman [2]. Prey refuge plays a substantial role on the co-existence of the prey-predator relationship. Some reported results have shown that prey refuge can enhance the dynamical behaviour of prey-predator systems. see [11, 14, 18, 24]. Consequently, this could bring about cannibalism amongst the predators. However, the existence of both refuge amongst the preys and cannibalism amongst the predators serve as a stabilizing mechanism on the prey-predator systems if certain conditions on the parameters describing the predator-prey system will hold.

As a new development which is fast growing in the recent years, cannibalism as a special factor in social life population nature has attracted the interest of many researchers. See [13, 23, 25, 26]. Studies have shown clearly that cannibalism has a significant effect on the dynamic behaviors of model describing populations [25-26]. We observe that the qualitative analysis of parameters describing such system are very complicated to analyze. For example, Magnusson [20] proposed and analyzed the destabilizing effect of cannibalism in a structure prey-predator model. Zhang et al [31] investigated a diffusive predator-prey models with predator cannibalism. Also, Deng et al [10] investigated a predator-prey model incorporating cannibalism with predator nonlinear cannibalism. Deng et al [10] investigated the global asymptomatic stability of the systems in [10] which has both positive and negative effect on the system and hence significant effect on the dynamic behaviors of the system. The results extended the result of Basheer et al [1]. Much more complicated still is the predator-prey model incorporating predator cannibalism and prey-refuge. Most of the studies mentioned above used numerical simulation and simple mathematical analysis while just few like Zhang et al [31], Deng et al [10] and Ma et al [19] in addition used suitable Lyapunov's function to obtain conditions for the stability of the boundary equilibria of the model system under consideration. These are interesting studies since the results showed that either predator cannibalism or prey refuge severally has positive and negative effect on the stability of the system.

This paper is concerned with problem of asymptotic stability and ultimate boundedness of the parameters defining the prey refuge and predator cannibalism of the Lokta Volterra Prey-Predator model of the form

$$\begin{aligned} \dot{y_1} = &\quad by_1 - ay_1^2 - e\Phi(y_1, y_2) + \lambda G(y_1) \\ \dot{y_2} = &\quad -\epsilon y_2 + \rho y_2 + gy_1y_2 - \mu F(y_2) \end{aligned} \tag{1}$$

where y_1 and y_2 are the densities of the prey and the predator at time t, respectively; ρ is the birth rate from the predator cannibalism, ϵ is the death rate of the predator, $\rho > \epsilon$; a and b are intraspecific competition and intrinsic growth rate of the prey respectively; e is the strength intraspecific interaction between prey and predator, g is the conversion efficiency of ingested prey into a new predator; λ is the rate at which the prey get protection from their refuge and μ is the rate of cannibalism. The continuous functions $\lambda G(y_1)$, $\mu F(y_2)$ and $e\Phi(y_1, y_2)$ are the prey refuge, predator cannibalism and competitive interaction between predator and prey respectively and the constants $a, b, \rho, \epsilon, e, g, \lambda, \mu$ are positive.

Analysis of qualitative behaviors of state parameters (y_1, y_2) describing such prey-predator model is usually complicated. The difficulty increases depending on the assumption made on the nonlinear functions Φ, G, F and the requirement for a complete Lyapunov function. The Lyapunov functionals used in Zhang et al [31], Deng et al [10] and Ma et al [19] and others mentioned therein do not possess a functional relationship to the original model considered and the effect of continuous nonlinear functions $e\Phi(y_1, y_2)$ showing the competitive interaction between prey and predator and $\lambda G(y_1)$, $\mu F(y_2)$ the prey refuge and cannibalism of the predator respectively raised this present analysis where a complete Lyapunov function is constructed which possess a functional relationship to same original predator-prey model system under consideration. The role which Lyapunov theory [16] plays in the analysis of stability and boundedness of dynamical systems and models of natural phenomena remains undisputed. Though it goes without saying that stability and boundedness are very important problem of dynamical systems, Lyapunov's theory is less visible. Lyapunov's second (or direct) method allows us to predict the stability in the large and boundedness behavior of parameters describing the systems of prey-predator models without any prior knowledge of the solutions. Lyapunov functional approach remains an excellent tool in the study of dynamical systems. see (Qin et al [22], Biryuk et al [3], Olutimo and Omoko [21]). However, the construction of these Lyapunov functional is indeed a general problem. ([4], [9], [28]).

