**TOPOLOGICAL DYNAMICAL SYSTEMS: SOME TENETS OF TRANSFORMATION BASED ON ITERATION AND HOMEOMORPHISMS**

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**ABSTRACT**

This work was motivated by the fact that using topological dynamical properties needs care and understanding of terms. Frequently asked questions in the said area urged the beginning of the paper and so it suffices to delve into some tenets. This work was purported to introduce some terms and ideas that would deepen interest in topological dynamical systems and their usage.In achieving the aims, treatment was given to fundamental issues by starting from iterations which results in dynamical systems.The outcomes were actually in the form of bringing to the fore knowledge ofhomeomorphisms, iteration and transformation.Examples were given to buttress the contents of the text. The result is important in that it will drum home the tenets of dynamical systems.

**Keywords**: topological dynamical systems, topological dynamical properties, tenets of transformation, iteration, homeomorphisms, transformation

**1.0 Introduction**

Dynamical Systems comprise Topological Dynamical Systems, Measure-Theoretic Preserving Systems or Ergodic Theory and Differentiable Dynamical Systems or Smooth Dynamical Systems. However, emphasis will be placed on Topological Dynamical Systems (Dontwi, Obeng – Denteh, Manu, Nyarko, 2013). The vicinity of mathematics known as dynamical systems has seen giant strides and growth over the years. A dynamical system is a meticulous type of function used to model time-varying processes. Instances of such processes show up in fluid mechanics, population growth, celestial mechanics, cardiac behaviour, and a myriad of scenarios where a physical system alters over time. In the study of a specific dynamic system, qualitative tools and techniques are often engaged, and concepts from topology often motivate these qualitative methods (Adams and Fransoza, 2008). Dynamical systems theory attempts to understand, or at least describe, the changes over time that occur in physical and artificial "systems". Examples of such systems include the solar system which is made up the sun and planets, the weather, the motion of billiard balls on a billiard table, sugar dissolving in a cup of coffee, the growth of crystals, the stock market, the formation of traffic jams, the behavior of the decimal digits of the square root of 2 (Hochman, …).

Henri Poincare (1854-1912), considered as one of the leading figures in the development of topology in the late 1800s and 1900s is also in the main regarded as the chief originator of the field of dynamical systems. Poincare studied the three-body problem, modelling the positions and velocities of three bodies in motion under each other’s gravitational sway. Because general solution formulas for the associated differential equations are difficult to obtain, he took the novel approach of qualitatively studying their structure within the space in which they are defined. Thus the field of dynamical systems was born (Adams and Fransoza, 2008).

A current definition of topological dynamics says that ‘it is the study of transformation groups with respect to those topological properties whose prototype occurred in classical dynamics’. If in this definition the adjective ‘topological’ is replaced by ‘measure-theoretic’ then one obtains a description of Ergodic Theory. Similarly, ‘differentiable’ or ‘smooth’ instead of ‘topological’ gives a description of Differentiable Dynamics. Thus in each of these three fields of mathematical research one studies groups and also semi-groups of transformations of a space that preserve the structure of the space which could be a topological, or a measurable, or a differentiable structure (de Vries, 1993).

The innermost entity of learning in topological dynamics is a topological dynamical system, namely, a [topological space](http://en.wikipedia.org/wiki/Topological_space) that is, together with a [continuous transformation](http://en.wikipedia.org/wiki/Continuous_map_%28topology%29), a continuous flow, or more generally, a [semigroup](http://en.wikipedia.org/wiki/Transformation_semigroup) of continuous transformations of that space. The beginning of topological dynamics stretch out in the learning of asymptotic properties of [trajectories](http://en.wikipedia.org/wiki/Trajectory) of systems of [autonomous](http://en.wikipedia.org/wiki/Autonomous_system_%28mathematics%29) [ordinary differential equations](http://en.wikipedia.org/wiki/Ordinary_differential_equation), in particular, the behaviour of [limit sets](http://en.wikipedia.org/wiki/Limit_set) and various manifestations of "repetitiveness" of the motion, such as periodic trajectories, recurrence and minimality, stability, [non-wandering points](http://en.wikipedia.org/wiki/Non-wandering_point). [George Birkhoff](http://en.wikipedia.org/wiki/George_Birkhoff) is considered to be the founder of the field. A structure theorem for minimal distal flows proved by [Hillel Furstenberg](http://en.wikipedia.org/wiki/Hillel_Furstenberg) in the early 1960s inspired much work on classification of minimal flows. A lot of research in the 1970s and 1980s was devoted to topological dynamics of one-dimensional maps, in particular, [piecewise linear](http://en.wikipedia.org/wiki/Piecewise_linear_function) self-maps of the interval and the circle (Topological Dynamics, 2013).