Our motivation comes from the papers by Zhang et al [31], Deng et al [10] and Ma et al [19]. We obtain sufficient conditions for the asymptotic stability and ultimate boundedness of (y_1, y_2)) that affect the dynamic behaviors of the Lokta Volterra Prey Predator model (1). The result obtained indeed becomes critical since the existence of both refuge amongst the preys and cannibalism amongst the predators play a significant role in determining the dynamic behavior of the prey-predator system (1). The result obtained is not only new but provide for the development of more general formulation. Also, numeric example and geometric argument are given to support our findings on the dynamic behaviours of the system.

2 Stability Analysis

Theorem 1. In addition to the basic assumptions imposed on functions $\Phi(y_1, y_2)$, $G(y_1)$ and $F(y_2)$ appearing in (1) and are continuous for all y_1, y_2, we further suppose that the functions $\Phi(0,0) = 0$, $G(0) = 0$ and $F(0) = 0$. We assume that there exist positive constants m, ν and l_1, l_2 such that the following conditions hold:

(i) $\frac{\Phi(y_1,y_2)}{y_1} \geq \nu$, $y_1 \neq 0$

(ii) $\frac{\Phi(y_1,y_2)}{y_2} \geq m$, $y_2 \neq 0$

(iii) $\int_0^{y_2} \Phi_{y_1}(y_1, \xi) d\xi \leq 0$,

(iv) $|\frac{y_1}{a+\lambda}| \leq l_1$, $|\frac{y_1}{b}| \leq l_2$, $l_1 > l_2$.

Then, the analyzed state parameters (y_1, y_2) describing the system (1) are asymptotically stable as $t \to \infty$ if

$$\frac{e}{b}\nu + al_1 > 1$$

and

$$\frac{\rho-\epsilon}{a+\lambda}+\frac{g}{ae+\mu}>gl_2. \tag{2}$$

Proof:

Our main tool in the proof of the result is the scalar function $V=V(y_1,y_2)$ defined as

$$V(y_1,y_2) \;=\; \frac{1}{2b}y_1^2+\frac{1}{2(a+\lambda)}y_2^2+\frac{g}{ae+\mu}\int_0^{y_2}\Phi(y_1,\xi)d\xi \tag{3}$$

From (3), we see that $\Phi(0,0)=0$.

By the hypothesis (ii) of Theorem 1, we have

$$\int_0^{y_2}(\Phi(y_1,\xi)d\xi \geq my_2^2$$

It follows that

$$V(y_1,y_2) \;\geq\; \frac{1}{2b}y_1^2+\left(\frac{1}{2(a+\lambda)}+\frac{gm}{ae+\mu}\right)y_2^2.$$

It is obvious that the function V defined in (3) is a positive definite function, that is,

$$V(y_1,y_2) \;\geq\; \zeta(y_1^2+y_2^2), \tag{4}$$

where $\zeta=\min\frac{1}{2}\{\frac{1}{b},(\frac{1}{(a+\lambda)}+\frac{2gm}{ae+\mu})\}$.

Now, we consider the case where (1) is homogeneous, that is, $\lambda G(y_1)=0$, $\mu F(y_2)=0$. The derivative of function $V(y_1,y_2)$ in (3) along system (1) with respect to t after simplification gives:

$$\begin{aligned}\frac{dV(y_1,y_2)}{dt} &= y_1^2 - a\frac{y_1}{b}y_1^2 - \frac{e}{b}y_1\Phi(y_1,y_2) - \frac{1}{a+\lambda}(\rho-\epsilon)y_2^2 \\ &+ g\frac{y_1}{a+\lambda}y_2^2 + \frac{g}{ae+\mu}\int_0^{y_2}\Phi_{y_1}(y_1,\xi)d\xi + \frac{g}{ae+\mu}y_2\Phi(y_1,y_2)\end{aligned}$$

$$\begin{aligned}\frac{dV(y_1,y_2)}{dt} &= y_1^2 - \frac{e}{b}\frac{\Phi(y_1,y_2)}{y_1}y_1^2 - \frac{1}{a+\lambda}(\rho-\epsilon)y_2^2 - a\left|\frac{y_1}{b}\right|y_1^2 + g\left|\frac{y_1}{a+\lambda}\right|y_2^2 \\ &+ \frac{g}{ae+\mu}\int_0^{y_2}\Phi_{y_1}(y_1,\xi)d\xi + \frac{g}{ae+\mu}\frac{\Phi(y_1,y_2)}{y_2}y_2^2\end{aligned}$$

Using the hypothesis (i), (ii) and (iv) of Theorem 1, we have,

$$\begin{aligned}\frac{dV(y_1,y_2)}{dt} &\leq y_1^2 - \frac{e}{b}\nu y_1^2 - \frac{1}{a+\lambda}(\rho-\epsilon)y_2^2 \\ &- al_1y_1^2 + gl_2y_2^2 + \frac{g}{ae+\mu}my_2^2\end{aligned}$$

$$\frac{dV(y_1,y_2)}{dt} \leq -(\frac{e}{b}\nu + al_1 - 1)y_1^2 - (\frac{\rho-\epsilon}{a+\lambda} + \frac{g}{ae+\mu} - gl_2)y_2^2$$

Since (2) is satisfied, we have that

$$\begin{aligned}\frac{dV(y_1,y_2)}{dt} &\leq -\delta_1y_1^2 - \delta_2y_2^2 \\ &\leq -\eta(y_1^2+y_2^2), \end{aligned} \tag{5}$$

for some $\delta_1 > 0, \delta_2 > 0$, where $\eta = \min\{\delta_1, \delta_2\}$.

It follows that

$$\frac{dV(y_1,y_2)}{dt} \leq 0.$$

The proof of Theorem 1 is complete.

3 Boundedness Analysis

As in Theorem 1, the boundedness analysis of the state parameters (y_1, y_2), describing the system (1) depends on the scalar differentiable Lyapunov function $V(y_1, y_2)$ defined in (3). Here, we consider the heterogenous case where $\lambda G(y_1) \neq 0$ and $\mu F(y_2) \neq 0$ in (1).

Theorem 2. Let all the conditions of Theorem 1 be satisfied and in addition we assume that there exist a positive constants q, n such that the following hold:

(i) $|G(y_1)| \leq q$;

(ii) $|F(y_2)| \leq n$,

uniformly for all y_1, y_2. Then, there exist a constant $D > 0$ such that the state parameters $(y_1(t), y_2(t))$ describing system (1) uniformly ultimately satisfies

$$|y_1(t)| \leq D, \quad |y_2(t)| \leq D,$$

for all sufficiently large t, where the magnitude of D depends only on $a, b, \epsilon, \rho, \lambda, \mu, m, \nu, q, n$ and l_1, l_2.

Proof: In view of (5),

$$\frac{dV(y_1, y_2)}{dt} \leq -\eta(y_1^2 + y_2^2) + \frac{1}{b}\lambda|G(y_1)|y_1 + \frac{1}{a}\mu|F(y_2)|y_2,$$

since $V(\dot{y_1}, y_2)_{(5)} \leq 0$ for all y_1, y_2.

By noting the the hypothesis of Theorem 2, we have that

$$\frac{dV(y_1, y_2)}{dt} \leq -\eta(y_1^2 + y_2^2) + \frac{1}{b}\lambda q|y_1| + \frac{1}{a}\mu n|y_2|$$

So that

$$\frac{dV(y_1,y_2)}{dt} \quad \leq \quad -\eta(y_1^2+y_2^2)+\delta_3|y_1|+\delta_4|y_2|,$$

where $\delta_3 = \frac{1}{b}\lambda q$ and $\delta_4 = \frac{1}{a}\mu n$.

$$\frac{dV(y_1,y_2)}{dt} \quad \leq \quad -\eta(y_1^2+y_2^2)+\delta_5(|y_1|+|y_2|),$$

where $\delta_5 = \max\{\delta_3, \delta_4\}$.

Using the fact that $2|y_1||y_2| \leq y_1^2 + y_2^2$, we have

$$\frac{dV(y_1,y_2)}{dt} \leq -\eta(y_1^2+y_2^2)+\sqrt{2}\delta_5(y_1^2+y_2^2)^{\frac{1}{2}}.$$

$$\frac{dV(y_1,y_2)}{dt} \leq -\eta(y_1^2+y_2^2)+\delta_6(y_1^2+y_2^2)^{\frac{1}{2}}, \tag{6}$$

where $\delta_6 = \sqrt{2}\delta_5$.