In [topology](http://en.wikipedia.org/wiki/Topology) and related branches of [mathematics](http://en.wikipedia.org/wiki/Mathematics), a topological space is a [set](http://en.wikipedia.org/wiki/Set_%28mathematics%29) of [points](http://en.wikipedia.org/wiki/Point_%28geometry%29), along with a set of [neighbourhoods](http://en.wikipedia.org/wiki/Neighbourhood_%28mathematics%29) for each point, that satisfy a set of [axioms](http://en.wikipedia.org/wiki/Axiom#Non-logical_axioms) relating points and neighbourhoods. The definition of a topological space relies only upon [set theory](http://en.wikipedia.org/wiki/Set_theory) and is the most general notion of a [mathematical space](http://en.wikipedia.org/wiki/Space_%28mathematics%29) that allows for the definition of concepts such as [continuity](http://en.wikipedia.org/wiki/Continuous_function_%28topology%29), [connectedness](http://en.wikipedia.org/wiki/Connected_space), and [convergence](http://en.wikipedia.org/wiki/Limit_of_a_sequence) (Topological space, 2013).

**2.0 Preliminary Notes**

A dynamical contains a space X, also called the phase-space, which is explained as the set of all possible states of the physical system. Added to that there is a ‘rule of evolution’ which describes how any state assumed by the system alters with time. Let $π(t,x)$ represent the state of the system reached after a time interval of length t when it starts in state x where $t \geq 0$ and $x \in X.$ Then $π(s,π(t,x))$ is the state reached after a time interval $s+t$ when starting in x, i.e.,

$$π\left(s,π\left(t,x\right)\right)=π(s+t,x)). (1)$$

By the definition of $π$

$$π\left(0,x\right)=x (2)$$

 For continuous systems $s\geq 0$ and $t\geq 0$.

For many systems too time can be reversed so that negative values of s and t are used. Then a mapping $π:R ×X \rightarrow X for all s,t \in R and x\in X. $Such a mapping is called an action of (the additive group) R on X.

**3. 0 Transformation**

Let *X* be a non-empty set (Simmons, 1963) and refer to the elements as points and *T:*:*X* →*X* is a transformation, function or operator. Having defined a certain type of structure on a set and having thus arrived at a mathematical system, the introduction of certain concepts and properties are done and then a study is made of the relations that exist between them. These entities are called *invariants*- which remain unchanged under one-one structure-preserving mapping called *isomorphisms* in algebra and *homeomorphisms* in topology (Thron, 1966).

**4.0 Dynamical systems from Iterations**

The set *X* can be thought of as a state spacefor some system and *T* as the evolution of some discrete deterministic (autonomous) dynamics on *X* : if *x* is a point in *X* , denoting the current state of a system, then *Tx* can be interpreted as the state of the system after one unit of time has elapsed.

Referring to *X* and *T* , we define the iterates *Tn : X*→*X* for a every non-negative integer n, actually induces a representation of either the additive semi-group Z+ or the additive group Z (Katok and Hassleblatt, 1995) and this is the mathematical manifestation of time from the dynamical facet. In terms of continuous time evolution, the real line can be considered. Assuming *T* to be invertible, the pair (*X,T*) is known as a cyclic dynamical system or without any loss of generality a dynamical system.

**Definition 4.1.** The dynamical system defined by $f:X \rightarrow X$ is the family of functions $\left\{f^{n}\right\}\_{n \in Z\_{+}}$, with each $f^{n}$ mapping X to X (Adams and Fransoza, 2008)..

**Definition 4.2.** The functions$f:X \rightarrow X$ and $g:Y \rightarrow Y$ (and the dynamical systems defined by them) are said to be topologically conjugate if there exists a homeomorphism $h:X \rightarrow Y$ such that $g ° h=h ° f$. The function h is called topological conjugacy between $f $and $g$ (Adams and Fransoza, 2008).

**Explanation 4.3.** Let *X* and *Y* be topological spaces. A continuous map $f:X \rightarrow X$

is a *homeomorphism* if it is one-to-one and the inverse is continuous.

**Explanation 4.4.** A *topological dynamical system* is a topological space *X* and either a continuous map$ f:X \rightarrow X$ or a continuous (semi)flow on *X*, i.e., a (semi)flow for which the underlying map is continuous (Brin and Stuck, 2002).

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**5.0 Main thrust and concluding remarks**

The main thrust and concluding remarks are enshrined in the ensuing buttressing examples structured in order to throw adequate light and colour in the concepts of topological dynamical systems.

**5.1 Buttressing Examples**

**5.2.** The dynamics of the functions$f\left(x\right)=3x$ and $g\left(x\right)=4x$ come into view qualitatively and are identical (See Definitions 4.1, 4.2). In both scenarios, there is a fixed point at 0 and all other orbits hang about either on the positive or negative side of 0 and be in motion outward from 0. A fixed point is obtained by setting $f\left(x\right)=x.$ In actuality, these two functions are topologically conjugate(See Definitions 4.1, 4.2). The functions $h:R \rightarrow R,$ defined by $h\left(x\right)= x^{log\_{3^{(4)}}}$is a homeomorphism(See Explanations 4.3, 4.4) that satisfies $g ° h=h ° f$. The orbit of $x$ $ $under $f$ is the sequence$ \left(x, f\left(x\right), f^{2}\left(x\right),…,f^{n}\left(x\right),…\right)$ and is denoted by $O\left(x\right).$

**5. 2**. Considering Finite Systems, $X=\{1, 2, 3, 4, 5\}$is a finite set, and *T* : *X* → *X* is a permutation on *X* (Dontwi *et al*, 2013).

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