If we choose

$$(y_1^2+y_2^2)^{\frac{1}{2}} \geq \delta_7 = \eta^{-1}\delta_6,$$

the inequality (6) implies that

$$\frac{dV(y_1,y_2)}{dt} \leq -\eta(y_1^2+y_2^2).$$

Then, there exist a δ_8 such that

$$\frac{dV(y_1,y_2)}{dt} \leq -\delta_8 \quad \textit{provided} \;\; (y_1^2+y_2^2) \geq \delta_8\eta^{-1}.$$

This completes the proof of Theorem 2.

4 Numerical Example

Consider equation (1) in the form

$$\begin{aligned} \dot{y_1} = &\ 2y_1 - 4y_1^2 - \frac{1}{3}(y_1 + \frac{y_1}{1+y_1^2} + y_2 + y_2^2) + \frac{1}{2}(1+y_1)^2 \\ \dot{y_2} = &\ -y_2 + 3y_2 + \frac{2}{5}y_1y_2 - \frac{1}{4}\left(\frac{y_2}{1+y_2^2} + 2\right) \end{aligned} \tag{7}$$

It is clear that $a = 4$, $b = 2$, $\epsilon = 1$, $\rho = 3$, $e = \frac{1}{3}$, $g = \frac{2}{5}$, $\lambda = \frac{1}{2}$, $\mu = \frac{1}{4}$ and

$$\Phi(y_1, y_2) = \left(y_1 + \frac{y_1}{1+y_1^2} + y_2 + y_2^2\right),$$

$$G(y_1) = (1+y_1)^2,$$

$$F(y_2) = \left(\frac{y_2}{1+y_2^2} + 2\right).$$

It is easy to check that the hypothesis in Theorem 1 and Theorem 2 are satisfied since

$$\frac{\Phi(y_1, y_2)}{y_2} \geq 1 = m$$

$$\frac{\Phi(y_1, y_2)}{y_1} \geq 1 = \nu$$

$$|G(y_1)| \geq 1 = q$$

$$|F(y_2)| \geq 2 = n$$

and if we pick $l_1 = \frac{3}{5}$ and $l_2 = \frac{1}{10}$, then the inequalities in (2) are also satisfied since $y_1 \neq 0$.

Hence, this shows that all the conditions of Theorem 1 and Theorem 2 are satisfied. Thus, we conclude that analyzed state parameters (y_1, y_2) describing the system (7) are asymptotically stable and ultimately bounded.

5 Stability and Boundedness Analysis of Nonlinear Prey-Predator System (7)

1. In Figure 1 and Figure 2, the population densities (y_1, y_2) of prey and predator respectively defining the prey-predator model (7) are asymptotically stable if the conditions of Theorem 1 are satisfied. In Figure 1, the population density of prey increases as a result of refuge from predators and and then remain stable as $t \to \infty$. In Figure 2, the population density of predators decreases initially as a result of prey refuge and then increases for a time due to the increase in prey population which remain stable as $t \to \infty$.

2. In Figure 3, visualizing how the trajectories of system (7) satisfying the conditions of Theorem 1 tends towards $(0, 0)$. Thus the state parameters (y_1, y_2) defining (7) are asymptotically stable as $t \to \infty$.

3. In Figure 3 and Figure 4, the prey and predator densities (y_1, y_2) describing the system (7) are ultimately bounded as $t \to \infty$. Prey refuge and predator cannibalism significantly affect dynamic behaviours of the system. The result obtained shows that the prey refuge and predator cannibalism must be controlled or restricted in order to achieve a stabilized dynamic system.

Conclusion

The results obtained show that prey refuge can lead to prey cannibalism in the long run if the ratio of prey density to growth rate and the ratio of prey density to intraspecific competition of prey is not controlled. Thus, the conditions obtained then serve as a stabilizing mechanism in the prey-predator system (7). Also, prey refuge can in the long run lead to prey cannibalism and predator cannibalism which can lead to extinction if prey and predator densities are not bounded by a single constant. Thus, the prey refuge and predator cannibalism must be controlled. The result obtained shows that even with the presence of prey refuge and predator cannibalism the two species will be persistent.

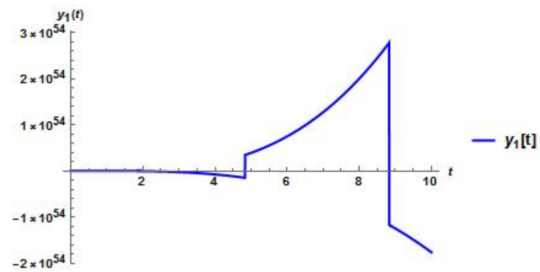

Figure 1: The parameter $y_1(t)$ in (blue) of (7) satisfying all the conditions of Theorem 1 as $t \to \infty$.

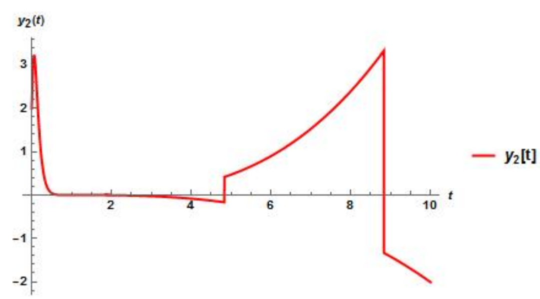

Figure 2: The parameter $y_2(t)$ in (red) of (7) satisfying all the conditions of Theorem 1 for as $t \to \infty$.

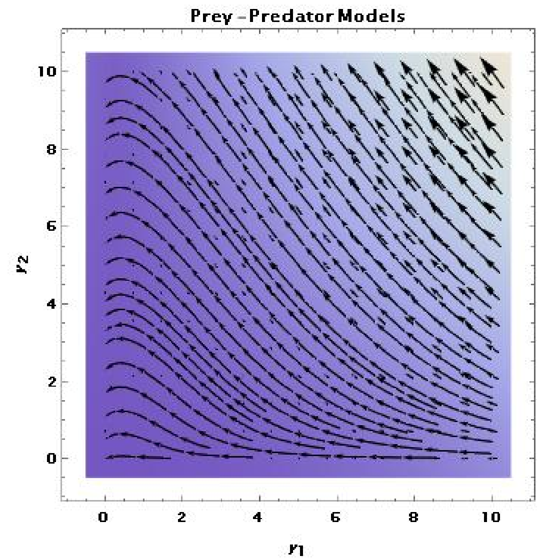

Figure 3: Visualizing how the solution paths satisfying the conditions of Theorem 1 for the stability of $y_1(t)$, $y_2(t)$ in (7) converge to $(0, 0)$.

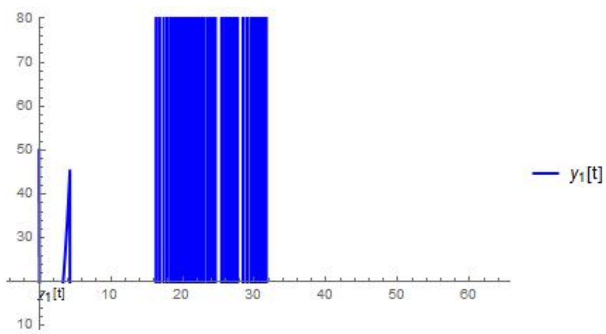

Figure 4: The parameter $y_1(t)$ in (blue) of (7) satisfying the conditions of Theorem 1 and Theorem 2 for is ultimately bounded by a single constant as $t \to \infty$.

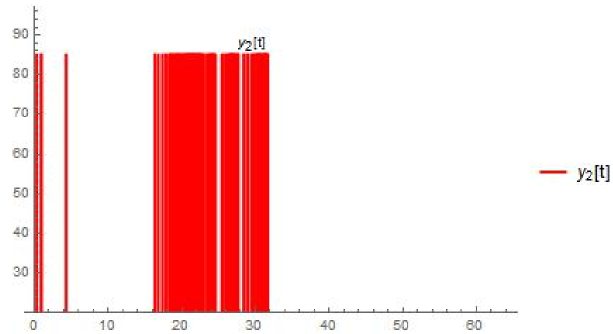

Figure 5: The parameter $y_2(t)$ in (red) of (7) satisfying the conditions of Theorem 1 and Theorem 2 for is ultimately bounded by a single constant as $t \to \infty$.

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